CIP-groups

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Introduction

We say that a group G has the "congruence intersection property", or more shortly that the group is a CIP group, if

$$H^G \cap K^G = (H \cap K)^G$$

for all $H, K \leq G$.

The term "congruence intersection property" originates from universal algebra [6]. The interested reader is referred to this paper for a more general definition. As we will see, CIP groups are closely related to Dedekind groups. A group is a Dedekind group if every subgroup is normal. These groups have been classified [3,4]. A group G is a Dedekind group if and only if G is either abelian or the direct product of a quaternion group of order 8 and an abelian torsion group without elements of order 4. It is obvious that every Dedekind group is a CIP group. Whether the converse holds is still an open question. Our main result is the following.

Theorem 1 *If G is a CIP group that is not a Dedekind group then G has a factor N with the following properties:*

1. *N is torsion free.*
2. *If $a, b \in N \setminus \{1\}$ then $\langle a \rangle \cap \langle b \rangle \neq \{1\}$.*

3. N is simple.

In Section 2 we will in fact get more detailed information about the structure of G, given that it exists. Clearly a group N with the properties above is a CIP group. The theorem therefore implies that the existence of a non-Dedekind CIP group is equivalent to the existence of a group N with those properties above. Both Adian[1,2] and Ol'shanskii[5] have constructed a nonabelian group with properties 1 and 2. These groups are however not simple. The author has spoken to Ol'shanskii who believes that such a group exists. But as far as we know a construction of such a group has not yet been made.

1 CIP groups which are finitely generated or torsion groups

In this section we will show that CIP groups that are either finitely generated or torsion groups are Dedekind groups. We will denote the normal closure of a subgroup H by $\overline{H}$.

Lemma 1 *Let G be a CIP group. If $a, b \in G$ do not commute then $\langle a \rangle \cap \langle b \rangle \neq \{1\}$.*

Proof We have $1 \neq [a, b] \in \overline{\langle a \rangle} \cap \overline{\langle b \rangle} = \overline{\langle a \rangle \cap \langle b \rangle}$. Therefore we must have $\langle a \rangle \cap \langle b \rangle \neq \{1\}$. □

Lemma 2 *If a, b are elements in a CIP group and $a \neq 1$, then there is an $n \in \mathbb{N}$ such that a^n commutes with b and $a^n \neq 1$.*

Proof If a and b commute then we can take $n = 1$. Suppose then that they do not commute. By Lemma 1 we have that $1 \neq a^n \in \langle a \rangle \cap \langle b \rangle$ for some $n \in \mathbb{N}$. □

Lemma 3 *If G is a CIP group and $a \in G$ has order p, where p is a prime, then $a \in Z(G)$.*

Proof Let $b \in G$. By Lemma 2 we have that $a^n = a^{nb}$ for some $n \not\equiv 0 \pmod{p}$. Let m be the inverse of n modulo p. Then $a = a^{nm} = a^{nmb} = a^b$. So a commutes with b. □

Proposition 1 *Let G be a CIP group. If $a \in G$ is of finite order then $\overline{\langle a \rangle} = \langle a \rangle$.*

Proof We prove this by induction on the order of a. This is true if the order is a prime by Lemma 3. Now suppose a has order $n = mp$ with $m > 1$ and suppose that the proposition is true when a is of smaller order. Then $a^m \in Z(G)$ by Lemma 3. Now consider $H = G/\langle a^m \rangle$. This is also a CIP group (easy to see) and we therefore have from the induction hypothesis that

$$\overline{\langle a \rangle}/\langle a^m \rangle = \langle a \rangle / \langle a^m \rangle$$

which implies that $\overline{\langle a \rangle} = \langle a \rangle$. □

Proposition 2 *If G is a finitely generated CIP group then also $\overline{\langle a \rangle} = \langle a \rangle$ when a is of infinite order.*

Proof Suppose $G = \langle b_1, b_2, \ldots, b_m \rangle$. By Lemma 2 there are integers $n_1, \ldots, n_m$ such that a^{n_i} commutes with b_i. Let $n = n_1 n_2 \cdots n_m$ then $a^n \in Z(G)$. Now consider $H = G/\langle a^n \rangle$. Since H is a CIP group and a is of finite order modulo $\langle a^n \rangle$ we get from Proposition 1 that $\overline{\langle a \rangle}/\langle a^n \rangle = \langle a \rangle / \langle a^n \rangle$ which implies that $\overline{\langle a \rangle} = \langle a \rangle$. □

It follows from Proposition 1 and Proposition 2 that torsion CIP groups and finitely generated CIP groups are Dedekind groups. It is also clear that every Dedekind group is a CIP group.

2 Non-Dedekind CIP groups

Let us now assume that G is a CIP group which is not a Dedekind group. In this section we will prove Theorem 1. We will divide the proof into few simple steps.

Step 1. All torsion elements of G are in $Z(G)$.

