A polynomial upper bound for the nilpotency classes of Engel-3 Lie algebras over a field of characteristic 2

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1 Introduction

A Lie algebra L is called an Engel-n Lie algebra if it satisfies the additional condition that $\mathrm{ad}(b)^n = 0$ for all b. Engel Lie algebras are related to the "restricted Burnside problem". This can be stated as follows: Given integers n and r, is it true that there is an upper bound on the orders of finite r-generator groups of exponent n?

The answer to this question is yes. In 1959 P. Hall and G. Higman[**1**] made, given some assumptions about finite simple groups, the following reduction. "It is sufficient to look at $B(r, n)$ where n is a power of a prime." Here $B(r, n)$ is the (relatively) free r-generator group of exponent n. From the classification of finite simple groups we have that the assumptions of Hall and Higman are valid.

The relationship with Lie algebras comes from the equivalence of the following two statements.

1. There is a largest finite r-generator group of exponent p^m;
2. The associated Lie-ring of $B(r, p^m)$ is nilpotent.

In 1959 Kostrikin[**3**] showed that statement 2 is true when $m = 1$ and therefore solved the restricted Burnside problem for groups of prime exponent. Here, in fact, the associated Lie algebra is an Engel-(p-1) Lie algebra over a field of characteristic p. It was then not until 1989 that E. I.

Zel'manov[**10,11**] completed the solution of the restricted Burnside problem by showing that certain class of Lie algebras are locally nilpotent. This class not only contains the Lie algebras in statement 2 for all prime powers but also all Engel-n Lie algebras over a field. So as a corollary to Zelmanov's proof we have that Engel-n Lie algebras over fields are locally nilpotent. For a more detailed discussion of the Burnside problem we refer to [**4,7**].

The natural question that now arises is what can be said about the nilpotency class of finitely generated Engel-n Lie algebras. How does the nilpotency class depend on the number of generators r and on n? In [**8**] Zel'manov and Vaughan-Lee give upper bounds. Before we state their results we introduce some notation. Define a function $T : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by induction in the following way: $T(m, 1) = m$, $T(m, r + 1) = m^{T(m,r)}$. Let L be an Engel-n Lie algebra generated by r elements. It follows from the work of Zel'manov and Vaughan-Lee that L is nilpotent of class at most $T(r, n^{n^n})$. When the characteristic of the field is greater than n they get smaller bounds. So if $25 \leq n < p$ then L is nilpotent of class at most $T(r, 2^n)$ and if $26 > n < p$ we have that L is nilpotent of class at most $T(r, 3^n)$. The authors nevertheless believe that these bounds are too high and make the conjecture that the class can always be bounded by a function which is polynomial in r.

There is still not much evidence for this conjecture to be true. But we have some supporting facts. From a theorem of Zel'manov[**9**] we have that for each n there is a constant n_0 such that every Engel-n Lie algebra over a field k with $\operatorname{char} k > n_0$ or $\operatorname{char} k = 0$ is nilpotent. Here the nilpotency class does not depend on r, so we have a constant upper bound. We then have some detailed information about Engel-n Lie algebras for small values of n. It is well known that Lie algebras satisfying the Engel-2 identity are nilpotent of class at most 3. In [**5**] it is shown that Engel-3 Lie algebras with $\operatorname{char} k \neq 2, 5$ are nilpotent of class at most 4. In that paper it is also shown that for Engel-4 Lie algebras of characteristic $\neq 2, 3, 5$ we have that the class is at most 7. For all these values of n it is known that we have a linear upper bound in r whenever $p \geq n$. For $n = 3$ we have that the class c is not more than $2r$ when $p = 5$[**5**] and for Engel-4 Lie algebras we have $c \leq 3r$ when $p = 3$[**5**] but $c \leq 6r$ when $p = 5$[**2**]. For Engel-5 Lie algebras we have also linear upper bounds when $\operatorname{char} k \geq 5$[**6**]. It may well be that this is true for all n.

When the characteristic p is less than n things are more complicated. In this article we will find a polynomial upper bound fo the nilpotency class of

Engel-3 Lie algebras over a field of characteristic 2. This can be used to get similar results for Engel-4 Lie algebras over a field of characteristic 2.

