Group Theory, 2016

Exercise sheet 9 (solutions)

Exercise 1. (a) We know that if $n(p) = n(q) = 1$ then the unique Sylow p-subgroup P and the unique Sylow q-subgroup Q are normal in G. (Follows in fact from the 2nd Sylow Theorem as the Sylow p-subgroups and the Sylow q-subgroup form each a single conjugacy class). As $|P \cap Q|$ divides $|P| = p$ and $|Q| = q$ it follows that $P \cap Q = \{1\}$. Finally we know from lectures that $|PQ| = |P| \cdot |Q|/|P \cap Q| = pq/1 = |G|$ and thus $G = PQ$ is the internal direct product of P and Q.

(b) Using the Sylow Theorems, we know that $n(5) = (1 + 5r)|3$ and $n(3) = (1 + 3s)|5$ for some non-negative integers r, s. The only possible values for r and s, where this holds, are $r = s = 0$. Thus $n(5) = n(3) = 1$ and by (a) we know that $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_{15}$. (The last fact that $\mathbb{Z}_3 \oplus \mathbb{Z}_5$ is cyclic follows from Question 4 on Sheet 5).

Exercise 2. Let P be a Sylow 3-subgroup of G. The core of P, P_G, is a subgroup of P that is normal in G. As P is not normal in G it follows that $P_G = \{1\}$. Theorem 5.10 from lectures tells us then that G is isomorphic to a subgroup of S_4 of order 12. This is then a subgroup of index 2 in S_4 and thus normal in S_4. From Exercise 5 on Sheet 5 we know that the only normal subgroup in S_4 of order 12 is A_4. Hence $G \cong A_4$.

Exercise 3. (a) We have that $36 = 3^2 \cdot 2^2$. By the 1st Sylow theorem, there is a Sylow 3 subgroup which here has order 9 and therefore index 4. If G were simple then by Poincaré's lemma, we would have that G is isomorphic to a subgroup of S_4. But then

$$36 = |G| \leq |S_4| = 24$$

which is absurd. Hence G can't be simple.

(b) Without loss of generality we can assume that $p < q$. By the 3rd Sylow Theorem we have

$$n(q) = 1 + qr \,|\, p^2$$

for some non-negative integer r. If G is simple then $n(q) > 1$ and must be either p or p^2. As $q > p$ this can only happen if $n(q) = p^2$. So we have

$$qr = (p - 1)(p + 1).$$

As $q > p$ we then must have that q divides $p + 1$ and thus equal to $p + 1$ (as $q > p$). This can only happen if $p = 2$ and $q = 3$. But then the order of G is $2^2 \cdot 3^2 = 36$. By (a) we know that G can't be simple. $\square$

Exercise 4. For both parts we apply the 2nd isomorphism theorem $PN/N \cong P/P \cap N$. As the orders of the two groups is then the same we get in particular $|PN| \cdot |P \cap N| = |N| \cdot |P|$ and thus

$$[N : P \cap N] = [PN : P].$$

(a) As $P \cap N \leq P$ it is clearly a p-group. Furthermore $[N : N \cap P] = [PN : P]$ and as $[PN : P]$ divides $[G : P] = [G : PN] \cdot [PN : P]$ it is not divisible by p. (Notice that as P is a Sylow p-subgroup of G, $[G : P]$ is coprime to p). It follows that $N \cap P$ is a Sylow p-subgroup of N.

(b) As $PN/N \cong P/N \cap P$, it is a p-group. Furthermore $[G/N : PN/N] = [G : PN]$ that divides $[G : P] = [G : PN] \cdot [PN : P]$. Hence $[G/N : PN/N]$ is coprime to p and PN/N is a Sylow p-subgroup of G/N.

Exercise 5. (a) The elements of A_4 are

order 1: id
order 2: $(1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)$
order 3: $(1\ 2\ 3),\ (1\ 3\ 2),\ (1\ 2\ 4),\ (1\ 4\ 2),\ (1\ 3\ 4),\ (1\ 4\ 3),\ (2\ 3\ 4),\ (2\ 4\ 3)$

The Sylow 3-subgroups are of order 3 and thus cyclic. These are clearly

$$H_1 = \langle (1\ 2\ 3)\rangle,\ H_2 = \langle (1\ 2\ 4)\rangle,\ H_3 = \langle (1\ 3\ 4)\rangle,\ H_4 = \langle (2\ 3\ 4)\rangle$$

Notice that the number of these is $4 = 1 + 3 \cdot 1$ which is in accordance with the third Sylow theorem. They also form a single conjugacy class as $H_2 = H_1^{(1\ 2)(3\ 4)}$, $H_3 = H_1^{(1\ 3)(2\ 4)}$ and $H_4 = H_1^{(1\ 4)(2\ 3)}$. There are only 4 elements that are of 2-power order which means that we have only one Sylow 2-subgroup

$$K_4 = \{\text{id},\ (1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)\}.$$

(b) The extra 12 elements of S_4, i.e. the odd elements are

order 2: $(1\ 2),\ (1\ 3),\ (1\ 4),\ (2\ 3),\ (2\ 4),\ (3\ 4)$
order 4: $(1\ 2\ 3\ 4),\ (1\ 2\ 4\ 3),\ (1\ 3\ 2\ 4),\ (1\ 3\ 4\ 2),\ (1\ 4\ 2\ 3),\ (1\ 4\ 3\ 2)$

Notice that none of these elements have order that is a power of 3. Hence the Sylow 3-subgroups remain the same as in (a). Notice that $|S_4| = 3 \cdot 8$ and so the Sylow 2-subgroups should have order 8. First let us see that every Sylow 2-subgroup must contain the normal subgroup K_4 of S_4 as a subgroup. Let G be any subgroup of order 8. Then $GK_4/K_4 \cong G/G \cap K_4$. This implies that

$$|GK_4| = |GK_4/K_4| \cdot |K_4| = \frac{|G| \cdot |K_4|}{|G \cap K_4|}$$

which is a power of 2. Since $G \leq GK_4$ it follows that $GK_4 = G$ and thus $K_4 \leq G$. By the correspondence theorem we now only need to find all subgroups in S_4/K_4 of order 2. But these are the cyclic groups of order 2 that correspond to the elements of 2-power order that are not in K_4. These are all the cosets aK_4 where a runs through those 12 extra odd elements. Inspection shows that we get the following groups:

$R_1 = K_4 \cup K_4(1\ 2) = K_4 \cup \{(1\ 2),\ (3\ 4),\ (1\ 4\ 2\ 3),\ (1\ 3\ 2\ 4)\}$
$R_2 = K_4 \cup K_4(1\ 3) = K_4 \cup \{(1\ 3),\ (2\ 4),\ (1\ 4\ 3\ 2),\ (1\ 2\ 3\ 4)\}$
$R_3 = K_4 \cup K_4(1\ 4) = K_4 \cup \{(1\ 4),\ (2\ 3),\ (1\ 3\ 4\ 2),\ (1\ 2\ 4\ 3)\}$

Notice that the number of these is $3 = 1 + 2 \cdot 1$ in accordance with the third Sylow theorem and that they form a single conjugacy class as

$$R_2 = R_1^{(2\ 3)},\ R_3 = R_1^{(2\ 4)}.$$