Engel Lie-algebras

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Introduction

In this paper we investigate Engel-n Lie-algebras. They are Lie-algebras which satisfy the additional condition that $\mathrm{ad}(b)^n = 0$ for all b. Our aim is to get some detailed information about the nilpotency classes of Engel-n Lie-algebras for $n \leq 4$.

Engel Lie-algebras arise naturally in group theory in connection with "the restricted Burnside problem".

> For what values of r and n is there an upper bound on the orders of finite r-generator groups of exponent n?

The answer turns out to be that such an upper bound exists for all r and n. From the classification of finite simple groups and the work of P. Hall and G. Higman [4] it follows that it is sufficient to consider n when n is a power of a prime. In 1959 Kostrikin [8] proved that there is an upper bound if n is a prime. And finally in 1989 Zelmanov [14,15] showed that an upper bound exists if n is a power of a prime.

The theorems of Kostrikin and Zelmanov are in fact theorems about Lie-algebras. The reason is that the following are equivalent.

1. There is a largest finite r-generator group of exponent p^m;
2. The associated Lie-ring of $B(r, p^m)$ is nilpotent.

Here $B(r, n)$ is the (relatively) free r-generator group of exponent n. Now the associated Lie-ring of $B(r, p)$ satifies Engel-$(p-1)$ identity and has characteristic p, and so we can think of it as an Engel Lie-algebra over the field $\mathbb{Z}/p\mathbb{Z}$. So it was sufficient for Kostrikin to prove the following.

Kostrikin's Theorem[8] *Let L be a finitely generated Engel-$(p-1)$ Lie-algebra over a field of characteristic p. Then L is nilpotent.*

The more general result, that the associated Lie-ring of $B(r, p^m)$ is nilpotent, follows from

Zelmanov's Theorem[14,15] *Let L be a finitely generated Lie-algebra. Suppose that there exist positive integers s, t such that:*

$$\sum_{\sigma \in Sym(s)} x x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(s)} = 0 \qquad (1)$$

for all $x, x_1, x_2, \ldots, x_s \in L$, and

$$ab^t = 0 \qquad (2)$$

for all $a \in L$, and all $b \in L$ such that b can be expressed as a Lie product of the generators.
Then L is nilpotent.

Identities (1) and (2) are closely related to an Engel identity. In fact Zelmanov's theorem gives an answer to one of the major questions about Engel Lie-algebras, whether every finitely generated Engel-n Lie-algebra is nilpotent. The answer is yes, because an Engel-n algebra satisfies conditions (1) and (2) in the theorem with $s = t = n$. But we still have the problem of determining how the nilpotency class varies in terms of r (number of generators), n, and the characteristic of the field.

There are infinitely generated Engel Lie-algebras which are non-nilpotent. But it is known [13] that if the characteristic of the field is 0 then the Engel Lie-algebra is nilpotent. As a logical consequence of this we have that for each n there is an integer m_n such that

> if L is an Engel-n Lie-algebra over a field k such that char $k \geq m_n$ then L is nilpotent.

On the other hand, Razmyslov [10,11] proved the following result.

Razmyslov's Theorem *If k is a field and $\operatorname{char} k = p$ where $p \geq 5$ then there exists a non-solvable Engel-$(p-2)$ Lie-algebra over k.*

This theorem implies that $m_n > n$ for some n. Zelmanov has conjectured that $m_n \leq 2n$ for all n.

As mentioned above, we will investigate Engel-n Lie-algebras for low values of n.

In section 1 we will summarise some basic properties of Engel algebras needed later on.

In section 2 we will look at Engel-3 Lie-algebras. It has been known for a long time that if L is an Engel-3 Lie-algebra over a field k, and if $\operatorname{char} k \neq 2, 5$ then L is nilpotent. P. J. Higgins [6] proved that the nilpotency class is less than or equal to 6. In fact his argument gives the correct upper bound, namely 4. We will give a proof of this using Higgins' argument. In the case when $\operatorname{char} k = 2, 5$ there exist non-nilpotent Engel-3 Lie-algebras. But in the case when $\operatorname{char} k = 5$ we will see that for all x in L we have $Id\langle x\rangle^3 = 0$, where $Id\langle x\rangle$ is the ideal generated by x. This implies that if the number of generators is r then the nilpotency class is $\leq 2r$. In a paper of S. Bachmuth, H. Mochizuki and D. Walkup [2] they prove that there exist Engel-3 Lie-algebras over a field of characteristic 5 with class $2r-1$. When the characteristic is 2 we will construct an Engel-3 Lie-algebra with an element x such that $Id\langle x\rangle$ is non-nilpotent.

Let us now turn to Engel-4 algebras. To deal with them we are going to use super algebras, so we will study super algebras in section 3. In [9] Kostrikin shows that if $\operatorname{char} k \geq 7$ then L is soluble of derived length ≤ 7 which implies nilpotency of class at most 5461. In section 4 we will prove that when $\operatorname{char} k \neq 2, 3, 5$ then L is nilpotent of class ≤ 7. We will also see that when $\operatorname{char} k = 3$ then for all x in L we have $Id\langle x\rangle^4 = 0$, which implies that the class of an r-generator Engel-4 Lie algebra is at most $3r$. In the case when $\operatorname{char} k = 5$ we have the Higman, Havas, Newman and Vaughan-Lee [5] result that $Id\langle x\rangle^7 = 0$ so then the class of an r-generator Engel-4 Lie algebra is at most $6r$. The examples of non-nilpotent Engel-3 Lie-algebras of characteristic 2 and 5 are also examples of non-nilpotent Engel-4 Lie-algebras and when the characteristic is 3 there are also non-nilpotent Engel-4 Lie-algebras. (See

[3] p. 12-13)

1 Basic concepts

In this section we will give some basic facts about Engel Lie-algebras over a field k which are needed later on.

1.1 Definitions

We let ab denote the Lie product of two elements a, b in a Lie algebra. In the following we will use left normed notation: thus if $a_1, a_2, \ldots, a_n$ are elements of L, then

$$a_1 a_2 \cdots a_n = ((\cdots((a_1 a_2)a_3)\cdots)a_{n-1})a_n.$$

This means that

$$ab^n = (\cdots((a\,\underbrace{b)b)\cdots)b}_{n}$$

We say that L is graded if we can express it in the form

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_i \oplus \cdots,$$

where each L_i is a k-linear subspace and we have

$$L_i L_j \leq L_{i+j}, \quad L_i L_1 = L_{i+1}$$

for all i, j.

If L_1 has basis $\{a_1, \ldots, a_r\}$ as a vector space, then we can give each Lie-product of basis elements a 'multiweight' in the following way: the product is given multiweight $W = (w_1, \cdots, w_r)$ if a_i appears w_i times in the product for $i = 1, 2, \ldots, r$.

Now let L_W be the subspace of L generated by all Lie-products of basis elements which have multiweight W. An element that lies in L_W is said to be homogeneous of multiweight W. If

$$L = \bigoplus_W L_W$$

where the sum ranges over all possible r-tuples of non-negative integers, then we say that L is multigraded.

We let L^n denote the n-th element of the lower central series of L and $L^{(n)}$ denote the n-th element of the derived series.

1.2 Linearizations of the Engel-identity

If we want to show that a certain algebra is an Engel Lie-algebra or we want to construct an Engel Lie-algebra, then it is convenient if it is enough to look at elements from a vector space basis of the Lie algebra. But because the Engel identity is not multilinear, the Engel identity does not necessarily follow if $ab^n = 0$ for all basis elements a and b. However, in most cases the Engel identity is equivalent to a multilinear identity.

Proposition 1.1 *Let L be an Engel-n Lie-algebra over a field k. If $\operatorname{char} k$ is zero or bigger than n then the Engel-n identity is equivalent to*

$$\sum_{\sigma \in Sym(n)} ab_{\sigma(1)} \cdots b_{\sigma(n)} = 0. \tag{3}$$

Proof To show that (3) implies the Engel identity let $b_1 = b_2 = \cdots = b_n = b$. Then we get $n!ab^n = 0$ which implies $ab^n = 0$ since $\operatorname{char} k$ does not divide $n!$. The converse holds for all fields k. Let $\lambda_1, \dots, \lambda_n$ be indeterminates and consider

$$U = a(\lambda_1 b_1 + \cdots + \lambda_n b_n)^n.$$

If we multiply this out we get

$$U = r_0 + \lambda_1 r_1 + \lambda_1 \lambda_2 r_2 + \cdots + \lambda_1 \lambda_2 \cdots \lambda_n \sum_{\sigma \in Sym(n)} ab_{\sigma(1)} \cdots b_{\sigma(n)}$$

where r_0 is the sum of the monomials not divisible by λ_1, $\lambda_1 r_1$ is the sum of those monomials from $U - r_0$ not divisible by λ_2, and so on. Now $U = 0$ for all $\lambda_1, \dots \lambda_n$. Putting $\lambda_1 = 0$, we get $r_0 = 0$. Then let $\lambda_1 = 1$, $\lambda_2 = 0$ and we get $r_1 = 0$. Continuing in this way we see finally that $\sum_\sigma ab_{\sigma(1)} \cdots b_{\sigma(n)}$ is 0. $\square$

We call identity (3), the 'full linearization' of the Engel identity. Clearly (3) is satisfied for all $a, b_1, b_2, \dots, b_n \in L$ if it is satisfied whenever $a, b_1, b_2, \dots, b_n$ are basis elements.

For low characteristics things are more complicated and we have to consider partial linearizations. Let us look at Engel-3 Lie-algebras. In an Engel-3 Lie-algebra

$$0 = a(\lambda b + c)^3 = \lambda(abc^2 + acbc + ac^2 b) + \lambda^2(ab^2 c + abcb + acb^2), \tag{4}$$

and

$$0 = \lambda a(b+c)^3 = \lambda(abc^2 + acbc + ac^2b) + \lambda(ab^2c + abcb + acb^2). \qquad (5)$$

Now if we subtract (5) from (4), we get

$$ab^2c + abcb + acb^2 = 0 \qquad (6)$$

if $\lambda \neq 0, 1$. So (6) holds if $|k| \geq 3$.

So if $|k| \geq 3$, the Engel-3 identity implies (6) and the full linearization

$$\sum_{\sigma \in Sym(3)} ab_{\sigma(1)}b_{\sigma(2)}b_{\sigma(3)} = 0. \qquad (7)$$

On the other hand it is easy to see that if (6), (7) and the identity $ab^3 = 0$ hold for all basis elements a, b, c, b_1, b_2, b_3 in some Lie-algebra L, then L is an Engel-3 Lie-algebra. Notice that all these identities are homogeneous.

So for each Engel-n Lie-algebra we have some partial linearizations of the Engel identity. The Engel-n identity implies these linearizations provided $|k| \geq n$. In the following we will be assuming that $|k| \geq 3$ for Engel-3 algebras and that $|k| \geq 4$ for Engel-4 algebras. Slightly different arguments are needed to deal with Engel-4 Lie algebras over the field of three elements.