2 Some preliminaries

Let L be an Engel-3 Lie algebra generated by $\{e_1, \ldots, e_r\}$ over a field k of characteristic 2. For $a \in L$ we have the adjoint operator $\mathrm{ad}(a)$, defined as follows. If $v \in L$ then $v \cdot \mathrm{ad}(a) = va$. It is easy to see that the set of all adjoint operators is a Lie algebra with the Lie product given by $[\mathrm{ad}(a), \mathrm{ad}(b)] := \mathrm{ad}(a)\mathrm{ad}(b) - \mathrm{ad}(b)\mathrm{ad}(a) = \mathrm{ad}(ab)$. This Lie algebra is denoted $\mathrm{ad}(L)$. Let $A(L)$ be the associative algebra generated by $\mathrm{ad}(L)$ then we can think of $A(L)$ as a Lie algebra with the Lie product $[u, v] := uv - vu$. Note that the Lie product of two elements of $\mathrm{ad}(L)$ is again in $\mathrm{ad}(L)$. The Engel-3 identity gives us that $a^3 = 0$ for all $a \in \mathrm{ad}(L)$. This gives

$$0 = (\lambda a + b)^3 = \lambda(ab^2 + bab + b^2a) + \lambda^2(a^2b + aba + ba^2),$$

and

$$0 = \lambda(a + b)^3 = \lambda(ab^2 + bab + b^2a) + \lambda(a^2b + aba + ba^2).$$

In particular

$$a^2b + aba + ba^2 + ab^2 + bab + b^2a = 0,$$

and when $|k| > 2$ we can choose λ not equal to 0 or 1 which gives

$$a^2b + aba + ba^2 = 0 \quad , \quad ab^2 + bab + b^2a = 0.$$

But we do not want to exclude the case $k = Z_2$. Therefore we shall only be assuming that the first of these three identities holds. In the case when $|k| > 2$ everything simplifies and we can in fact get stronger results. We will come back to this later.

3 A polynomial upper bound

In [**5**] there is an example of an Engel-3 Lie algebra which shows that $\mathrm{Id}\langle x \rangle$ need not be nilpotent when $\mathrm{char}\, k = 2$. It is therefore unlikely that there is an linear upper bound for the nilpotency class. In this section we shall show that we can still get an upper bound for the nilpotency class which is a polynomial in the number of generators.

Let L be an Engel-3 Lie algebra generated by $\{e_1, \ldots, e_r\}$ over a field k of characteristic 2. We will be working in the associative algebra $A(L)$ generated by $\mathrm{ad}(L)$.

Lemma 1 *Let $a, b \in ad(L)$ then*

$$b^2ab + bab^2 = 0, \qquad b^2ab^2 = 0.$$

Proof We have

$$0 = [a, b, b, b] = bab^2 + b^2ab.$$

We get the second identity by multiplying this identity with b from the left. □

Lemma 2 *Let $a, b \in$ ad (L) then*

$$b^2a^2b^2 = 0.$$

Proof We have from the partial linearization of the Engel identity

$$\begin{aligned}(ab^2 + bab + b^2a)a &= (ba^2 + aba + a^2b)a \\ &= aba^2 + a^2ba \\ &= [b, a, a, a] \\ &= 0,\end{aligned}$$

and similarly $a(ab^2 + bab + b^2a) = 0$. So

$$ab^2a + baba + b^2a^2 = 0, \tag{1}$$

$$ab^2a + abab + a^2b^2 = 0. \tag{2}$$

If we interchange a and b in (1) we get from (2)

$$ab^2a = ba^2b. \tag{3}$$

Then using (1),(2) and (3) we have

$$[a,b]^2 = abab + ab^2a + baba + ba^2b = a^2b^2 + b^2a^2, \tag{4}$$

and then

$$\begin{aligned}
[a,b]b[a,b] &= ab^2ab + babab + bab^2a \\
&= b^2a^2b + bab^2a \quad \text{(by (1))} \\
&= b \cdot 0 \qquad\qquad\quad \text{(by (3))} \\
&= 0.
\end{aligned}$$