Proof Since G is not a Dedekind group it must contain an element a of infinite order by Proposition 1.

We first show that the torsion elements commute with elements of infinite order. So suppose b is of finite order and c of infinite order. If b and c do not commute then we have by Lemma 1 that $\langle b\rangle \cap \langle c\rangle \neq 1$ which is absurd since the intersection would contain an element which is both of finite order and infinite order.

Now suppose b, c are of finite order. By last paragraph we have that c and a commute which implies that ca is of infinite order. Then b commutes with both ca and a and then also $c = ca \cdot a^{-1}$.

It follows that all torsion elements are in $Z(G)$. □

Step 2. By step 1 the torsion elements form a group $\tau(G)$. Let $H = G/\tau(G)$. H is a torsion-free CIP group that is not a Dedekind group.

Proof It is clear that H is a torsion-free CIP group. Let us show that it is not a Dedekind group.

Suppose H is a Dedekind group. Let $a, b \in H$ and suppose $a^b = a^r$. By Lemma 2 there is an $n \in \mathbb{N}$ with $a^{nb} = a^n$. Then

$$a^n = a^{nb} = a^{nr}.$$

But since a is of infinite order this implies that $r = 1$ and thus H must be abelian if it is a Dedekind group.

Since G is not a Dedekind group we must have some non-commuting elements $c, d \in G$. By Step 1 we have that c, d are both of infinite order. Then

$$1 \neq [c,d] \in \overline{\langle [c,d]\rangle} \cap \overline{\langle c\rangle} = \overline{\langle [c,d]\rangle \cap \langle c\rangle}$$

which implies that $\langle [c,d]\rangle \cap \langle c\rangle \neq \{1\}$. Therefore $[c,d]$ is of infinite order since c is of infinite order. But if H is a Dedekind group, and therefore abelian, we have $[c,d] \in \tau(G)$. Therefore H is not a Dedekind group. □

We remind the reader that a group is an Engel group if for each ordered pair (x, y) of elements in the group there is a positive integer $n(x, y)$ such that $[\underbrace{x, \cdots, [x, [x}_{n(x,y)}, y]]] = 1$.

Step 3. H is not an Engel group.

Proof Since H is not a Dedekind group there are $a, b \in H$ such that $[a, b] \neq 1$. If H was an Engel group there would be an integer $n \geq 1$ such that

$$c := [\underbrace{a, \dots, [a, [a}_{n}, b]]] \neq 1$$

but with $[a, c] = 1$. By Lemma 2 there is an $m \in \mathbb{N}$ such that

$$[a^m, [\underbrace{a, \dots, [a, [a}_{n-1}, b]]]] = 1.$$

But it is easy to see that the left hand side is equal to $c^{a^{m-1}+\cdots+a+1}$. Since a commutes with c, it follows that

$$1 = c^{a^{m-1}+\cdots+a+1} = c^m$$

which is absurd since H is torsion free. □

Step 4. H has the property that every two (non-trivial) cyclic subgroups intersect non-trivially.

Proof Let $I = H \setminus \{1\}$. We define an equivalence relation on I as follows

$$a \sim b \text{ iff } \langle a \rangle \cap \langle b \rangle \neq \{1\}.$$

Let us first see why this is an equivalence relation. The only thing that is non-trivial is the transitivity property. Suppose $a \sim b$ and $b \sim c$, say $\langle a \rangle \cap \langle b \rangle = \langle b^r \rangle$ and $\langle b \rangle \cap \langle c \rangle = \langle b^s \rangle$, then $\langle b^{rs} \rangle \subseteq \langle a \rangle \cap \langle b \rangle \cap \langle c \rangle$. Since b is of infinite order this implies that $\langle a \rangle \cap \langle c \rangle \neq \{1\}$, that is $a \sim c$.

Suppose $c \sim d$ and $c^r \in \langle d \rangle$. We will show that either $cd \sim d$ or $cd = 1$. If c commutes with d then

$$(cd)^r = c^r d^r \in \langle d \rangle$$

so $cd \sim d$ or $cd = 1$, since H is torsion free. If c does not commute with d then $[cd, d] \neq 1$ and Lemma 1 implies that $cd \sim d$.

Now take some non-commuting elements $a, b \in I$. Let $[a]$ be the equivalence class of a and let $K = [a] \cup \{1\}$. It is obvious that an inverse of an element in K is in K. It then follows from last paragraph that K is a subgroup of H.

We now show that $K = H$. Since $[a, b] \neq 1$ we have by Lemma 1 that $a \sim b$. Suppose $c \in H \setminus K$ then we have that c commutes with a by Lemma 1 ($a \in K \Rightarrow \langle a \rangle \cap \langle c \rangle = \{1\}$). It follows that a does not commute with bc. Then it follows from Lemma 1 that $bc \sim a$ so $bc \in K$. It follows that $c = b^{-1} \cdot bc \in K$ which is a contradiction.