2 Engel-3 algebras

In his paper "Lie rings satisfying the Engel condition" [6], P. J. Higgins proves that every Engel-3 algebra over a field k with $\mathrm{char}\, k \neq 2, 5$ is nilpotent of class less than or equal to 6. In fact his argument can be used to get the correct upper bound, namely that the class is less than or equal to 4. In this section we will give a proof of this result. In the case when $\mathrm{char}\, k = 2, 5$ there exists non-nilpotent Engel-3 algebras. But in the case when $\mathrm{char}\, k = 5$ we will see that for all x in L we have $\mathrm{Id}\langle x\rangle^3 = \{0\}$, where $\mathrm{Id}\langle x\rangle$ is the ideal generated by x. This implies local nilpotency. When the characteristic is 2 we will construct an Engel-3 Lie-algebra with an element x such that $\mathrm{Id}\langle x\rangle$ is non-nilpotent.

2.1 A counterexample for $\operatorname{char} k = 2$

We will now construct an example of an Engel-3 Lie-algebra which has an element x such that $\mathrm{Id}\langle x\rangle$ is non-nilpotent.

Remark: To show that an algebra L is an Engel-3 Lie-algebra it is sufficient to show that for all basis elements a, b, c, b_1, b_2, b_3 we have

1) $a^2 = 0, \quad ab = -ba;$
2) $abc + bca + cab = 0;$
3) $ab^3 = 0;$
4) $abc^2 + acbc + ac^2b = 0;$
5) $\sum_{\sigma\in Sym(3)} ab_{\sigma(1)}b_{\sigma(2)}b_{\sigma(3)} = 0.$

We need 3, 4 and 5 to show that the Engel-3 identity holds.

Example Let k be a field such that $\operatorname{char} k = 2$. We define an algebra L over k in the following way.

For each nonempty finite $A \subseteq \mathbb{N}$ we let U_A, V_A and W_A be basis elements. To these we add one other basis element that we will call x. So we define L to be the vector space over k with this basis. We define a product on L in the following way. We let $ab = ba$ and $aa = 0$ for all basis elements a and b. Furthermore we let

$$U_A x = 0, \qquad V_A x = 0, \qquad W_A x = U_A,$$

$$U_A U_B = V_A V_B = W_A W_B = 0,$$

$$U_A V_B = \begin{cases} U_{A\cup B} & \text{if } A\cap B = \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

$$U_A W_B = \begin{cases} V_{A\cup B} & \text{if } A\cap B = \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

$$V_A W_B = \begin{cases} W_{A\cup B} & \text{if } A\cap B = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

We extend this product linearly to all L.

Proposition 2.1 *L is a non-soluble Engel-3 algebra. Furthermore $\mathrm{Id}\langle x\rangle$ is non-nilpotent.*

Proof Since $\operatorname{char} k = 2$, we have $ab = ba = -ba$ for all a, b in L. Every product of basis elements such that the corresponding integer sets are not pairwise disjoint is zero. We therefore only have to check the Jacobi and Engel identities over the basis elements where the corresponding integer sets are pairwise disjoint. There are 4 different types of basis elements. It is sufficient to look at the Jacobi and Engel identities over every possible combination of these types. Therefore there are only a finite number of different cases to check and it is not difficult to show that we get 0 in all these cases. As an example we have

$$U_A V_B W_C + V_B W_C U_A + W_C U_A V_B = 2V_{A \cup B \cup C} = 0,$$

and

$$\begin{array}{l} W_A W_B U_C x + W_A W_B x U_C + W_A U_C W_B x + W_A U_C x W_B + W_A x W_B U_C + \\ W_A x U_C W_B = 0 + 0 + U_{A \cup B \cup C} + 0 + U_{A \cup B \cup C} + 0 = 2U_{A \cup B \cup C} = 0 \end{array}.$$

So L is an Engel-3 Lie-algebra. Now it is clear from the definition of the product that $L^{(n)}$ contains U's, V's and W's for all n, and therefore it is non-solvable. Furthermore

$$W_{\{1\}} x W_{\{2\}} W_{\{3\}} x W_{\{4\}} W_{\{5\}} \cdots x W_{\{2n\}} W_{\{2n+1\}} = W_{\{1, \cdots, 2n+1\}}$$

So for each n there is a non-zero product containing n appearances of x. This implies that $\operatorname{Id}\langle x \rangle^n \neq \{0\}$ for all n, so $\operatorname{Id}\langle x \rangle$ is non-nilpotent. □

2.2 Nilpotency of Id$\langle x \rangle$ when char $k \neq 2$

Let L be an Engel-3 algebra over a field k such that $\operatorname{char} k \neq 2, 3$. In this subsection we will show that $\operatorname{Id}\langle x \rangle^3 = \{0\}$. This implies that if L is finitely generated then L is nilpotent of class $\leq 2r$, where r is the number of generators. It will follow from next subsection that this is also true if $\operatorname{char} k = 3$.

Lemma 2.1 *For all x, a_1, a_2 in L we have*

$$x a_1 x a_2 x = 0.$$

Proof From the Jacobi identity we get

$$(a_2xa_1)x = (a_1xa_2)x + (a_2a_1x)x.$$

By using this, the Jacobi identity, and then the Engel-3 identity we have

$$\begin{aligned} 0 &= (a_1x)a_2x + a_2x(a_1x) + x(a_1x)a_2 \\ &= a_1xa_2x + a_2xa_1x - a_2x^2a_1 - a_1x^2a_2 \\ &= 2a_1xa_2x + a_2a_1x^2 + xa_2xa_1 + xa_1xa_2, \end{aligned} \tag{8}$$

$$\begin{aligned} 0 &= a_1xa_2x + a_1a_2x^2 + a_1x^2a_2 \\ &= a_1xa_2x - a_2a_1x^2 - xa_1xa_2, \end{aligned} \tag{9}$$

$$\begin{aligned} 0 &= xa_2a_1x + xa_1a_2x + xa_2xa_1 + xa_1xa_2 \\ &= -2a_1xa_2x - a_2a_1x^2 + xa_2xa_1 + xa_1xa_2. \end{aligned} \tag{10}$$

Since $\operatorname{char} k \neq 2$, (8) and (10) implies $xa_2xa_1 + xa_1xa_2 = 0$. Therefore $2\cdot(9)-(8)$ gives $2xa_1xa_2 = -3a_2a_1x^2$. Thus $2xa_1xa_2x = -3a_2a_1x^3 = 0$ which implies $xa_1xa_2x = 0$. $\square$

Lemma 2.2 *For all x, a_1, a_2 in L we have*

$$xa_1a_2x^2 = 0.$$

Proof By the Engel-3 identity we have

$$\begin{aligned} 0 &= xa_1a_2x^2 + xa_1xa_2x + xa_1x^2a_2 \\ &= xa_1a_2x^2 - a_1x^3a_2 \qquad \text{(Lemma 2.1)} \\ &= xa_1a_2x^2. \qquad \square \end{aligned}$$

Lemma 2.3 *For all $n \in \mathbb{N}$ and for all $x, a_1, \ldots, a_n \in L$, we have*

$$xa_1 \cdots a_nx^2 = 0.$$

Proof We use induction over n. The case $n = 1$ follows at once from the Engel identity and the case $n = 2$ follows from Lemma 2.2. So now suppose this is true for some $n \geq 2$. Then for every $i \in \{1, \cdots, n\}$ we have by the induction hypothesis

$$xa_1 \cdots a_{i-1}(a_i a_{i+1})a_{i+2} \cdots a_{n+1}x^2 = 0,$$

so

$$xa_1 \cdots a_{i-1}a_i a_{i+1}a_{i+2} \cdots a_{n+1}x^2 = xa_1 \cdots a_{i-1}a_{i+1}a_i a_{i+2} \cdots a_{n+1}x^2.$$

It follows that $xa_1a_2a_3a_4 \cdots a_{n+1}x^2 = xa_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)}a_4 \cdots a_{n+1}x^2$ for every $\sigma \in S(3)$. Therefore

$$3!xa_1a_2a_3a_4 \cdots a_{n+1}x^2 = \sum_{\sigma \in S(3)} xa_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)}a_4 \cdots a_{n+1}x^2 = 0,$$

where the last identity follows from the Engel identity. So $xa_1 \cdots a_{n+1}x^2 = 0$ since $\operatorname{char} k \neq 2, 3$. (In fact it is also true if $\operatorname{char} k = 3$ because then L is nilpotent of class 4, as we will see in the next subsection.) The lemma now follows. $\square$

So we can now prove the assertion made at the beginning of the section.

Theorem 2.1 *If L is as above then for all $i, j \in \mathbb{N}$ we have*

$$xa_1 \cdots a_i x b_1 \cdots b_j x = 0$$

for all $x, a_1, \ldots, a_i, b_1, \ldots b_j \in L$.

Proof We use induction over j. The case $j = 0$ follows from Lemma 2.3. Now look at the case $j = 1$. Using the Engel identity we get

$$\begin{aligned} 0 &= xa_1 \cdots a_i x b_1 x + xa_1 \cdots a_i x^2 b_1 + xa_1 \cdots a_i b_1 x^2 \\ &= xa_1 \cdots a_i x b_1 x \quad \text{(Lemma 2.3)}. \end{aligned}$$

Now suppose this is true for some $j \geq 1$. Then using the Engel identity again we get

$$\begin{aligned} 0 &= xa_1 \cdots a_i x b_1 \cdots b_{j-1} b_j b_{j+1} x + xa_1 \cdots a_i x b_1 \cdots b_{j-1} b_{j+1} b_j x \\ &\quad + xa_1 \cdots a_i x b_1 \cdots b_{j-1} b_j x b_{j+1} + xa_1 \cdots a_i x b_1 \cdots b_{j-1} b_{j+1} x b_j \\ &\quad + xa_1 \cdots a_i x b_1 \cdots b_{j-1} x b_j b_{j+1} + xa_1 \cdots a_i x b_1 \cdots b_{j-1} x b_{j+1} b_j \\ &= xa_1 \cdots a_i x b_1 \cdots b_{j-1} b_j b_{j+1} x + xa_1 \cdots a_i x b_1 \cdots b_{j-1} b_{j+1} b_j x. \end{aligned}$$

Where the last identity follows from the induction hypothesis. But

$$\begin{aligned}&xa_1\cdots a_ixb_1\cdots b_{j-1}b_jb_{j+1}x - xa_1\cdots a_ixb_1\cdots b_{j-1}b_{j+1}b_jx\\ &= xa_1\cdots a_ixb_1\cdots b_{j-1}(b_jb_{j+1})x = 0 \qquad \text{(by the induction hypothesis).}\end{aligned}$$

Therefore $2xa_1\cdots a_ixb_1\cdots b_{j+1}x = 0$ which implies that $xa_1\cdots a_ixb_1\cdots b_{j+1}x = 0$ since $\operatorname{char} k \neq 2$. Now the theorem follows. $\square$

Corollary 2.1 *If* $\operatorname{char} k \neq 2, 3$ *then* $\operatorname{Id}\langle x\rangle^3 = \{0\}$ *for every* x *in* L.