It now follows from the identities above that

$$\begin{aligned}
0 &= ([a,b]^2b + [a,b]b[a,b] + b[a,b]^2 + [a,b]b^2 \\
&\quad + b[a,b]b + b^2[a,b])b \\
&= b^2a^2b^2 + b^2ab^2 + b^2ab^2 \\
&= b^2a^2b^2
\end{aligned}$$

and we have the lemma. □

Lemma 3 *Let $a, b, c \in ad(L)$ then*

$$[b,c]ac^2 + c^2a[b,c] = 0, \qquad [b,c]a[b,c] = b^2ac^2 + c^2ab^2.$$

Proof From the Engel identity we have

$$\begin{aligned}
0 &= bc[a,c,c] + cb[a,c,c] + [a,c,c]bc + [a,c,c]cb \\
&\quad + b[a,c,c]c + c[a,c,c]b \\
&= bcac^2 + cbac^2 + cbc^2a + ac^2bc + c^2abc + c^2acb \\
&\quad + bc^2ac + cac^2b \\
&= [b,c]ac^2 + c^2a[b,c] + c(bc^2a + ac^2b) + (bc^2a + ac^2b)c.
\end{aligned}$$

But using Lemma 1 and the Engel identity we get

$$\begin{aligned}
c(bc^2a + ac^2b) + (bc^2a + ac^2b)c &= c^2(bca + acb) + (bca + acb)c^2 \\
&= c^2(bac + abc) + (cba + cab)c^2 \\
&= c^2[a,b]c + c[a,b]c^2 \\
&= 0.
\end{aligned}$$

So we have got the first identity. Now we have from Lemma 1

$$\begin{aligned}
0 &= (b+c)^2a(b+c)^2 \\
&= (b^2+c^2+[b,c])a(b^2+c^2+[b,c]) \\
&= b^2a[b,c]+[b,c]ab^2+c^2a[b,c]+[b,c]ac^2+[b,c]a[b,c] \\
&\quad +b^2ac^2+c^2ab^2+b^2ab^2+c^2ac^2 \\
&= [b,c]a[b,c]+b^2ac^2+c^2ab^2
\end{aligned}$$

which gives the second identity. □

Lemma 4 *Let* $x_1 = a^2$*,* $x_2 = [b,c]$ *and* $x_3 = d$ *where* $a,b,c,d \in ad(L)$ *then*

$$\sum_{\sigma \in \mathrm{Sym}(3)} x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)} = 0.$$

Proof Let $u = [b,c]$. From the Engel identity we have

$$\begin{aligned}
0 &= a[d,u,u]d+ad[d,u,u]+d[d,u,u]a+[d,u,u]da \\
&\quad +da[d,u,u]+[d,u,u]ad \\
&= ad^2u^2+au^2d^2+d^2u^2a+u^2d^2a+[d,u,u]ad+da[d,u,u]. \qquad (5)
\end{aligned}$$

But since $u^2 = b^2c^2+c^2b^2 = [b,c,c,b] \in \mathrm{ad}(L)$, we have

$$\begin{aligned}
0 &= (d+u^2)^2a(d+u^2)+(d+u^2)a(d+u^2)^2 \\
&= d^2ad+dad^2+[d,u,u]au^2+u^2a[d,u,u] \\
&\quad +[d,u,u]ad+da[d,u,u]+d^2au^2+u^2ad^2.
\end{aligned}$$

And because

$$\begin{aligned}
[d,u,u]au^2+u^2a[d,u,u] &= du^2au^2+u^2au^2d+u^2dau^2+u^2adu^2 \\
&= u^2[d,a]u^2 \\
&= 0
\end{aligned}$$

we have then by Lemma 3

$$
\begin{aligned}
[d,u,u]ad+da[d,u,u] &= d^2au^2+u^2ad^2 \\
&= [d,u]a[d,u],
\end{aligned}
$$

so (5) gives

$$0=a[d,u]^2+[d,u]^2a+[d,u]a[d,u].$$

From the Engel identity we now have

$$
\begin{aligned}
[d,u]a^2+a[d,u]a+a^2[d,u] &= a[d,u]^2+[d,u]^2a+[d,u]a[d,u] \\
&= 0.
\end{aligned}
$$

