Group Theory, 2016

Exercise sheet 10 (solutions)

Exercise 1. Consider the map $f : H \ltimes_\psi N \to G$, that takes (h, n) to hn. By Lemma 6.1 from lectures we know that each g in G has a unique representation $g = hn$ with $h \in H$ and $n \in N$. Therefore the map f is bijective. It remains to see that f is a homomorpism but this follows from

$$\begin{aligned} f((x_1, y_1) \cdot (x_2, y_2)) &= f(x_1x_2, y_1^{\Psi(x_2)}y_2) \\ &= f(x_1x_2, y_1^{x_2}y_2) \\ &= x_1x_2y_1^{x_2}y_2 \\ &= x_1x_2x_2^{-1}y_1x_2y_2 \\ &= x_1y_1x_2y_2 \\ &= f(x_1, y_1) \cdot f(x_2, y_2) \end{aligned}$$

Exercise 2. (a) As G is not cyclic, there is no element of order 8. Hence any non-trivial element must have order 2 or 4. If every non-trivial element had order 2 then G would be abelian ($x^2 = 1 \Rightarrow x = x^{-1}$ for all $x \in G$ and then $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$). Hence we must have an element a of order 4. As $\langle a \rangle$ is of index 2 it must be normal in G.

(b) G is generated by a and b and as G is not abelian we can't have $a^b = a$ but a^b must be an element in $\langle a \rangle$ of same order as a and thus $a^b = a^{-1}$.

(c) Notice that

$$\begin{aligned} a^r \cdot a^s &= a^{r+s} \\ ba^r \cdot a^s &= ba^{r+s} \\ a^r \cdot ba^s &= b(a^b)^r a^s = ba^{s-r} \\ ba^r \cdot ba^s &= b^2a^{s-r} \end{aligned}$$

Thus the structure of G depends only on what b^2 is.

Let $N = \langle a \rangle$. Now bN is of order 2 and thus $b^2 \in N$. But G is not cyclic and thus we can't have $b^2 = a$ or $b^2 = a^{-1}$. Hence there are two possibilites $b^2 = 1$ or $b^2 = a^2$. So there are at most two non-abelian group of order 8. There are in fact exactly two groups. D_8 and Q. These are not isomorphic as the only element of order 2 in Q is -1 whereas D_8 has 5 elements of order 2. (Check it)

Exercise 3. (a) By Poincaré's Lemma, G is isomorphic to a subgroup K of S_3. But this is absurd as $|G| = 60 < 6 = |S_3|$.

(b) By Poincaré's Lemma, G is isomorphic to a subgroup K of S_5. Then K is of index 2 in S_5 and thus normal in S_5. We have seen on an earlier sheet that the only normal subgroup of S_5, apart from $\{1\}$ and S_5 is A_5. Thus $K = A_5$.

Exercise 4. Let $P = \langle a \rangle$. Suppose first that Q is cyclic, say $Q = \langle b \rangle$. In order to determine G

we need only to determine a^b. As G is non-abelian $a^b = a^{-1}$. So we get at most one such group. This group can be realised as an external semidirect product. This is the group T from lecturers.

We are only left with the case when Q is the diect product of two cyclic groups of order 2, say $Q = \langle b \rangle \cdot \langle c \rangle$. Now each of a^b, a^c, a^{bc} is either a or a^{-1}. I claim that at least one is a. If not then we get the contradiction that $a^{bc} = (a^b)^c = (a^{-1})^c = (a^c)^{-1} = a$. Without loss of generality we can suppose that $a^b = a$. The structure of G then only depends on a^c and as G is non-abelian we must have $a^c = a^{-1}$. Let $x = ab$ then x is of order 6 and $G = \langle ab \rangle \cdot \langle c \rangle$ where $x^c = a^c b^c = a^{-1}b = b^{-1}a^{-1} = x^{-1}$ and $G \cong D_{12}$.

Exercise 5. The numbers $2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, 37, 41, 43, 47, 49, 53, 59$ are all prime powers and $6 = 2 \cdot 3, 10 = 2 \cdot 5, 14 = 2 \cdot 7, 15 = 3 \cdot 5, 21 = 3 \cdot 7, 22 = 2 \cdot 11, 26 = 2 \cdot 13, 33 = 3 \cdot 11, 34 = 2 \cdot 17, 35 = 5 \cdot 7, 38 = 2 \cdot 19, 39 = 3 \cdot 13, 46 = 2 \cdot 23, 51 = 3 \cdot 17, 55 = 5 \cdot 11, 57 = 3 \cdot 19, 58 = 2 \cdot 29$ are all of the form pq. Then $12 = 4 \cdot 3, 18 = 9 \cdot 2, 20 = 4 \cdot 5, 28 = 4 \cdot 7, 44 = 4 \cdot 11, 45 = 9 \cdot 5, 50 = 25 \cdot 2, 52 = 4 \cdot 13$ are all of type p^2q whereas $36 = 4 \cdot 9$ was dealt with in last exercise.

The only orders that have not been dealt with are $24 = 3 \cdot 8$, $30 = 2 \cdot 3 \cdot 5$, $40 = 5 \cdot 8$, $42 = 2 \cdot 3 \cdot 7$, $48 = 3 \cdot 16$, $54 = 2 \cdot 27$ and $56 = 7 \cdot 8$.

Firstly there can't be any simple group of order 54. Such a group must have a sylow subgroup of order 27 that has then index 2 and is therefore normal. Secondly for a group of order 24 or 48 the Sylow 2-subgroup is of index 3. If the group was simple, Poincaré's Lemma would imply that the order of the group divides $3! = 6$ which is aburd. Similarly for a group of order 42, the Sylow 7-subgroup has index 6. If the group was simple then the order would have to divide 6!. In particular 7 would divide 6! but this is absurd.

We next turn to groups of order 40. By the Sylow theorems we have that the number $n(5)$ of Sylow 5 subgroups is of the from

$$n(5) = 1 + 5r, \quad r \geq 0$$

and divides $40/5 = 8$. The only possibility is $n(5) = 1$. Hence there is only one Sylow 5-subgroup P and is therefore normal by the 2nd Sylow theorem.

Suppose that G is a group of order 30. The numbers $n(3)$ and $n(5)$ of Sylow 3-subgroups and Sylow 5-subgroups satisfy

$$n(5) = 1 + 5r, \; n(3) = 1 + 3s, \quad r, s \geq 0.$$

Also $n(5)|6$ and $n(3)|10$. We argue by contradiction and suppose that G is simple. Then we must have $n(5), n(3) > 1$ and the only possibilities satisfying the criteria above are $n(5) = 6$ and $n(3) = 10$. It follows that we have $6 \cdot 4 = 24$ elements of order 5 and $10 \cdot 2 = 20$ elements of order 3. But this gives us 44 elements in total which is absurd as the group has only 30 elements. Hence G can't be simple.

We are now only left with the case when G has $56 = 7 \cdot 8$ elements. The number $n(7)$ of Sylow 7 subgroups is of the form

$$n(7) = 1 + 7r$$

and divides 8. If $n(7) = 1$ then there is a normal subgroup of order 7. The only other possibility is that $n(7) = 8$. Then there are $8 \cdot 6 = 48$ elements of order 7. Now there is also a Sylow 2-subgroup H of order 8. By Lagrange's theorem H can't have any elements of order 7. Hence there are only $56 - 48 = 8$ candidates left for elements in H. Hence there is a unique Sylow 2-subgroup that must then be normal, by Sylow's 2nd theorem, as before.