Suppose now that L has r generators. We can express every element in L as a sum of products of the generators. But because $\operatorname{Id}\langle u\rangle^3 = 0$ for every generator u, every generator can at most appear twice in a non-zero product. So the maximal length of a non-zero product is $2r$. This implies that the nilpotency class is less than or equal to $2r$. This is in particular true in the case when $\operatorname{char} k = 5$. In the case when $\operatorname{char} k \neq 2, 5$ we are now going to prove a stronger result.

2.3 Nilpotency of L when char $k \neq 2, 5$

Let $\operatorname{ad}(u)$ be the endomorphism from $L \to L$ with $\operatorname{ad}(u)(z) = zu$. To simplify the notation we will let U denote $\operatorname{ad}(u)$. Similarly we let X denote $\operatorname{ad}(x)$, Y denote $\operatorname{ad}(y)$, and so on. For Engel-2 algebras we have the following well known result. (See Higgins [6] or Vaughan-Lee [12] for example.)

Theorem 2.2 *If* L *is an Engel-*2 *algebra over some field* k *then* $L^3 = \{0\}$ *if* $\operatorname{char} k \neq 3$ *and* $L^4 = \{0\}$ *if* $\operatorname{char} k = 3$.

So we are now ready for the main result of this section.

Theorem 2.3 *If* L *is an Engel-*3 *algebra over some field* k *such that* $\operatorname{char} k \neq 2, 5$ *then* $L^5 = \{0\}$.

Proof The proof of the result is almost identical to Higgin's proof that $L^7 = \{0\}$. But we repeat the argument since it is very short and actually proves the stronger result that $L^5 = \{0\}$.

From the Engel-3 identity we have

$$XY^2 + YXY + Y^2X = 0. \tag{11}$$

We also have

$$\begin{aligned}
0 &= -xay^2 - xyay - xy^2a \\
&= axy^2 + a(xy)y + a(xy^2) \\
&= 3axy^2 - 3ayxy + ay^2x,
\end{aligned}$$

and therefore

$$3XY^2 - 3YXY + Y^2X = 0. \tag{12}$$

If $\operatorname{char} k = 3$ we thus have $Y^2X = 0$. Consider the subspace

$$I = \mathrm{Sp}\langle uv^2 | u, v \in L\rangle.$$

It is not difficult to see that I is an ideal. The quotient algebra L/I is an Engel-2 algebra. It follows from Theorem 2.2 that every product of four elements lies in I. If $\operatorname{char} k = 3$ then the fact that $Y^2X = 0$ implies that $ay^2x = 0$ for all $a, y, x \in L$, and so every product of five elements is zero. Now suppose $\operatorname{char} k \neq 2, 3, 5$. From (11) and (12) we have

$$XY^2 = 2YXY \quad \text{and} \quad Y^2X = -3YXY. \tag{13}$$

It also follows that $3XY^2 = -2Y^2X$. If we interchange X and Y in (13) we get

$$YX^2 = 2XYX \quad \text{and} \quad X^2Y = -3XYX. \tag{14}$$

Now multiply (13) by X on the left and (14) on the right by Y. We then get

$$X^2Y^2 = 2XYXY \quad \text{and} \quad X^2Y^2 = -3XYXY.$$

It follows that $5X^2Y^2 = 0$ so $X^2Y^2 = 0$ since $\operatorname{char} k \neq 5$. From Theorem 2.2 we then have $X_1X_2Y^2 = 0$. But since $3XY^2 = -2Y^2X$ we get

$$4Y^2X_1X_2 = -9X_1X_2Y^2 = 0$$

and then again from Theorem 2.2

$$Y_1Y_2X_1X_2 = 0$$

so $L^5 = 0$ □

It is easy to construct Engel-3 Lie algebra of class 4, and so this result is best possible.

3 Superalgebras

In Section 4 we are going to study Engel-4 algebras. There we shall see that if L is an Engel-4 algebra over a field k such that $\operatorname{char} k \neq 2, 3, 5$ then L is nilpotent of class≤ 7. This means that we have to show that every left normed product of 8 elements is zero. If we were to use the same approach as for the Engel-3 algebras, we would have to work in 8-generator Engel-4 algebras. Fortunately it will be sufficient to work only with 4 generators. But instead of looking at Engel-4 algebras we will use superalgebras. In this section we will describe this reduction to superalgebras.

3.1 Reduction for Engel-4 algebras

Let L be the free Engel-4 algebra generated by $x_1, x_2, \ldots, x_8$. Let M be the subspace generated by all left normed products of weight $(1, 1, \ldots, 1)$ in $x_1, x_2, \ldots, x_8$. Let $S(8)$ be the symmetric group on $\{1, 2, \ldots, 8\}$. We can think of M as a $kS(8)$-module with an action defined as follows: if $K(x_1, x_2, \ldots, x_8)$ is some left normed product of $x_1, \ldots, x_8$ in some order and $\sigma \in S(8)$ then

$$\sigma K(x_1 \ldots, x_8) = K(x_{\sigma(1)}, \ldots, x_{\sigma(8)}).$$

Example $(125)x_1x_2x_3x_4x_5x_6x_7x_8 = x_2x_5x_3x_4x_1x_6x_7x_8.$

Definition 3.1 *We call an element μ in $kS(8)$ symmetrized if*

$$\mu = \lambda(\, id + (12) + (13) + (23) + (123) + (132)\,)\lambda^{-1}\sigma$$

for some $\lambda, \sigma \in S(8)$. We call μ skew-symmetrized if

$$\mu = \lambda(\, id - (12) - (13) - (23) + (123) + (132)\,)\lambda^{-1}\sigma.$$

We also say that μ is [skew-]symmetrized in $\lambda(1)$, $\lambda(2)$ and $\lambda(3)$. We say that U in M is [skew-]symmetrized if $U = \mu K$ where μ is a [skew-]symmetrized element in $kS(8)$ and K is a left normed product of $x_1 \ldots x_8$ in some order. We say that $U = \mu K$ is [skew-]symmetrized in $x_{i_1}, x_{i_2}, x_{i_3}$ if μ is [skew-]symmetrized in i_1, i_2, i_3.

Example Let $\sigma = (135)$, $\lambda = (14)(25)$ and $\mu = \lambda(\,id + (12) + (13) + (23) + (123) + (132)\,)\lambda^{-1}\sigma$. Then

$$\sigma x_1x_2x_3x_4x_5x_6x_7x_8 = x_3x_2x_5x_4x_1x_6x_7x_8,$$

and $\lambda(\,id + (12) + (13) + (23) + (123) + (132)\,)\lambda^{-1} = (\,id + (45) + (43) + (53) + (453) + (435)\,)$. So

$$\mu x_1x_2x_3x_4x_5x_6x_7x_8 = \sum_{\tau \in Sym\{3,4,5\}} x_{\tau(3)}x_2x_{\tau(5)}x_{\tau(4)}x_1x_6x_7x_8.$$

If μ had been skew-symmetrized in the example above, then we would have had to put a minus sign in front of every product in the sum that we got from odd permutations before adding. That is, we should have

$$\mu x_1x_2x_3x_4x_5x_6x_7x_8 = \sum_{\tau \in Sym\{3,4,5\}} sign(\tau)x_{\tau(3)}x_2x_{\tau(5)}x_{\tau(4)}x_1x_6x_7x_8.$$

We can think of symmetrized and skew-symmetrized elements of M in the following way. Start with any left normed product of $x_1, \ldots, x_8$ in some order. Then take some three elements inside the product and permute them in all possible ways. What we get are 6 products. If we add these products together we get a symmetrized element and if we form the alternating sum of these products we get a skew-symmetrized element.

The key to the reduction is the fact that $id \in kS(8)$ is in the linear span of the symmetrized and skew-symmetrized elements provided that $\mathrm{char}\, k \neq 2, 3, 5$. To be more precise we should talk about 3-symmetrized elements and 3-skew-symmetrized elements. More generally if we are looking at $S(n)$ and m is the least integer greater than or equal to $\sqrt{n}$ then it follows from the representation theory of the symmetric groups that id is in the linear span of m-symmetrized elements and m-skew-symmetrized elements in $kS(n)$ provided that $\mathrm{char}\, k > n$. (See [7])

For $\sigma \in S(5)$ let S_σ be the sum of all elements in the conjugacy class of σ. Let $\mu_+ = (\,id + (12) + (13) + (23) + (123) + (132)\,)$ and $\mu_- = (\,id - (12) - (13) - (23) + (123) + (132)\,)$. We have

$$\sum_{\sigma \in S(5)} \sigma\mu_+\sigma^{-1} \;=\; 120id + 36S_{(12)} + 12S_{(123)},$$

$$
\begin{aligned}
\sum_{\sigma\in S(5)} \sigma\mu_-\sigma^{-1} &= 120id - 36S_{(12)} + 12S_{(123)},\\
\sum_{\sigma\in S(5)} \sigma\mu_+(45)\sigma^{-1} &= 12S_{(12)} + 24S_{(12)(34)} + 12S_{(12)(345)},\\
\sum_{\sigma\in S(5)} \sigma\mu_-(45)\sigma^{-1} &= 12S_{(12)} - 24S_{(12)(34)} + 12S_{(12)(345)},\\
\sum_{\sigma\in S(5)} \sigma\mu_+(14)\sigma^{-1} &= 12S_{(12)} + 12S_{(123)} + 8S_{(12)(34)} + 8S_{(1234)},\\
\sum_{\sigma\in S(5)} \sigma\mu_-(14)\sigma^{-1} &= 12S_{(12)} - 12S_{(123)} - 8S_{(12)(34)} + 8S_{(1234)},
\end{aligned}
$$

where we think of $S(5)$ as a subgroup of $S(8)$ in the usual way. From this it follows that

$$
\begin{aligned}
720id &= 3\sum_\sigma \sigma\mu_+\sigma^{-1} + 3\sum_\sigma \sigma\mu_-\sigma^{-1} - 3\sum_\sigma \sigma\mu_+(14)\sigma^{-1} + \\
&\quad 3\sum_\sigma \sigma\mu_-(14)\sigma^{-1} + \sum_\sigma \sigma\mu_+(45)\sigma^{-1} - \sum_\sigma \sigma\mu_-(45)\sigma^{-1}.
\end{aligned}
$$

Considered as permutations in $S(8)$, all the permutations in this sum fix 6, 7 and 8. Clearly for any $r, s, t \in \{1, 2, \dots, 8\}$ we can find similar expression for $720id$ as a linear combination of symmetrized and skew-symmetrized elements which fix r, s, t.