Since from (3) we have $a[d,u]a=da^2u+ua^2d$, the lemma now follows. $\square$

Lemma 5 *Let* $x_1=a^2$, $x_2=[b,c]$ *and* $x_3=d^2$ *where* $a,b,c,d\in ad\ (L)$ *then*

$$\sum_{\sigma\in\mathrm{Sym}(3)} x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}=0.$$

Proof Let $u=[b,c]$. Because $u^2\in\mathrm{ad}(L)$ we have by (2.6) and the Engel identity

$$
\begin{aligned}
[a,d]u^2+u^2[a,d]+u[a,d]u &= adu^2+dau^2+u^2ad+u^2da \\
&\quad +au^2d+du^2a \\
&= 0.
\end{aligned}
$$

Therefore by the Engel identity

$$
\begin{aligned}
u[a,d]^2+[a,d]u[a,d]+[a,d]^2u &= [a,d]u^2+u^2[a,d]+u[a,d]u \\
&= 0.
\end{aligned}
$$

But from Lemma 3 we have $[a,d]u[a,d]=a^2ud^2+d^2ua^2$ and therefore, using (4) for the second identity, we get

$$
\begin{aligned}
0 &= u[a,d]^2+[a,d]^2u+a^2ud^2+d^2ua^2 \\
&= ua^2d^2+ud^2a^2+a^2d^2u+d^2a^2u \\
&\quad +a^2ud^2+d^2ua^2
\end{aligned}
$$

and hence we have the lemma. □

Consider now the Lie product

$$u = [x_1, x_2, \ldots, x_n]$$

where $n \geq 2$ and $x_i = a_i$ or a_i^2 with $a_i \in$ad(L). We then have $u \in$ad(L). This follows from

$$[a^2, b] = [b, a, a] \text{ and } [a^2, b^2] = [a, b, b, a].$$

We prove the second identity. We have

$$\begin{aligned} [a, b, b, a] &= [a, b, b]a + a[a, b, b] \\ &= ab^2a + b^2a^2 + a^2b^2 + ab^2a \\ &= a^2b^2 + a^2a^2 \\ &= [a^2, b^2]. \end{aligned}$$

The first identity is easily proved by similar calculations. It is therefore clear that if $b \in$ad(L) then $b^2ub^2 = 0$ when $n \geq 2$. But from Lemma 2, we have that this is also true if $n = 1$. Note also the following fact, although we will not have to use it.

$$[b_1^2, b_2^2, \ldots, b_n^2] = [b_1, b_2, \ldots, b_n]^2$$

which follows by induction from $[a^2, b^2] = [a, b]^2$.

Proposition 1 *Let* $\mathbb{H}= \{x_1, x_2, \ldots, x_{2s}\} = \{a_1, \ldots, a_s, a_1^2, \ldots, a_s^2\}$ *where* $a_i \in$*ad*(L). *Suppose that*

$$t \geq 4s^3 + 2s^2 + s + 1,$$

that c *is some Lie product in ad*(L) *of length greater than one, and that* $u_1, u_2, \ldots u_j$ *are Lie products of elements in* $\mathbb{H}$. *If* $b \in$*ad*(L) *then all products of* $t+1$ b^2*'s and* $u_1, \ldots, u_j, c$ *are zero.*

Proof We prove this by induction on j. If $0 \leq j \leq 2t-2$ this is clearly true since $b^2u_ib^2 = 0$ and $b^4 = 0$. So assume $j \geq 2t-1$ and that the proposition is true when j is smaller. Now if c, u_i or u_i, u_j are adjacent then the product is symmetric in these two elements. This follows from

$$\cdots [c, u_i] \cdots = 0 \text{ and } \cdots [u_i, u_j] \cdots = 0 \quad \text{(by the inductive hypothesis)}$$

