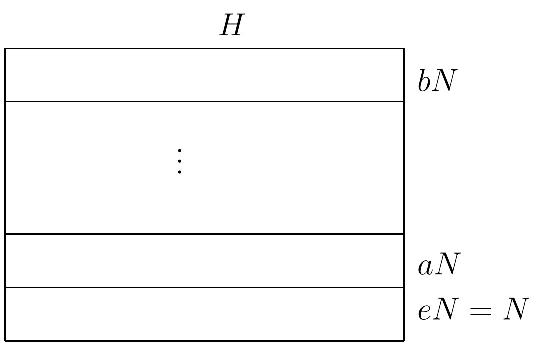
We have that $H \trianglelefteq G$ iff $g^{-1}Hg = H$ for all $g \in G$. As Ψ is a bijection this holds iff $\Psi(g^{-1}Hg) = \Psi(H)$ for all $g \in G$. In view of the identity above this holds iff $\phi(g)^{-1}\Psi(H)\phi(g) = \Psi(H)$ for all $g \in G$. But as ϕ is surjective this is true iff $r^{-1}\Psi(H)r = \Psi(H)$ for all $r \in G/N$ that is iff $\Psi(H) \trianglelefteq G/N$. $\square$

The picture that is good to keep in mind is the following.



$$\Psi(H) = H/N$$

$$\left\{ \begin{matrix} bN \\ \vdots \\ aN \\ eN \end{matrix} \right\}$$

$\Psi(H)$ is the collection of all the cosets of N in H and H is the pairwise disjoint union of these cosets. Thus if we know H we get $\Psi(H)$ as the cosets of N in H and if we know $\Psi(H)$ we get H as the union of the cosets in $\Psi(H)$.

Definition. Let $\phi : G \to H$ be a group homomorphism. The *image* of ϕ is

$$\operatorname{im}\phi = \{\phi(g) : \ g \in G\}$$

and the *kernel* of ϕ is

$$\ker\phi = \{g \in G : \phi(g) = 1\}.$$

Notice that as $G \leq G$, it follows from Lemma 1.8 that $\operatorname{im}\phi = \phi(G)$ is a subgroup of H. Also, as $\{1\} \trianglelefteq H$ it follows from Lemma 1.8 that $\ker\phi = \phi^{-1}(\{1\})$ is a normal subgroup of G.

Theorem 1.10 *(1st Isomorphism Theorem). Let $\phi : G \to H$ be a homomorphism. Then $Im\,\phi \leq H$, $Ker\,\phi \trianglelefteq G$ and*

$$G/Ker\,\phi \cong Im\,\phi.$$

Proof As we have noted previously, it follows from Lemma 1.8 that $\operatorname{Im}\phi \leq H$ and $\operatorname{Ker}\phi \trianglelefteq G$. Define a map $\Phi : G/\operatorname{Ker}\phi \to \operatorname{Im}\phi$ by setting $\Phi([a]) = \phi(a)$. This map is clearly surjective. We next show that it is well defined and injective. This follows from

$$\begin{aligned} \Phi([a]) = \Phi([b]) &\Leftrightarrow \phi(a) = \phi(b) \\ &\Leftrightarrow \phi(a^{-1}b) = \phi(a)^{-1}\phi(b) = 1 \\ &\Leftrightarrow a^{-1}b \in \operatorname{Ker}\phi \\ &\Leftrightarrow [a] = [b] \end{aligned}$$

To show that Φ is an isomorphism, it remains to show that Φ is a homomorphism. This follows from

$$\Phi([a] \cdot [b]) = \Phi([ab]) = \phi(ab) = \phi(a)\phi(b) = \Phi([a]) \cdot \Phi([b]).$$