Now suppose we want to show that $x_1 \cdots x_8$ is zero. Using the formula above we see that $720x_1 \cdots x_8$ is a sum of symmetrized and skew symmetrized products. So because the only divisors of 720 are 2, 3 and 5, it is enough to show that every symmetrized and skew symmetrized product is zero if $\operatorname{char} k \neq 2, 3, 5$.

So suppose $\operatorname{char} k \neq 2, 3, 5$ and look at some symmetrized element $U \in M$, say $U = \lambda\mu_+\lambda^{-1}\sigma K$. We can express id in $kS(8)$ as a sum of symmetrized and skew-symmetrized elements $\gamma\mu_+\gamma^{-1}\tau$ or $\gamma\mu_-\gamma^{-1}\tau$ where τ fixes $\lambda(1), \lambda(2), \lambda(3)$, and where $\{\gamma(1), \gamma(2), \gamma(3)\} \cap \{\lambda(1), \lambda(2), \lambda(3)\} = \emptyset$. So U is in the linear span of elements of the form

$$
(\gamma\mu_+\gamma^{-1}\tau)(\lambda\mu_+\lambda^{-1}\sigma)K \quad \text{or} \quad (\gamma\mu_-\gamma^{-1}\tau)(\lambda\mu_+\lambda^{-1}\sigma)K
$$

where τ fixes $\lambda(1), \lambda(2), \lambda(3)$ and where $\{\gamma(1), \gamma(2), \gamma(3)\} \cap \{\lambda(1), \lambda(2), \lambda(3)\} = \emptyset$.

Example Let $\gamma = (14)(25)(37)$. Then if $\mu_1 = \gamma\mu_+\gamma^{-1}\mu_+$ we have

$$\mu_1 x_1x_2x_3x_4x_5x_6x_7x_8 = \sum_{\alpha,\beta} x_{\alpha(1)}x_{\alpha(2)}x_{\alpha(3)}x_{\beta(4)}x_{\beta(5)}x_6x_{\beta(7)}x_8,$$

and if $\mu_2 = \gamma\mu_-\gamma^{-1}\mu_+$ then

$$\mu_2 x_1x_2x_3x_4x_5x_6x_7x_8 = \sum_{\alpha,\beta}(sign\beta)x_{\alpha(1)}x_{\alpha(2)}x_{\alpha(3)}x_{\beta(4)}x_{\beta(5)}x_6x_{\beta(7)}x_8,$$

where α runs through $Sym\{1,2,3\}$ and β through $Sym\{4,5,7\}$.
We can similarly look at skew-symmetrized elements in M. We can express them as linear combination of elements of the form

$$(\gamma\mu_+\gamma^{-1}\tau)(\lambda\mu_-\lambda^{-1}\sigma)K \quad \text{or} \quad (\gamma\mu_-\gamma^{-1}\tau)(\lambda\mu_-\lambda^{-1}\sigma)K,$$

where τ fixes $\lambda(1)$, $\lambda(2)$ and $\lambda(3)$,and $\{\gamma(1),\gamma(2),\gamma(3)\}\cap\{\lambda(1),\lambda(2),\lambda(3)\} = \emptyset$.

Example Let $\gamma = (14)(25)(37)$ as in the last example, and let $\mu_3 = \gamma\mu_+\gamma^{-1}\mu_-$ and $\mu_4 = \gamma\mu_-\gamma^{-1}\mu_-$. Then

$$\mu_3 x_1x_2x_3x_4x_5x_6x_7x_8 = \sum_{\alpha,\beta}(sign\alpha)x_{\alpha(1)}x_{\alpha(2)}x_{\alpha(3)}x_{\beta(4)}x_{\beta(5)}x_6x_{\beta(7)}x_8$$

and

$$\mu_4 x_1x_2x_3x_4x_5x_6x_7x_8 = \sum_{\alpha,\beta}(sign\alpha)(sign\beta)x_{\alpha(1)}x_{\alpha(2)}x_{\alpha(3)}x_{\beta(4)}x_{\beta(5)}x_6x_{\beta(7)}x_8.$$

Definition 3.2 *We call an element $U \in M$ a (sym, sym) element if*

$$U = (\gamma\mu_+\gamma^{-1}\tau)(\lambda\mu_+\lambda^{-1}\sigma)K$$

where K is some left normed product of $x_1,\ldots,x_8$, and $\gamma,\lambda,\tau,\sigma \in Sym(8)$ are such that τ fixes $\lambda(1)$, $\lambda(2)$ and $\lambda(3)$, and $\{\gamma(1),\gamma(2)\gamma(3)\}\cap\{\lambda(1),\lambda(2),\lambda(3)\} = \emptyset$. We call U a $(skew, skew)$ element if

$$U = (\gamma\mu_-\gamma^{-1}\tau)(\lambda\mu_-\lambda^{-1}\sigma)K$$

and we call U a $(skew, sym)$ element if

$$U = (\gamma\mu_-\gamma^{-1}\tau)(\lambda\mu_+\lambda^{-1}\sigma)K \quad \text{or} \quad U = (\gamma\mu_+\gamma^{-1}\tau)(\lambda\mu_-\lambda^{-1}\sigma)K,$$

where $\gamma,\lambda,\tau,\sigma \in Sym(8)$ satisfy the same conditions.

Remark It is not difficult to see that the class of all elements of the form $(\gamma\mu_-\gamma^{-1}\tau)(\lambda\mu_+\lambda^{-1}\sigma)K$ is equal to the class of all elements of the form $(\gamma\mu_+\gamma^{-1}\tau)(\lambda\mu_-\lambda^{-1}\sigma)K$.

We conclude that to show that $x_1 \cdots x_8$ is zero it is enough to show that all (sym, sym), $(skew, sym)$ and $(skew, skew)$ elements are zero. And now it is time to introduce superalgebras.

Let U, V, E and F be the following associative algebras with unity.

$$U = \langle u_1, u_2, u_3 \,|\, u_i^2 = 0, \quad u_i u_j = u_j u_i \ \text{ for all } i, j\rangle,$$

$$V = \langle v_1, v_2, v_3 \,|\, v_i^2 = 0, \quad v_i v_j = v_j v_i \ \text{ for all } i, j\rangle,$$

$$E = \langle e_1, e_2, e_3 \,|\, e_i^2 = 0, \quad e_i e_j = -e_j e_i \ \text{ for all } i, j\rangle,$$

$$F = \langle f_1, f_2, f_3 \,|\, f_i^2 = 0, \quad f_i f_j = -f_j f_i \ \text{ for all } i, j\rangle.$$

Now look first at the algebra

$$\tilde{L} = L \otimes U \otimes V.$$

Clearly $\tilde{L}$ is an Engel-4 Lie algebra. If we let $x = x_1 \otimes u_1 \otimes 1 + x_2 \otimes u_2 \otimes 1 + x_3 \otimes u_3 \otimes 1$ and $y = x_4 \otimes 1 \otimes v_1 + x_5 \otimes 1 \otimes v_2 + x_6 \otimes 1 \otimes v_3$ then, for example,

$$x^3 y^2 x_7 y x_8 = \big(\sum_{\alpha,\beta} x_{\alpha(1)} x_{\alpha(2)} x_{\alpha(3)} x_{\beta(4)} x_{\beta(5)} x_7 x_{\beta(6)} x_8\big) \otimes u_1 u_2 u_3 \otimes v_1 v_2 v_3.$$

(Here we are identifying x_7 with $x_7 \otimes 1 \otimes 1$ and x_8 with $x_8 \otimes 1 \otimes 1$.) Every product of $x, x, x, y, y, y, x_7, x_8$ in $\tilde{L}$ corresponds to a (sym, sym) element in this way. And it is clear that if we want to show that every (sym, sym) element in L is zero it is sufficient to show that every product of $x, x, x, y, y, y, x_7, x_8$ in $\tilde{L}$ is zero. (This is because L is free.)

To treat $(skew, sym)$ elements we consider the algebra

$$S = L \otimes U \otimes E.$$

This is not a Lie-algebra. We shall call it the superalgebra associated with the Lie algebra $L \otimes U$. If we let $x = x_1 \otimes 1 \otimes e_1 + x_2 \otimes 1 \otimes e_2 + x_3 \otimes 1 \otimes e_3$ and $y = x_4 \otimes u_1 \otimes 1 + x_5 \otimes u_2 \otimes 1 + x_6 \otimes u_3 \otimes 1$, then to show that all

$(skew, sym)$ elements in L are zero it is again sufficient to show that all products of $x, x, x, y, y, y, x_7, x_8$ are zero. Here

$$x^3y^2x_7yx_8 = (\sum_{\alpha,\beta} sign(\alpha)x_{\alpha(1)}x_{\alpha(2)}x_{\alpha(3)}x_{\beta(4)}x_{\beta(5)}x_7x_{\beta(6)}x_8) \otimes u_1u_2u_3 \otimes e_1e_2e_3.$$

Finally to handle $(skew, skew)$ elements we look at the *associated colour algebra*

$$C = L \otimes E \otimes F.$$

Here we let $x = x_1 \otimes e_1 \otimes 1 + x_2 \otimes e_2 \otimes 1 + x_3 \otimes e_3 \otimes 1$ and $y = x_4 \otimes 1 \otimes f_1 + x_5 \otimes 1 \otimes f_2 + x_6 \otimes 1 \otimes f_3$ and as before we see that if every product of $x, x, x, y, y, y, x_7, x_8$ is zero it follows that every $(skew, skew)$ element in L is zero.

3.2 Calculations in superalgebras and colour algebras

Let L be an Engel-4 Lie algebra over some field k and let E be the following associative algebra with unity.

$$E = \langle e_1, e_2, e_3 \,|\, e_i^2 = 0,\ e_ie_j = -e_je_i \text{ for all } i, j\rangle.$$

E has basis $\{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\}$. We can write

$$E = E_0 \oplus E_1,$$

where

$$E_0 = sp\langle 1, e_1e_2, e_1e_3, e_2e_3\rangle$$

and

$$E_1 = sp\langle e_1, e_2, e_3, e_1e_2e_3\rangle.$$

An element in $E_0[E_1]$ is called even[odd]. Let us now look at the superalgebra $S = L \otimes E$. Then $S = S_0 \oplus S_1$ where $S_0 = L \otimes E_0$ and $S_1 = L \otimes E_1$. We call S_0 the even part and S_1 the odd part of S. An element that lies in $S_0[S_1]$ will be called even[odd]. It is clear that elements in E_0 commute with all elements in E. But if e and f are odd elements in E then $ef = -fe$. Using the fact that the Lie product in L satisfies $ab = -ba$ for all $a, b \in L$ we see that this implies that if u is some even element in S, then we have $uv = -vu$ for every v in S. But if both u and v are odd then $uv = vu$.