From lemmas 4 and 5 we have that the product is also symmetric in c, u_i if they are separated by b^2, since

$$\begin{aligned} 0 &= (\cdots u_i c b^2 \cdots) + (\cdots c u_i b^2 \cdots) + (\cdots b^2 c u_i \cdots) + (\cdots b^2 u_i c \cdots) \\ &\quad + (\cdots u_i b^2 c \cdots) + (\cdots c b^2 u_i \cdots) \\ &= 2(\cdots u_i c b^2 \cdots) + 2(\cdots b^2 c u_i \cdots) + (\cdots c b^2 u_i \cdots) + (\cdots u_i b^2 c \cdots) \\ &= (\cdots c b^2 u_i \cdots) - (\cdots u_i b^2 c). \end{aligned}$$

It follows that we can bring any two u's together without changing the value of the product. Here c acts as a "lift" to bring elements over b^2. As an example let us see how we can bring u_1 and u_5 together in the product $b^2 u_1 u_2 b^2 u_3 c b^2 u_4 u_5 b^2$.

$$\begin{aligned} b^2 u_1 u_2 b^2 u_3 c b^2 a_4 a_5 b^2 &= b^2 u_2 u_1 b^2 c u_3 b^2 u_4 u_5 b^2 \\ &= b^2 u_2 c b^2 u_1 u_3 b^2 u_4 u_5 b^2 \\ &= b^2 u_2 c b^2 u_3 u_1 b^2 u_4 u_5 b^2 \\ &= b^2 u_2 u_3 b^2 c u_1 b^2 u_4 u_5 b^2 \\ &= b^2 u_2 u_3 b^2 u_1 c b^2 u_5 u_4 b^2 \\ &= b^2 u_2 u_3 b^2 u_1 u_5 b^2 c u_4 b^2. \end{aligned}$$

We argue by contradiction and assume that there is non-zero product with j u's. We choose such a product with a maximal number of u's of length 1 and with a maximal number of u's of length more than 1 ending in a square. Now $u_1, u_2, \ldots, u_j$ have one of the following forms

$$x_f, \ [x_f, x_g], \ [x_f, x_g, x_h], \text{ or } [u, x_f, x_g, x_h]$$

where u is some Lie product of elements in $\mathbb{H}$. Since $j \geq (2s)^3 + (2s)^2 + (2s) + 1$, two of the products $u_1, \ldots, u_j$ must have the same three last elements (could be $\emptyset, x_i, x_j$ or $\emptyset, \emptyset, x_i$). Call these two elements v_1, v_2. We can now bring these elements together and get the subproduct

$$v_1 v_2 = [w_1, x_i][w_2, x_i].$$

We consider two cases.

Case 1. x_i is a square.

Then since $x_i u x_i = 0$ for all products u of elements in $\mathbb{H}$, we have that both w_1, w_2 must be non-empty if $v_1 v_2$ is not to be zero. So

$$v_1 v_2 = x_i w_1 w_2 x_i = [x_i, w_1, w_2] x_i$$

and we can replace v_1, v_2 with $[x_i, w_1, w_2], x_i$ and get a product with more u's of length one. Therefore the product which we chose must have been zero which is a contradiction.

Case 2. x_i is not a square.

Now if one of the w's were a square in $\mathbb{H}$, then $v_1 = [x_i, w_1]$ and we would have a product with more u's ending in a square. But then the product would have to be zero. We can therefore assume that the w's are either empty or in $\mathrm{ad}(L)$. If one of the w's is empty, and therefore both, we would have

$$v_1 v_2 = x_i^2.$$

But then we could replace v_1, v_2 with a square from $\mathbb{H}$. That is we would have a product with fewer u's and hence it would be zero by the induction hypothesis. We can therefore assume that both the w's are non-empty. If both w_1, w_2 have length ≤ 2 then $w_1 = w_2$ so

$$\begin{aligned} v_1 &= w_1 x_i w_1 x_i + x_i w_1 x_i w_1 + x_i w_1^2 x_i + w_1 x_i^2 w_1 \\ &= w_1^2 x_i^2 + x_i^2 w_1^2 \quad \text{(by (1),(2))} \\ &= [w_1, x_i^2, w_1] \end{aligned}$$