This finishes the proof. □

Theorem 1.11 *(2nd Isomorphism Theorem). Let $H \le G$ and $N \trianglelefteq G$. Then $HN \le G$, $H \cap N \trianglelefteq H$ and*

$$H/(H \cap N) \cong HN/N.$$

Proof We apply the 1st Isomorphism Theorem. Consider the homomorphism

$$\phi : G \to G/N,\ a \mapsto aN.$$

Let ψ be the restriction of ϕ on H. This gives us a homomorphism $\psi : H \to G/N$. By the 1st Isomorphism Theorem we have that $\operatorname{Im}\psi = \{hN :\ h \in H\}$ is a subgroup of G/N. By the correspondence theorem we have that this subgroup is of the form U/N, where U is a subgroup of G that is given by

$$U = \bigcup_{h \in H} hN = HN.$$

Thus $\operatorname{Im}\psi = HN/N$. It remains to identify the kernel. The identity of G/N is the coset $eN = N$. Then for $h \in H$, we have

$$\begin{aligned} \psi(h) = N \quad &\Leftrightarrow \quad hN = N \\ &\Leftrightarrow \quad h \in N. \end{aligned}$$

As $h \in H$ this shows that the kernel of ψ is $H \cap N$. Thus by the 1st Isomorphism Theorem, $H \cap N \trianglelefteq H$ and

$$H/H \cap N = H/\operatorname{Ker}\psi \simeq \operatorname{Im}\psi = HN/N$$

This finishes the proof. □

Theorem 1.12 *(3rd Isomorphism Theorem). Suppose that $H, N \trianglelefteq G$ and $N \le H$. Then $H/N \trianglelefteq G/N$ and*

$$(G/N)/(H/N) \cong G/H.$$

Proof Again we apply the 1st Isomorphism Theorem. This time on the map

$$\phi : G/N \to G/H\, aN \mapsto aH.$$

Let us first see that this is well defined. If $aN = bN$ then $a^{-1}b \in N \subseteq H$ and thus $aH = bH$. It is also a homomorphism as

$$\phi(aN \cdot bN) = \phi(abN) = abH = aH \cdot bH = \phi(aN) \cdot \phi(bN).$$

We clearly have that $\operatorname{Im}\phi = G/H$ and it remains to identify the kernel. The identity in G/H is the coset $eH = H$ and then

$$\begin{aligned} \phi(aN) = H \quad &\Leftrightarrow \quad aH = H \\ &\Leftrightarrow \quad a \in H. \end{aligned}$$

The kernel thus consists of the cosets aN of G/N where $a \in H$. That is the kernel is H/N. The 1st Isomorphism Theorem now gives us that $H/N \trianglelefteq G/N$ (that we had proved already in the proof of the correspondence theorem anyway) and that

$$(G/N)/(H/N) = (G/N)/\operatorname{Ker}\phi \cong \operatorname{Im}\phi = G/H.$$

This finishes the proof. □

2 Direct products and abelian groups

I. Direct products.

<u>Closure properties for the set of normal subgroups of G.</u>

(1) If $H, K \trianglelefteq G$ then $H \cap K \trianglelefteq G$. To see that this is a subgroup notice that $1 \in H$ and $1 \in K$ as both are subgroups and hence $1 \in H \cap K$. Now let $a, b \in H \cap K$. As $H \leq G$ and $a, b \in H$ we know that $ab, a^{-1} \in H$. Similarly as K is a subgroup, containing a, b, we have $ab, a^{-1} \in K$. Thus $ab, a^{-1} \in H \cap K$. To see $H \cap K$ is normal notice that for $g \in G$, we have $(H \cap K)^g \subseteq H^g = H$ and $(H \cap K)^g \subseteq K^g = K$ and thus $(H \cap K)^g \subseteq H \cap K$.

(2) We also have that if $H, K \trianglelefteq G$ then $HK \trianglelefteq G$: It follows from the 2nd Isomorphism Theorem that $HK \leq G$. To see that HK is normal notice that we have $(HK)^g = H^g K^g = HK$ for $g \in G$.

Normal products. We have seen that if $H, K \trianglelefteq G$ then $HK \trianglelefteq G$. Inductively it follows that if $H_1, \ldots, H_n \trianglelefteq G$, then $H_1 \cdots H_n \trianglelefteq G$. Since $H_i H_j = H_j H_i$ for $1 \leq i < j \leq n$, we have

$$H_{\sigma(1)} \cdots H_{\sigma(n)} = H_1 \cdots H_n$$

for all $\sigma \in S_n$.