Now consider the full linearization of the Engel-4 identity

$$K(x_1, x_2, x_3, x_4, x_5) = \sum_{\sigma \in Sym(4)} x_5 x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} = 0.$$

If for each $i \in \{1, \ldots, n\}$ we substitute $h_i = a_i \otimes g_i$ for x_i where $a_i \in L$ and $g_i \in E$ then

$$K(h_1, h_2, h_3, h_4, h_5) = \sum_{\sigma} a_5 a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)} a_{\sigma(4)} \otimes g_5 g_{\sigma(1)} g_{\sigma(2)} g_{\sigma(3)} g_{\sigma(4)}.$$

If g_1, g_2, g_3 and g_4 are all even then

$$\begin{aligned} K(h_1, h_2, h_3, h_4, h_5) &= (\sum_{\sigma} a_5 a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)} a_{\sigma(4)}) \otimes g_5 g 1 g_2 g_3 g_4 \\ &= 0 \otimes g_5 g_1 g_2 g_3 g_4 \\ &= 0. \end{aligned}$$

So this identity still holds in S in this case. But unfortunately it is not necessarily true if some of g_1, g_2, g_3, g_4 are odd. 'Cross out' all the even g's and look only at the permutation induced on the odd g's. We have that

$$g_{\sigma(1)} g_{\sigma(2)} g_{\sigma(3)} g_{\sigma(4)} = g_1 g_2 g_3 g_4,$$

if the odd g_i are permuted evenly and

$$g_{\sigma(1)} g_{\sigma(2)} g_{\sigma(3)} g_{\sigma(4)} = -g_1 g_2 g_3 g_4$$

if the odd g_i are permuted oddly. Denote the correct sign with $sign(\sigma, \mathrm{odd}\, g)$. Then

$$K(h_1, h_2, h_3, h_4, h_5) = (\sum_{\sigma} sign(\sigma, \mathrm{odd}\, g) a_5 a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)} a_{\sigma(4)}) \otimes g_5 g_1 g_2 g_3 g_4$$

which is not always zero. More generally, let $h_1, h_2, \ldots h_5$ be elements of S, where each is either even or odd. We define $sign(\sigma, \mathrm{odd}\, h)$ as we defined $sign(\sigma, \mathrm{odd}\, g)$. That is we cross out all the even h's and look just at the permutation induced on the odd h's. Now let

$$K^*(x_1, x_2, x_3, x_4, x_5) = \sum_{\sigma \in Sym(4)} sign(\sigma, \mathrm{odd}\, x) x_5 x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}.$$

Then $K^*(h_1, h_2, h_3, h_4, h_5) = 0$. So in S we will be working with identities like $K^* = 0$ instead of $K = 0$. But the identities are similar and there is not much complication.

Example Let $b, x_1, x_2, y \in L$. The Engel identity gives

$$\begin{aligned} 0 &= y(x_1 b)x_2 y^2 + y(x_1 b)yx_2 y + y(x_1 b)y^2 x_2 + \\ & \quad yx_2(x_1 b)y^2 + yx_2 y(x_1 b)y + yx_2 y^2(x_1 b). \end{aligned}$$

Now let g_1 and g_2 be odd elements in S. Then identifying b and y with the elements $b \otimes 1, y \otimes 1 \in S$, we get the following identity

$$\begin{aligned} 0 &= y(g_1 b)g_2 y^2 + y(g_1 b)yg_2 y + y(g_1 b)y^2 g_2 - \\ & \quad yg_2(g_1 b)y^2 - yg_2 y(g_1 b)y - yg_2 y^2(g_1 b). \end{aligned}$$

If for example $g_1 = g_2 = x = x_1 \otimes e_1 + x_2 \otimes e_2 + x_3 \otimes e_3$, then we have the identity

$$\begin{aligned} 0 &= y(xb)xy^2 + y(xb)yxy + y(xb)y^2 x - \\ & \quad yx(xb)y^2 - yxy(xb)y - yxy^2(xb). \end{aligned}$$

Now take another copy of E, say

$$F = \langle f_1, f_2, f_3 \,|\, f_i^2 = 0 \;\; f_i f_j = -f_j f_i \text{ for all } i, j \rangle$$

and consider the colour algebra

$$C = L \otimes E \otimes F.$$

We can also split F into an even part and an odd part, $F = F_0 \oplus F_1$. Then

$$C = C_{(0,0)} \oplus C_{(0,1)} \oplus C_{(1,0)} \oplus C_{(1,1)},$$

where $C_{(i,j)} = L \otimes E_i \otimes E_j$. Let us look again at the identity

$$K(x_1, x_2, x_3, x_4, x_5) = \sum_{\sigma \in S(4)} x_5 x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} = 0,$$

which holds in L. If we substitute h_k for each x_k where each $h_k \in C_{(i,j)}$ for some i, j, we get

$$K(h_1, h_2, h_3, h_4, h_5) = (\sum_{\sigma} sign(\sigma, \mathrm{odd}_E h) sign(\sigma, \mathrm{odd}_F h) h_5 h_{\sigma(1)} h_{\sigma(2)} h_{\sigma(3)} h_{\sigma(4)}),$$

where we think of h as an odd element with respect to E if it is a sum of elements $a \otimes r \otimes s$ where each r is odd in E. Similarly h would be odd with respect to F if each s was odd. So here we get $K^*(h_1, h_2, h_3, h_4, h_5) = 0$ in C, where

$$K^*(x_1, x_2, x_3, x_4, x_5) = \sum_{\sigma \in Sym(4)} sign(\sigma, \mathrm{odd}_E x) sign(\sigma, \mathrm{odd}_F x) x_5 x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}.$$

So again calculations will be quite similar to calculations in the Lie-algebra.

4 Engel-4 algebras

In his paper [6] P. Higgins gives a short elegant proof of global nilpotency of Engel-4 algebras when $\mathrm{char}\, k \neq 2, 3, 5, 7$. Kostrikin [9] shows that the same method works when $\mathrm{char}\, k = 7$. Here follows a sketch of the proof.

As before we let U denote $\mathrm{ad}(u)$. We shall use the bracket symbol to denote the Lie product in $\mathrm{ad}(L)$. So $[XY] = \mathrm{ad}(xy) = XY - YX$ and $[XY^3] = \mathrm{ad}(xy^3)$. First we get the following identities (for details see Higgins [6]).

$$\begin{array}{ll}
X^3Y^3 = -Y^3X^3 = 13T & 2X^2Y^2XY = -2Y^2X^2YX = -25T \\
X^2Y^3X = -Y^2X^3Y = -T & XYXYXY = -YXYXYX = -3T \\
XY^3X^2 = -YX^3Y^2 = 7T & 2XY^2XYX = -2YX^2YXY = -5T \\
2X^2YXY^2 = -2Y^2XYX^2 = T & 2XY^2X^2Y = -2YX^2Y^2X = 17T \\
2XYX^2Y^2 = -2YXY^2X^2 = -13T & 2XYXY^2X = -2YXYX^2Y = -7T
\end{array}$$

where $T = -X^2Y^3X$. Now let $V = \langle U^3 |\, U = \mathrm{ad}(u), u \in L\rangle$ be the linear space generated by the cubes of the adjoint operators. Easy calculations using the equations above give

$$[XY]^3 = -21T$$

and

$$[YX^3]Y^2 + Y[YX^3]Y + Y^2[YX^3] = 31T.$$

Since 21 and 31 are coprime, this implies that T lies in V. Therefore $X^3Y^3 = 13T$ is also in V. Note also that the product of cubes is anti commutative. That is $X^3Y^3 = -Y^3X^3$. Therefore we have

$$(X^3Y^3)Z^3 = -Z^3(X^3Y^3) = X^3Z^3Y^3 = -X^3Y^3Z^3$$

so $X^3Y^3Z^3 = 0$. Since for $\operatorname{char} k \neq 2, 5$ Engel-3 algebras are nilpotent of class less than or equal to 4 and hence soluble of derived length at most 3, we get $L^{(3)}Y^3Z^3 = 0$. From this it follows in the same way that $L^{(6)}Z^3 = 0$ and then finally $L^{(9)} = 0$. So we have that L is soluble and therefore by a well known theorem of Higgins [6] it follows that L is nilpotent.

Now Kostrikin goes even further and proves that $L^{(7)} = 0$, which implies nilpotency of class ≤ 5461. In this section we shall prove that the correct upper bound is 7 using a computer to deal with the complicated calculations involved. It is not hard to see that this bound is best possible. If L is the relatively free Engel-4 Lie algebra over k, freely generated by x, y, z, the Higgins calculations show that $T = -X^2Y^3X \neq 0$. So $L^7 \neq \{0\}$.

We shall also see that $Id\langle x\rangle$ is nilpotent of class at most 3 for all x in the algebra if the characteristic of the field is not 2 or 5. In the case when characteristic is 5 we have the Higman, Havas, Newman and Vaughan-Lee [5] result that $Id\langle x\rangle^7 = 0$ for all x in the algebra. In Chapter 2 we showed that for a graded Engel-4 Lie algebra there is a upper bound for the nilpotency class which is polynomial in the number of generators. Notice that every Engel-3 algebra is also an Engel-4 algebra. So all the examples of non-nilpotent algebras given in Section 2 are also examples of Engel-4 algebras. This means that there exist non-nilpotent Engel-4 algebras when the characteristic of the field is 2 or 5. J. A. Bahturin [3] gives an example of a non-nilpotent Engel-4 algebra over a field of characteristic 3.

4.1 Nilpotency of $Id\langle x\rangle$ when $\operatorname{char} k \neq 2, 5$

In this section L will be an Engel-4 algebra over a field k such that $\operatorname{char} k \neq 2, 5$ and x will be a fixed element of L. We shall prove that $Id\langle x\rangle^4 = \{0\}$. Unfortunately the proof given here is not valid in the case when $|k| = 3$, because then we can not use the identity

$$XY^3 + YXY^2 + Y^2XY + Y^3X = 0.$$

(See the discussion about linearizations from Section 1.) Nevertheless, the result is still true when $|k| = 3$, though a slightly different proof is needed. We omit the proof for this case since it is very similar to the proof given for the case $|k| > 3$.