and we would have a product with fewer u's. We can therefore assume that at least one of the w's has length ≥ 3. Let us say it is w_2, the other case can be treated similarly. If $w_2 = [w, x_j]$ then

$$\begin{aligned} & x_i^2 w_2 + x_i w_2 x_i + w_2 x_i^2 = \\ & \qquad x_i^2 [w, x_j] + w x_i^2 x_j + x_j x_i^2 w + [w, x_j] x_i^2 = 0 \end{aligned} \tag{6}$$

where we are using (3) for the first identity and Lemmas 4 and 5 for the second. Now

$$v_1v_2 = w_1x_iw_2x_i + x_iw_1x_iw_2 + x_iw_1w_2x_i + w_1x_i^2w_2.$$

Then

$$\begin{aligned} w_2w_1x_i^2 &= [w_1,w_2]x_i^2 + w_1w_2x_i^2 \\ &= [w_1,w_2]x_i^2 + w_1x_i^2w_2 + w_1x_iw_2x_i \quad \text{(by (6))} \\ &= [w_1,w_2]x_i^2 + w_1x_i^2w_2 + w_1[w_2,x_i^2] \quad \text{(by (6))} \end{aligned}$$

and

$$\begin{aligned} w_1x_iw_2x_i + w_2x_iw_1x_i &= x_i[w_1,w_2]x_i + [w_1,w_2]x_i^2 \\ &= x_i^2[w_1,w_2] \end{aligned}$$

where the first identity follows from the Engel identity and the second is true for the same reason as (6). Therefore

$$\begin{aligned} [x_i,w_1,w_2]x_i &= x_iw_1w_2x_i + w_1x_iw_2x_i + w_2x_iw_1x_i + w_2w_1x_i^2 \\ &= x_iw_1w_2x_i + x_i^2[w_1,w_2] + [w_1,w_2]x_i^2 \\ &\quad + w_1[w_2,x_i^2] + w_1x_i^2w_2 \end{aligned}$$

which implies that

$$x_iw_1w_2x_i + w_1x_i^2w_2 = [w_1,w_2,x_i^2] + w_1[w_2,x_i^2] + [x_i,w_1,w_2]x_i.$$

So

$$\begin{aligned} v_1v_2 &= w_1[w_2,x_i^2] + [w_1,w_2,x_i^2] + w_1[w_2,x_i^2] + x_iw_1x_iw_2 \\ &\quad + [x_i,w_1,w_2]x_i && \text{(using (6))} \\ &= [w_1,w_2,x_i^2] + x_i^2[w_1,w_2] + x_i[w_1,w_2]x_i + x_iw_2x_iw_1 \\ &\quad + [x_i,w_1,w_2]x_i && \text{(using the Engel id.)} \\ &= [w_1,w_2,x_i^2] + [w_1,w_2]x_i^2 + [x_i,w_1,w_2]x_i + [w_2,x_i^2]w_1. && \text{(by (6))} \end{aligned}$$

The first elements gives a product with fewer u's. Products 2 and 3 give products with more u's of length 1, and the last element is product with more u's ending in a square. So therefore the product we chose must have been zero. □

Corollary 1 *Let* $\mathbb{H}= \{x_1, x_2, \ldots, x_{2s}\} = \{a_1, \ldots, a_s, a_1^2, \ldots, a_s^2\}$ *where* $a_i \in ad(L)$. *Suppose that*

$$t \geq 4s^3 + 2s^2 + s + 1,$$

and that $u_1, u_2, \ldots, u_j$ *are Lie products of elements in* $\mathbb{H}$. *If* $b \in ad(L)$ *then all products of* $t+2$ b^2*'s and* $u_1, u_2, \ldots u_j$ *are zero.*

Proof From the proposition it follows that if b^2 and u_i are adjacent in such a product then the product is symmetric in b^2 and u_i. This follows since

$$\cdots [u_i, b^2] \cdots$$

would be a product of $t+1$ occurrences of b^2, $c = [u_i, b^2]$ and $j-1$ u's and would therefore be zero because of the proposition. We can therefore bring two b^2's together and get b^4. Therefore the product is zero. □