Lemma 2.1 *Let H and K be finite subgroups of G where K is normal. Then*

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}.$$

Proof By the 2nd Isomorphism Theorem, we have

$$HK/K \cong H/H \cap K.$$

Taking the orders on both sides gives. $|HK|/|K| = |H|/|H \cap K|$. The result follows immediately from this. $\Box$.

Remark. In particular it follows that $|HK| = |H| \cdot |K|$ if and only if $H \cap K = \{1\}$.

Definition Let $H_1, \ldots, H_n \trianglelefteq G$. The product $H_1 \cdots H_n$ is said to be an (internal) *direct product* of $H_1, \ldots, H_n$ if

$$H_i \cap \prod_{j \neq i} H_j = \{1\}$$

for $i = 1, \ldots, n$.

Remark. Suppose $1 \leq i < j \leq n$. As $H_j \leq \prod_{k \neq i} H_k$, we know in particular that

$H_i \cap H_j = \{1\}$ it follows from Exercise 1 on sheet 2 that all the elements in H_i commute with all the elements in H_j. So if $x_i \in H_i$ then

$$x_{\sigma(1)} \cdots x_{\sigma(n)} = x_1 \cdots x_n$$

for all $\sigma \in S_n$.

Proposition 2.2 *Let $H_1, \ldots, H_n \trianglelefteq G$ and suppose that $H_1 H_2 \cdots H_n$ is an internal direct product.*

(a) Every element $a \in H_1 \cdots H_n$ is of the form

$$a = x_1 x_2 \cdots x_n$$

for unique $x_i \in H_i$, $i = 1, \ldots, n$.
(b) If $x_i, y_i \in H_i$ for $i = 1, \ldots, n$ then

$$x_1 \cdots x_n \cdot y_1 \cdots y_n = (x_1 y_1) \cdots (x_n y_n).$$

Proof (a) If $x_1 \cdots x_n = y_1 \cdots y_n$ for some $x_i, y_i \in H_i$, then for each $1 \leq i \leq n$

$$x_i \prod_{j \neq i} x_j = y_i \prod_{j \neq i} y_j$$

and thus

$$y_i^{-1} x_i = (\prod_{j \neq i} y_j) \cdot (\prod_{j \neq i} x_j)^{-1}$$

and thus $y_i^{-1} x_i$ is in $H_i \cap \prod_{j \neq i} H_j = \{1\}$ and $x_i = y_i$.

(b) Using the fact that y_i commutes with x_j when $j > i$ we have

$$\begin{aligned} x_1 x_2 \cdots x_n y_1 y_2 \cdots y_n &= x_1 y_1 x_2 \cdots x_n y_2 \cdots y_n \\ &\vdots \\ &= (x_1 y_1)(x_2 y_2) \cdots (x_n y_n). \end{aligned}$$

This finishes the proof. $\square$

Remarks. (1) The last Proposition shows that the structure of the internal direct product $H_1 H_2 \cdots H_n$ only depends on the structure of $H_1, \ldots, H_n$. Each element is like an n-tuple $(x_1, \ldots, x_n)$ and we multiply two such componentwise. Later we will formalise this when we introduce the external direct product.

(2) Notice that it follows from part (a) of last proposition that for an internal direct product $H_1 H_2 \cdots H_n$, we have

$$|H_1 \cdots H_n| = |H_1| \cdots |H_n|.$$

The internal direct products are useful for helping us sorting out the structure of a given group. Next we discuss external direct products that are useful for constructing new groups from old groups.

Definition. Let $H_1, \dots, H_n$ be groups. The (external) *direct product* of $H_1, \dots, H_n$ is the cartesian set product

$$H_1 \times \cdots \times H_n$$

with multiplication

$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = (a_1 b_1, \dots, a_n b_n).$$

Remark. Since each H_i is a group it is immediate that the direct product is also a group with identity $(1_{H_1}, \dots, 1_{H_n})$. The inverse of $(a_1, a_2, \dots, a_n)$ is $(a_1^{-1}, a_2^{-1}, \dots, a_n^{-1})$. The associatative law follows from the fact that it holds in each component.

Next result tells us that the internal direct product is the same as the external direct product.

Lemma 2.3 *Suppose G is the internal direct product of $H_1, \dots, H_n$. Then*

$$G \cong H_1 \times \cdots \times H_n.$$

Proof (See sheet 4)