Lemma 4.1 *Let $a, b \in L$ and suppose that $ab = 0$. Then every left normed product of x, x, x, a, b is symmetric in a and b and is a multiple of $xaxbx$. In particular*

$$xabx^2 = xaxbx, \qquad xax^2b = -2xaxbx.$$

Proof It is easy to see that every such product lies in the linear span of

$$xaxbx, \quad xbxax, \quad xax^2b, \quad xbx^2a, \quad xabx^2.$$

From the Engel identity we have

$$ax^3b = -ax^2bx - axbx^2 = xaxbx + xabx^2,$$

and then using the Jacobi identity we get

$$\begin{aligned} 0 &= (xa)(xb)x + (xb)x(xa) + x(xa)(xb) \\ &= xaxbx - xabx^2 - bx^3a - xbxax + ax^3b + xaxbx \\ &= 3xaxbx - 2xbxax - xbax^2. \end{aligned}$$

If we interchange a and b we get (because $xbax^2 = xabx^2$)

$$\begin{aligned} 3xbxax - 2xaxbx &= 3xaxbx - 2xbxax \\ \Rightarrow \qquad 5xbxax &= 5xaxbx \\ \Rightarrow \qquad xbxax &= xaxbx. \end{aligned}$$

This clearly implies that $xabx^2 = xaxbx$ and that $xax^2b = -2xaxbx$. Therefore every product is a multiple of $xaxbx$ and is symmetric in a and b. $\square$

Lemma 4.2 *Let $a, b \in L$ and suppose that $ab = 0$. Then every left normed product of x, x, a, a, b is a multiple of $xaxab$. In particular*

$$xabxa = xaaxb = xbxaa = xaxab, \qquad xa^2bx = -2xaxab.$$

Proof It is clear that every such product is in the linear span of

$$L_a = xaxab, \quad L_b = xbxaa, \quad R_a = xabxa, \quad R_b = xaaxb, \quad U = xaabx.$$

From the Jacobi and Engel identities we get

$$\begin{array}{llll} (J1) & 0 & = & (xa)(xb)a + (xb)a(xa) + a(xa)(xb) \\ & & = & L_a - R_a + R_a - U - R_b + U \\ & & = & L_a - R_b, \end{array}$$

$$\begin{array}{llll} (J2) & 0 & = & (xb)(xa)a + (xa)a(xb) + a(xb)(xa) \\ & & = & L_b - R_a + R_b - U - R_a + U \\ & & = & L_b + R_b - 2R_a, \end{array}$$

$$\begin{array}{llll} (E1) & 0 & = & 2ax^2ab + axaxb + axbxa + 2axabx \\ & & = & -2L_a - R_b - R_a - 2U, \end{array}$$

$$\begin{array}{llll} (E2) & 0 & = & bx^2a^2 + bxaxa + bxa^2x \\ & & = & -L_b - R_a - U. \end{array}$$

From (J1) and (E2) we get $R_b = L_a$ and $R_a = -L_b - U$. Using this in (J2) and (E1) gives

$$(i) \quad 0 = 3L_b + L_a + 2U$$

and

$$(ii) \quad 0 = L_b - 3L_a - U.$$

From (i) and (ii) we have $5L_b = 5L_a$, that is, $L_b = L_a$ since $\operatorname{char} k \neq 5$. Then (ii) implies that $U = -2L_a$ and then we get from (E2) that $R_a = L_a$. So

$$R_a = R_b = L_b = L_a \quad \text{and} \quad U = -2L_a. \qquad \Box$$

Lemma 4.3 *Let $a, b, c \in L$ and suppose that $ab = ac = bc = 0$. Then all left normed products of x, x, a, b, c are symmetric in a, b, c and are multiples of $xaxbc$.*

Proof From Lemma 4.2 we have

$$-2xbxaa = -2xaaxb = xaabx.$$

So if we substitute $a + c$ for a and divide by 2 we get

$$-2xbxac = -2xacxb = xacbx.$$

But the last product is symmetric in a, b, c, so the other two are also symmetric. This clearly implies that all products of x, x, a, b, c are multiples of $xaxbc$. $\square$

Lemma 4.4 *Let $a, b \in L$ and suppose that $ab = 0$. Then every left normed product of x, x, x, a, a, b is a multiple of $xaxaxb$: indeed*

$$xaxabx = xaxbxa = xaxaxb.$$

Proof Suppose we have a left normed product of x, x, x, a, a, b. There are three possibilities. It ends in a, b or x. By Lemmas 4.1 and 4.2 it then lies in the linear span of $xaxbxa$, $xaxaxb$ and $xaxabx$. Now using Lemma 4.1 we have

$$2xaxbxa = -xax^2ba = -xax^2ab = 2xaxaxb,$$

so $xaxbxa = xaxaxb$. And then using the Jacobi identity we have

$$\begin{aligned}
0 &= (xax)(xa)b + (xa)b(xax) + b(xax)(xa) \\
&= -3xaxaxb + xab(xa)x - xabx(xa) - xaxb(xa) \quad \text{by Lemma 4.1} \\
&= -3xaxaxb + 3xaxabx - xaxbxa + xaxabx \\
&\quad -xaxbxa + xaxabx \quad \text{by Lemma 4.1 and Lemma 4.2} \\
&= -3xaxaxb - 2xaxbxa + 5xaxabx \\
&= -5xaxaxb + 5xaxabx.
\end{aligned}$$

Hence $xaxabx = xaxaxb$ since $\operatorname{char} k \neq 5$. $\square$

Lemma 4.5 *Let $a, b, c \in L$ such that $ab = ac = bc = 0$. Then every left normed product of x, x, x, a, b, c is symmetric in a, b, c and a multiple of $xaxbxc$.*

Proof If the product ends in x, then by Lemma 4.3 it is a multiple of $xaxbcx$ and symmetric in a, b and c. If it ends in c, then it is a multiple of $xaxbxc$ and symmetric in a and b by Lemma 4.1. But using Lemma 4.1 we get

$$2xaxbxc = -xax^2bc = -xax^2cb = 2xaxcxb.$$

So it is symmetric in a, b, c. This also implies that every product that ends in a, b or c is a multiple of $xaxbxc$. But from Lemma 4.4 we have

$$xaxabx = xaxaxb,$$

which by substituting $a + c$ for a implies that

$$2xaxcbx = 2xaxcxb,$$

and therefore

$$xaxbcx = xaxbxc.$$

So every product of x, x, x, a, b, c is a multiple of $xaxbxc$. $\square$

Lemma 4.6 *Let $a \in L$, then every product of x, x, x, x, a, a is 0.*

Proof Because L is an Engel-4 algebra, every product of x, x, x, x, a is zero. So we can assume that the product ends in x. But then by Lemma 4.1 it is a multiple of $xaxax^2$. It is therefore sufficient to show that $xaxax^2 = 0$. But

$$\begin{aligned} 0 &= (xax)(xa)x + (xa)x(xax) + x(xax)(xa) \\ &= xax^2ax - xaxax^2 + 0 - xax^3a + xax^2ax \\ &= -2xaxax^2 - xaxax^2 - 0 - 2xaxax^2 \quad \text{by Lemma 4.1} \\ &= -5xaxax^2, \end{aligned}$$

so $xaxax^2 = 0$. $\square$

Lemma 4.7 *Let $a, b \in L$ and suppose that $ab = 0$. Then every product of x, x, x, x, a, b is 0.*

Proof If the product is not to be zero, it must end in x because of the Engel condition. By Lemma 4.1 it is then a multiple of $xaxbx^2$ and is symmetric in a and b. But Lemma 4.6 implies that it is also anti-symmetric in a and b. Therefore it must be 0. $\Box$

Lemma 4.8 *Let $a, b \in L$ and suppose that $ab = 0$. Then every left normed product of x, x, x, x, a, a, b is 0.*

Proof From Lemmas 4.1, 4.2 and 4.4 we have

$$xbxaxa = xaxbxa = xaxaxb = xaxabx = xbxaax \quad (*).$$

By Lemma 4.7 we can assume that the product ends in x, and then by Lemma 4.4 it is a multiple of $xaxaxbx$. So we only have to show that this element is 0.

Now if U is a product of x, x, a, a and V of x, b then $UV = 0$:

$$(xaxa)(xb) = xaxaxb - xaxabx = 0 \text{ by } (*),$$

and

$$(xaax)(xb) = xa^2x^2b - xa^2xbx = xaxaxb - xaxabx = 0$$

by Lemma 4.1, Lemma 4.2 and (*). We shall use this in the following calculations (all underlined products are zero), and also the fact (Lemma 4.7) that every product of x, x, x, x, a, a, b which ends in a or b is 0. So in the following calculations we will not write down products that end in a or b. Now by the Engel identity we have

$$\begin{aligned}
0 &= x(xbx)a^2x + xa(xbx)ax + xa^2(xbx)x + xa^2x(xbx) + xaxa(xbx) \\
&= -xbx^2a^2x - xbx(xa)ax + xa^2(xb)x^2 - \underline{xa^2x(xb)x} \\
&\quad + \underline{xa^2x(xb)x} - xa^2x^2(xb) + \underline{xaxa(xb)x} - xaxax(xb) \\
&= 2xbxaxax + 3xbxaxax + 3xaxabx^2 + xaxaxbx \\
&\quad + xaxaxbx \quad \text{by Lemma 4.1 and Lemma 4.2} \\
&= 10xaxaxbx \quad \text{by } (*),
\end{aligned}$$

so $xaxaxbx = 0$, since $\operatorname{char} k \neq 2, 5$. $\Box$

Lemma 4.9 *Let $a, b, c \in L$ such that $ab = ac = bc = 0$. Then every product of x, x, x, x, a, b, c is 0.*

Proof By Lemma 4.7 we can assume that the product end in x. By Lemma 4.5 it is then a multiple of $xaxbxcx$ and symmetric in a, b, c. But from Lemma 4.8 we have $xaxbxax = 0$, so if we substitute $a+c$ for a we get $2xaxbxcx = 0$ and therefore $xaxbxcx = 0$. $\Box$

So we are now ready for more general results.

Lemma 4.10 *Let $a_1, \ldots, a_n \in L$ such that $n \geq 3$ and $a_ia_j = 0$ for all i, j. Then all products of $x, x, a_1, \ldots, a_n$ are symmetric in the a's and multiples of $xa_1xa_2 \cdots a_n$.*

Proof We will use induction on n. The case $n = 3$ follows from Lemma 4.3. So assume this is true for some $n \geq 3$.
Let $a_1, \ldots a_{n+1} \in L$ and suppose that $a_ia_j = 0$ for all i, j. By the inductive hypothesis every product of $x, x, a_1, \ldots a_{n+1}$ that ends in a_{n+1} is symmetric in $a_1, \ldots, a_n$ and a multiple of $xa_1xa_2 \ldots a_{n+1}$. But

$$xa_1xa_2 \cdots a_{n-1}a_na_{n+1} = xa_1xa_2 \cdots a_{n-1}a_{n+1}a_n.$$

So the product is symmetric in $a_1, \ldots, a_{n+1}$. This also implies that every product of $x, x, a_1, \ldots, a_{n+1}$ that ends in some a is a multiple of $xa_1xa_2 \cdots a_{n+1}$. It is therefore only left to show that $xa_1 \cdots a_{n+1}x$ is a multiple of $xa_1xa_2 \cdots a_{n+1}$. It is now clear that all products are symmetric in the a's. We consider two cases.
Case 1 If n is odd we have

$$\begin{aligned} 0 &= (xa_1 \cdots a_{n-1})(xa_n)a_{n+1} + (xa_n)a_{n+1}(xa_1 \cdots a_{n-1}) \\ &\quad +a_{n+1}(xa_1 \cdots a_{n-1})(xa_n) \\ &= (-1)^{n-1}xa_1 \cdots a_{n+1}x + xa_1 \cdots a_{n+1}x \\ &\quad + \text{a multiple of } xa_1xa_2 \cdots a_{n+1} \text{ (ind. hyp)} \\ &= 2xa_1 \cdots a_{n+1}x + \text{a multiple of } xa_1xa_2 \cdots a_{n+1}. \end{aligned}$$