Corollary 2 *Let* $b \in ad(L)$ *then*

$$\mathrm{Id}\langle b^2 \rangle^{4r^3+2r^2+r+2} = \{0\}.$$

Proof Let $\mathbb{H}= \{\mathrm{ad}(e_1), \ldots, \mathrm{ad}(e_r), \mathrm{ad}(e_1)^2, \ldots, \mathrm{ad}(e_r^2\}$. The corollary now follows from the last corollary. □

Let $\mathbb{K}= \{\mathrm{ad}(e_i)^2, \mathrm{ad}(e_i + e_j)^2 : 1 \leq i \leq r, 1 \leq j \leq r-1 \text{ and } j < i\}$. Let J be the ideal of $A(L)$ generated by $\mathbb{K}$. In the product

$$\mathrm{ad}(e_{i_1}) \cdots \mathrm{ad}(e_{i_{r+1}}) \qquad (*)$$

one of the generators must appear twice. Since

$$[\mathrm{ad}(e_{i_k}), \mathrm{ad}(e_{i_{k+1}})] = \mathrm{ad}(e_{i_k})^2 + \mathrm{ad}(e_{i_{k+1}})^2 + \mathrm{ad}(e_{i_k} + e_{i_{k+1}})^2$$

we have that the elements in $(*)$ commute modulo J. So modulo J we have

$$\mathrm{ad}(e_{i_1}) \cdots \mathrm{ad}(e_{i_{r+1}}) = \mathrm{ad}(e_{i_{\sigma(1)}}) \cdots \mathrm{ad}(e_{i_{\sigma(r+1)}})$$

for all $\sigma \in \mathrm{Sym}(r+1)$. Suppose $e_{i_l} = e_{i_m}$ then for a suitable σ we have

$$\mathrm{ad}(e_{i_1}) \cdots \mathrm{ad}(e_{i_{r+1}}) = \cdots \mathrm{ad}(e_{i_l})\mathrm{ad}(e_{i_m}) \cdots$$

which is in J. This implies that products of length $(r+1)s$ can be written as a sum of products including s elements from $\mathbb{K}$. Since $|\mathbb{K}| = r + r(r-1)/2 = r(r+1)/2$ and $4r^3 + 2r^2 + r + 1 \leq 4(r+1)^3$ it follows now from Corollary 2 that if

$$s \geq \frac{r(r+1)}{2} 4(r+1)^3 + 1$$

then products of length $(r+1)s$ would be zero. This implies that the nilpotency class of L is less than $2(r+1)^6$. We have therefore proved the following theorem.

Theorem 1 *If L is an Engel-3 algebra over a field k with characteristic 2 generated by r elements, then it is nilpotent of class less than*

$$2(r+1)^6.$$

As we said earlier, everything simplifies when we add the assumption that $|k| > 2$. By going through the same type of arguments as above one can prove that if we have this extra condition then the nilpotency class is less than

$$\frac{(r+1)^4}{2}.$$

From this we get the following result for Engel-4 Lie algebras

Corollary 3 *Let L be a Engel-4 Lie algebra over a field k of characteristic 2 and with $|k| \geq 4$. Suppose L is generated by r elements. Then L is nilpotent of class not more than*

$$\frac{(r+1)^4}{2}.$$

Proof Since $\operatorname{char} k = 2$ we have

$$\begin{aligned} b(ax^3) &= bax^3 - 3bxax^2 + 3bx^2ax - bx^3a \\ &= bax^3 + bxax^2 + bx^2ax + bx^3a \\ &= 0. \end{aligned}$$

Where the last equality follows from the Engel-4 identity since L is multigraded. Let I be the ideal in L generated by $\{ax^3 \,|a, x \in L\}$. Then L/I is an

Engel-3 Lie algebra and since $|k| > 2$ we have that L/I is nilpotent of class less than $(r+1)^4/2$ and from the calculations above we have that $Ib = 0$ in L for all $b \in L$. Therefore the corollary follows. $\Box$

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