Therefore $H = K$ and has therefore the claimed property. □

Step 5. Let $N = \bigcap_{a \in H \setminus \{1\}} \overline{\langle a \rangle}$ then $N \neq \{1\}$.

Proof Let $b \in H \setminus \{1\}$ and consider $H/\overline{\langle b \rangle}$. Since $\langle a \rangle \cap \langle b \rangle \neq \{1\}$ for all $a \in H \setminus \{1\}$ we have that $H/\overline{\langle b \rangle}$ is a torsion CIP group. By Proposition 1 it is then a Dedekind group. Let $a, c \in H$ since $H/\overline{\langle b \rangle}$ is a Dedekind group we have

$$a^c \equiv a^r \pmod{\overline{\langle b \rangle}}$$

for some r and thus $[a, [a, c]] \equiv [a, a^{r-1}] \equiv 1 \pmod{\overline{\langle b \rangle}}$.

By last paragraph $[a, [a, c]] \in \overline{\langle b \rangle}$ for all $b \in H \setminus \{1\}$. It follows that $[a, [a, c]] \in N$. So H/N is an 2-Engel group. It then follows from Step 3 that $N \neq \{1\}$. □

Step 6. For every element $a \in N$ we have that $H = C_H(a)N$.

Proof We have

$$H = \prod_{b \in H} \overline{\langle b \rangle} = \prod_{b \in H} \overline{\langle b \rangle} \cap \overline{\langle b^a \rangle} = \prod_{b \in H} \overline{\langle b \rangle \cap \langle b^a \rangle}.$$

Suppose that $\langle b \rangle \cap \langle b^a \rangle = \langle b^{n_a(b)} \rangle$. We first show that $b^{n_a(b)}$ commutes with a for all $b \in H$. Suppose $b^{n_a(b)} = b^{ra}$. By Lemma 2 there is an $m \in \mathbb{N}$ such that $b^{ma} = b^m$. Then (if $b \neq 1$)

$$b^{mn_a(b)} = b^{rma} = b^{mr}$$

which implies that $r = n_a(b)$. Let $A = \langle b^{n_a(b)} : b \in H \rangle$. We have proved that every element of A commutes with a and that $\overline{A} = H$.

Since N is a minimal normal subgroup of H and H/N is a Dedekind group, we have $\overline{A} = AN$. Hence $H = \overline{A} = AN$.

Theorem 1 *If G is a CIP group that is not a Dedekind group then G has factor N with the following properties:*

1. *N is torsion free.*
2. *If $a, b \in N \setminus \{1\}$ then $\langle a \rangle \cap \langle b \rangle \neq \{1\}$.*
3. *N is simple.*

Proof Since N is a non-trivial subgroup of H we have that 1 and 2 follow form steps 2 and 4. Let us prove 3.

Let $a \in N \setminus \{1\}$. By Step 6 we have that $H = C_H(a)N$. Then since N is a minimal normal subgroup we have $N = a^H$. Therefore

$$N = a^H = a^{C_H(a)N} = a^N.$$

Since this is true for all $a \in N \setminus \{1\}$ we have that N is simple. □

3 Simple subgroups of groups with the intersection property

From now on suppose G is a group satisfying property 2 in Theorem 1. If such a group is finitely generated and torsion free it can't have a simple subgroup. If $G = \langle a_1, \ldots, a_m \rangle$ then $\bigcap \langle a_i \rangle$ would be an abelian normal subgroup. Because of the intersection property 2 this would imply that every simple subgroup would be contained in this abelian normal subgroup which is absurd. It is therefore clear that if G is torsion free and contains a simple subgroup then G must be infinitely generated.

Proposition 3 *If $\{H_i : i \in I\}$ is a family of simple subgroups of G. Then the join of them $\bigvee_{i \in I} H_i$ is also a simple subgroup.*

Proof Let $H = \bigvee_{i \in I} H_i$. Suppose R is a non-trivial normal subgroup of H. Then $H_i \cap R$ is a non-trivial normal subgroup of H_i for all $i \in I$. Therefore $H_i \leq R$ for all $i \in I$ and thus $H = R$. □

Proposition 4 *If G has a simple subgroup then G has a unique maximal simple subgroup $S(G)$. Furthermore, $S(G)$ is the unique minimal normal subgroup of G.*

Proof Let L be a simple subgroup of G and R be a normal subgroup. Since $L \cap R$ is a non-trivial normal subgroup of L we have that $L \leq R$.

By assumption there exists a simple subgroup. Let $S(G)$ be its normal closure. By Proposition 3 we have that $S(G)$ is simple. By the last paragraph we have that $S(G)$ is contained in every normal subgroup so it is the minimal normal subgroup. It also follows from the last paragraph that $S(G)$ contains all simple subgroups (since it is normal) and thus it is the unique maximal simple subgroup. □

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