So $xa_1 \cdots a_{n+1}x$ is a multiple of $xa_1xa_2 \cdots a_{n+1}$.
Case 2 If n is even we have from the Engel identity

$$\begin{aligned} 0 &= 2a_{n+1}(xa_1 \cdots a_{n-2})a_{n-1}a_nx + 2a_{n+1}xa_{n-1}a_n(xa_1 \cdots a_{n-2}) \\ &\quad +\text{a multiple of } xa_1xa_2 \cdots a_{n+1} \text{ (ind. hyp)} \end{aligned}$$

$$\begin{aligned} &= -2xa_1\cdots a_{n+1}x - (-1)^{n-2}2xa_1\cdots a_{n+1}x \\ &\quad + \text{a multiple of } xa_1xa_2\cdots a_{n+1} \text{ (ind. hyp)} \\ &= -4xa_1\cdots a_{n+1}x + \text{a multiple of } xa_1xa_2\cdots a_{n+1}, \end{aligned}$$

so $xa_1\cdots a_{n+1}x$ is a multiple of $xa_1xa_2\cdots a_{n+1}$.
Hence the lemma follows. □

Lemma 4.11 *Let $a_1,\ldots,a_n \in L$ and suppose that $a_ia_j = 0$ for all i,j and $n \geq 3$. Then all products of $x,x,x,a_1,\ldots a_n$ are symmetric in the a's and multiples of $xa_1xa_2xa_3\cdots a_n$.*

Proof We will use induction over n. The case $n = 3$ follows from Lemma 4.5. So let $n \geq 3$ and assume this is true for n.
Let $a_1,\ldots,a_{n+1} \in L$ and suppose that $a_ia_j = 0$ for all i,j. By the same argument as in the proof of Lemma 4.10 it follows that these products are symmetric in the a's and that every product that ends in some a is a multiple of $xa_1xa_2xa_3\cdots a_{n+1}$. We can therefore assume that the product ends in x. From Lemma 4.10 we then have that it is a multiple of $xa_1xa_2\cdots a_{n+1}x$. So it is only left to show that $xa_1xa_2\cdots a_{n+1}x$ is a multiple of $xa_1xa_2xa_3\cdots a_{n+1}$. Now from the Jacobi identity we have

$$\begin{aligned} 0 &= (xa_1xa_2\cdots a_{n-1})(xa_n)a_{n+1} + (xa_n)a_{n+1}(xa_1xa_2\cdots a_{n-1}) \\ &\quad + a_{n+1}(xa_1xa_2\cdots a_{n-1})(xa_n) \\ &= xa_1xa_2\cdots a_{n+1}x + (-1)^{n-2}xa_{n+1}\cdots a_2(xa_1x) \\ &\quad + \text{a multiple of } xa_1xa_2xa_3\cdots a_{n+1} \text{ (ind. hyp)} \end{aligned}$$

and from the Engel identity

$$\begin{aligned} 0 &= 2a_{n+1}(xa_1xa_2\cdots a_{n-2})a_{n-1}a_nx + 2a_{n+1}xa_{n-1}a_n(xa_1xa_2\cdots a_{n-2}) \\ &\quad + \text{a multiple of } xa_1xa_2xa_3\cdots a_{n+1} \text{ (ind. hyp)} \\ &= -2xa_1xa_2\cdots a_{n+1}x + (-1)^{n-2}2xa_{n+1}\cdots a_2(xa_1x) \\ &\quad + \text{a multiple of } xa_1xa_2xa_3\cdots a_{n+1} \text{ (ind. hyp).} \end{aligned}$$

Now these equations imply that $xa_1xa_2\cdots a_{n+1}x$ is a multiple of $xa_1xa_2xa_3\cdots a_{n+1}$, so we have proved the lemma. □

Theorem 4.1 *Let L be an Engel-4 algebra over a field k such that* $\operatorname{char} k \neq 2, 5$ *and let L be generated by $x, a_1, a_2, \ldots$. Suppose that $a_i a_j = 0$ for all i, j, then*

$$Id\langle x\rangle^4 = \{0\}.$$

Proof We will prove by induction over n that every product of 4 x's and n a's is 0. The case $n = 1$ is trivial and if $n = 2$ or 3 this follows from lemmas 4.7 and 4.9. So assume this is true for some $n \geq 3$.

Every product of $x, x, x, x, a_1, \ldots, a_{n+1}$ that ends in a is 0 by the inductive hypothesis. So we can assume that it ends in x. But then it is a multiple of $xa_1xa_2xa_3 \cdots a_{n+1}x$ by Lemma 4.11. So we only have to show that this product is 0. Now using the Jacobi identity we have

$$\begin{aligned} 0 &= (xa_1xa_2xa_3 \cdots a_{n-1})(xa_n)a_{n+1} + (xa_n)a_{n+1}(xa_1xa_2xa_3 \cdots a_{n-1}) \\ &\quad + a_{n+1}(xa_1xa_2xa_3 \cdots a_{n-1})(xa_n) \\ &= xa_1xa_2xa_3 \cdots a_{n+1}x + (-1)^{n-3}xa_{n+1} \cdots a_3(xa_1xa_2x), \text{ (ind. hyp)} \end{aligned}$$

and from the Engel identity combined with the inductive hypothesis

$$\begin{aligned} 0 &= 2a_{n+1}(xa_1xa_2xa_3 \cdots a_{n-2})a_{n-1}a_nx + 2a_{n+1}xa_{n-1}a_n(xa_1xa_2xa_3 \cdots a_{n-2}) \text{ (ind. hyp)} \\ &= -2xa_1xa_2xa_3 \cdots a_{n+1}x + (-1)^{n-3}2xa_{n+1} \cdots a_3(xa_1xa_2x). \end{aligned}$$

From these equations we have $xa_1xa_2xa_3 \cdots a_{n+1}x = 0$. Hence the theorem follows. □

And we can now remove the restriction that $a_i a_j = 0$.

Theorem 4.2 *If L is an Engel-4 algebra over a field k such that* $\operatorname{char} k \neq 2, 5$*, then*

$$Id\langle x\rangle^4 = \{0\} \qquad \textit{for all } x \in L.$$

Proof Let L be the (relatively) free Engel-4 algebra over k generated by $x, a_1, a_2, \ldots$ and let I be the ideal generated by $a_i a_j$ for $i, j \in \mathbb{N}$. We will prove the following statement by induction on n.

> 'If $u_5, \ldots, u_n$ are elements of L and U is a left normed product of $x, x, x, x, u_5, \cdots, u_n$ in some order, then $U = 0$.'

The case $n = 5$ is trivial. Suppose this is true for $n = m$. Let a be a left normed product of $x, x, x, x, a_5, \ldots, a_{m+1}$. Now L and I are multigraded and $a_5 + I, a_6 + I, \cdots$ commute in L/I. Therefore it follows from Theorem 4.1 that a is a linear combination of elements in $L(x, x, x, x, a_5, \ldots, a_{m+1}) \cap I$, where $L(x, x, x, x, a_5, \ldots, a_{m+1})$ is the homogeneous component of L of weight $(4, 1, \cdots, 1)$ in $x, a_5, \ldots a_{m+1}$. Now I is spanned, as a vector space, by products $b_1 b_2 \cdots b_r$ where one of the b's is some $a_i a_j$ and the other b's are generators of the algebra. Since L is multigraded $L(x, x, x, x, a_5, \ldots, a_{m+1}) \cap I$ is spanned by products $b_1 b_2 \cdots b_m$ where 4 of the b's are x's, one b is some $a_i a_j$ and the other b's are the other a's in $\{a_5, \ldots, a_{m+1}\}$. But these are all left normed products of 4 x's and m u's and hence zero by the induction hypothesis. Now let U be a product of $x, x, x, x, u_5, \ldots, u_{m+1}$. Let θ be a homomorphism from L to L which sends x to x and a_i to u_i. Then $U = \theta(a)$ where a is a product of $x, x, x, x, a_5, \ldots, a_{m+1}$. But then $a = 0$ and therefore also $U = 0$. So we have proved the theorem. □

4.2 Nilpotency of L when $\mathrm{char}\, k \neq 2, 3, 5$

In this section we shall prove that if L is an Engel-4 algebra over a field k such that $\mathrm{char}\, k \neq 2, 3, 5$ then L is nilpotent of class ≤ 7. In the last chapter we reduced this problem to three smaller ones; the (sym,sym) case, the (sym,skew) case and the (skew,skew) case.

An example of a non-nilpotent Engel-$(p+1)$ Lie-algebra over a field of characteristic p may be found in [3]. Taking $p = 3$, this example shows that there exist non-nilpotent Engel-4 Lie-algebras of characteristic 3.

Let us now turn to the proof of the nilpotency when $\mathrm{char}\, k \neq 2, 3, 5$. We will consider the (sym,skew) case. The other cases are treated similarly. We let S be the superalgebra $L \otimes U \otimes E$ defined in Section 3 and we let $x = x_1 \otimes u_1 \otimes 1 + x_2 \otimes u_2 \otimes 1 + x_3 \otimes u_3 \otimes 1$, $y = x_4 \otimes 1 \otimes e_1 + x_5 \otimes 1 \otimes e_2 + x_6 \otimes 1 \otimes e_3$, $a = x_7 \otimes 1 \otimes 1$ and $b = x_8 \otimes 1 \otimes 1$. We need to show that all products of x, x, x, y, y, y, a, b are zero. So consider the subalgebra A of S generated by a, b, x, y. Note that a, b and x are even, and that y is odd. A is spanned (as a vector space over k) by left-normed products of the generators a, b, x, y. If v is a product of the generators then v is even if an even number of y's occur in v, and v is odd if an odd number of y's occur in v. The fact that L is an Engel-4 Lie-algebra implies that A satisfies the following identities for all products v_1, v_2, v_3, v_4, v_5 of the generators a, b, x, y.

$$v_1 v_2 = -\mathrm{sign}(\sigma, \mathrm{odd}\, v) v_2 v_1, \tag{15}$$

where $\sigma = (12)$. Notice that if $v_1 = v_2 = v$ and both are even then this formula gives $vv = -vv$ and hence $vv = 0$, but if v is odd we have $vv = vv$ which gives nothing.

$$v_1 v_2 v_3 + \mathrm{sign}(\tau, \mathrm{odd}\, v) v_2 v_3 v_1 + \mathrm{sign}(\tau^2, \mathrm{odd}\, v) v_3 v_1 v_2 = 0, \tag{16}$$

where $\tau = (123)$.

$$\sum_{\sigma \in \mathrm{Sym}(4)} \mathrm{sign}(\sigma, \mathrm{odd}\, v) v_5 v_{\sigma(1)} v_{\sigma(2)} v_{\sigma(3)} v_{\sigma(4)} = 0. \tag{17}$$

The subalgebra A could possibly satisfy more relations, but it satisfies at least these relations.

Now look at the largest algebra generated by a, b, x, y subject to the relations (15) and (16) (but not (17)). Note that this algebra is multigraded. We let B be the algebra we get from this one by factoring out the ideal consisting of all left normed products in a, b, x, y with more than one a, or one b, or three x's, or three y's. Since we are just interested in the left normed products of weight $(1, 1, 3, 3)$ in a, b, x, y and we will only work with multilinear identities, we do not need the elements from this ideal and therefore we can ignore them. The 'Nilpotent Quotient algorithm' for Lie-algebras is described in [12]. The corresponding algorithm for superalgebras is similar. We can use this algorithm to get a weighted product presentation for the algebra B. Now B is a graded algebra and

$$B = B_1 \oplus B_2 \oplus \cdots \oplus B_8$$

where B_i is the linear span of products of weight i in the generators a, b, x, y. The notion of a weighted product presentation is described in [12]. We have a vector space basis $\{b_1, \ldots, b_n\}$ for B consisting of left normed products in a, b, x, y. We have a weight $w_i \in \mathbb{N}$ assigned to every basis element b_i. This basis has the following properties:

0) $b_1 = a$, $b_2 = b$, $b_3 = x$, and $b_4 = y$;

1) $1 = w_1 = w_2 = w_3 = w_4 < w_5 \leq \ldots \leq w_n = 8$;

2) $\{b_i \,|\, w_i = j\}$ is a basis for B_j for $j = 1, \ldots, 8$;

3) if $1 \leq i \leq n$ and $w_i > 1$ then b_i has a definition of the form $b_i = b_u b_v$ for some $1 \leq v \leq u \leq n$ such that $w_i = w_u + 1, w_v = 1$;

4) b_i is odd/even if y occurs an odd/even number of times in the expression for it as a left normed product of a, b, x, y.

We also obtain relations

$$b_i b_j = c_{ij}, \quad b_j b_i = -\mathrm{sign}(\,(ij), \mathrm{odd}\, b) b_i b_j$$

for $1 \leq j \leq i \leq n$ where c_{ij} is a linear combination of basis elements of weight $w_i + w_j$. Now given such a presentation we can express every element in B uniquely as a linear combination of the basis elements and we can multiply elements together using the relations.

B_8 is the subspace of B spanned by all left normed products of weight $(1, 1, 3, 3)$ in a, b, x, y. As for Lie-algebras, it can be shown that B_8 has a basis consisting of all left normed products of weight $(1, 1, 3, 3)$ that start in a. This basis consists of 140 elements. We obtained a weighted product presentation for B which had these 140 elements as the basis elements of weight 8. Now take some basis elements $v_1, \ldots, v_5$ such that the sum of the weights is $(1, 1, 3, 3)$ in a, b, x, y. Then we use the identity (17) in A that corresponds to the full linearization of the Engel identity in L, namely

$$\sum_{\sigma} \mathrm{sign}(\sigma, \mathrm{odd}\, v) v_5 v_{\sigma(1)} v_{\sigma(2)} v_{\sigma(3)} v_{\sigma(4)} = 0,$$

as described in Section 3.2. Now use the presentation to multiply this out. We get a relation on the basis elements $m_1, \ldots, m_{140}$ of B_8

$$\alpha_1 m_1 + \cdots + \alpha_{140} m_{140} = 0.$$

If we do this for every 5-tuple $(v_1, \ldots, v_5)$ we get a set of homogeneous equations in $m_1, \ldots, m_{140}$.

The presentation for B and the relations on the basis elements of B_8 were obtained with the aid of a computer program. For each $i = 1, 2, \ldots, 8$ the homogeneous component B_i of B is spanned by left-normed products of weight i in the generators a, b, x, y. The relations between these spanning elements are all consequences of relations (15) and (16) where v_1, v_2, v_3 are left-normed products of a, b, x, y, together with the relations which specify that a left-normed product is zero if it has more than one a, one b, three x's, or three y's. The computer program computed the dimension of each B_i, and computed a basis for each B_i using these spanning products, and these relations. The program for the Nilpotent Quotient Algorithm which was used is able to produce a presentation for B and a set of homogeneous equations in $m_1, m_2, \ldots, m_{140}$ (as described above) provided the characteristic of the ground field k is a prime. However we needed a presentation for B over the field of rationals $\mathbb{Q}$, and the program was unable to do that directly. Fortunately there is a way round this problem. The coefficients in the weighted product presentation for B and in the relations $\alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_{140} m_{140} = 0$ are all integers. Furthermore it is possible to bound the integers that can occur in these relations. By choosing a prime p greater than twice the bound, and by using the program to evaluate the coefficients modulo p, it was possible to compute the values of the coefficients over $\mathbb{Q}$. To see how we can bound the coefficients look at the full linearizations of the identities

$$dvu^3 + duvu^2 + du^2vu + du^3v = 0$$

and

$$4dvu^3 - 6duvu^2 + 4du^2vu - du^3 = 0.$$

We get the second identity from the first by interchanging d and v. Now the sum of the absolute values of the coefficients in the identities above is at most 15. So in a full linearization we have at most $15 \cdot 6 = 90$ products with plus or minus sign. Each product is a left normed product in 5 variables, say v_1, v_2, v_3, v_4, v_5. We get the relations $\alpha_1 m_1 + \cdots + \alpha_{140} m_{140}$ from these identities by substituting basis elements of B for the variables such that the sum of the weights is $(1, 1, 3, 3)$ in a, b, x, y. By going through all possible combinations we get all the relations. It is clear that the coefficients are integers. Also it is not difficult to verify that when we substitute basis elements of B in a product of $v_1, \cdots, v_5$ we can get the same basis element of B_8 at most four times. For example,

$$ayyy(bx^3) = ay^3bx^3 - 3ay^3xbx^2 + 3ay^3x^2bx - ay^3x^3b,$$

and

$$ay(y^2)x(bx^2) = 2ay^3xbx^2 - 4ay^3x^2bx + 2ay^3x^3b.$$

Now since there are at most 90 such left normed products in an identity, we have the same basis element at most $90 \cdot 4 = 360$ times. So 360 is an upper bound for the coefficients.

So by evaluating the coefficients modulo the prime 997 we we were able to obtain 227 relations in $m_1, \cdots, m_{140}$ with coefficients in $\mathbb{Z}$.

The output was in such a form that one could check it by hand. First we have from the presentation for B a multiplication table for the basis elements

$$b_i b_j = \sum c_{ij}.$$

Some of the relations are defining relations $b_i b_j = b_k$ and from those the rest of the multiplication table can in principle be calculated by hand. So it is possible to check by hand that the weighted product presentation for B is correct. However the number of relations to check is enormous, and so only a representative sample was checked by hand in this way. Next the 227 relations were calculated using the presentation. Each of these relations corresponds to some equation

$$\sum_{\sigma} \mathrm{sign}(\sigma, \mathrm{odd}\, v) v_5 v_{\sigma(1)} \cdots v_{\sigma(4)} = 0$$

for some basis elements $v_1, \ldots, v_5$ with sum of weights $(1,1,3,3)$ in a, b, x, y. In the output not only the relations were given but also these $v_1, \ldots, v_5$ from which the relation was derived. So one can check a relation by calculating the left hand side of the equation using the presentation. Again, a representative sample of relations were checked in this way.

To solve the (sym,skew) case, we want to show that these relations force $m_1, \ldots, m_{140}$ to be zero.

Suppose we picked out 140 linearly independent (over $\mathbb{Z}$) relations and calculated the determinant d. Then if

$$d = 2^{\alpha} 3^{\beta} 5^{\gamma}$$

for some positive integers α, β, γ we would be done, because that would mean that $d \neq 0$ for every characteristic $\neq 2, 3, 5$.

We chose 140 independent relations but the determinant turned out to be too big to compute using a short Fortran program. But instead of calculating the determinant directly we used the Chinese Remainder Theorem:

$$\mathbb{Z}/p_1 \cdots p_m\mathbb{Z} \cong \mathbb{Z}/p_1\mathbb{Z} \times \cdots \mathbb{Z}/p_m\mathbb{Z}$$

if $p_1, \ldots, p_m$ are distinct primes.

We can calculate the determinant modulo $p_1, \ldots, p_m$ using an elementary Gauss elimination method and get m values $d_1, \ldots, d_m$. From these values we can then calculate the corresponding value d^* in $\mathbb{Z}/p_1 \cdots p_m\mathbb{Z}$. If $p_1 \cdots p_m$ is known to be bigger than the determinant d then from d^* we would get d. ($d = d^*$ or $d = d^* - p_1 \cdots p_m$).

Now a crude estimate of the determinant gave

$$|d| \leq 10^{227},$$

so $p_1 \cdots p_m > 10^{228}$ would be sufficient. In computing the values of the $d_1, d_2, \ldots, d_m$ of the determinant modulo the primes $p_1, p_2, \ldots, p_m$ we needed to avoid the possibility of integer overflow. So 70 prime numbers $p_1, \ldots, p_{70}$ between 10000 and 20000 were chosen and the determinants $d_1, \ldots, d_{70}$ in the fields $\mathbb{Z}/p_1\mathbb{Z},\cdots,\mathbb{Z}/p_{70}\mathbb{Z}$ were calculated using a short Fortran program. To get the corresponding value d^* in $\mathbb{Z}/p_1 \cdots p_{70}\mathbb{Z}$, a function from a package called Maple was used. A program from this package was used to factorize the determinant. It turned out to be

$$d_a = -2^{57} \cdot 3^{18} \cdot 5^{81} \cdot 11 \cdot 13^3 \cdot 17 \cdot 19 \cdot 31 \cdot 41 \cdot 163 \cdot 2753 \cdot 5217299.$$

Then we verified directly that $d_a = d_i \; mod \, p_i$ for $i = 1, 2, \ldots, 70$ as a check. Now this includes other prime factors than 2, 3 and 5, so another 140 relations were chosen with determinant

$$d_b = 2^{40} \cdot 3^{12} \cdot 5^{67} \cdot 7^9 \cdot 8025680120903.$$

Since the common divisor has only 2, 3 and 5 as divisors this solves the (sym,skew) case.

The (sym,sym) case and the (skew,skew) case were handled in the same way. Again two determinants were calculated with common divisor only divisible by 2, 3 and 5. So we have finally proved the following theorem.

Theorem 4.3 *If L is an Engel-4 algebra over a field k such that* $\mathrm{char}\, k \neq 2, 3, 5$ *then L is nilpotent of class* ≤ 7.

In fact the (sym,sym) case was also done by hand. But unfortunately in the other cases some of the symmetries were lost and therefore it was too complicated to do those cases by hand.

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