Group Theory, 2016

Exercise sheet 7 (solutions)

Exercise 2. Let $N \trianglelefteq S_n$. Then $N \cap A_n \trianglelefteq A_n$ and as A_n is simple it follows that there are two possibilities. Either $A_n \cap N = A_n$ or $A_n \cap N = \{1\}$.

In the first case we have $A_n \leq N$ and as $N/A_n \leq S_n/A_n$ we have by Lagrange's Theorem that $|N/A_n|$ is either 1 or 2, i.e. $N = S_n$ or $N = A_n$. We turn next to the latter case. As $N \cap A_n = \{1\}$, we have that id is the only even element in N. We will show that the only possibility is that $N = \{1\}$. Suppose not and let $1 \neq \alpha \in N$ where α is odd. If we had another $1 \neq \beta \in N$ then $\beta\alpha^{-1}$ is even and thus $\beta\alpha^{-1} = \mathrm{id}$ which implies that $\beta = \alpha$. It follows that $N = \{\alpha, \mathrm{id}\}$. From this one can get a contradiction as follows. As N is normal we must also have $\alpha^\beta = \alpha$ for all $\beta \in S_n$. As α is of order 2 it must have the following cycle structure

$$\alpha = (i_1\ j_1)(i_2\ j_2)\cdots(i_r\ j_r)$$

with r a odd number. We will apply the formula from lectures that

$$(i_1\ i_2\ \ldots\ i_r)^{\beta^{-1}} = (\beta(i_1)\ \ldots\ \beta(i_r)).$$

If $r \geq 3$ then (using $(i_1\ i_2)^{-1} = (i_1\ i_2)$)

$$\alpha^{(i_1\ i_2)} = (i_2\ j_1)(i_1\ j_2)\cdots(i_r\ j_r) \neq \alpha$$

and if $r = 1$ and $i \notin \{i_1, j_1\}$ then

$$\alpha^{(i_1\ i)} = (i\ j_1) \neq \alpha$$

and N is not normal in G. This shows that the only normal subgroups of S_n are $\{1\}$, S_n and A_n.

Exercise 3. (a) As

$$(i_1\ \cdots\ i_{r-1}\ i_r) = (i_1\ i_r)(i_1\ i_{r-1})\cdots(i_1\ i_2)$$

we have that a cycle is an even/odd element iff it has odd/even length. Now if we have a permutation $\alpha = \beta_1 \cdots \beta_r$, where the beta's are disjoint cycles, then α is even/odd iff there is an even/odd number of cycles that are odd elements. This happens iff there is an even/odd number of cycles that are of even length.

The possible cycle structures for an even element in S_5 are thus are (5), $(3,1,1)$, $(2,2,1)$ and $(1,1,1,1,1)$. So we have that A_5 is a union of four conjugacy classes in S_5. By Exercise 4, we have that the one with cycle structure (5) is the only one that breaks into two conjugacy classes in A_5 (as this is the only cycle structure where we have that all orbits are of odd length and these lengths are distinct). Hence we have 5 conjugacy classes in A_5. The first two w.r.t. to cycle structure (5) have $4!/2 = 12$ elements. The one with cycle structure $(3,1,1)$ has $\binom{5}{3} \cdot 2 = 20$ elements and the one with cycle structure $(2,2,1)$ has $5 \cdot 3 = 15$ elements. Then there is the conjugacy class containing the identity element that has only 1 element.

(b) We argue by contradiction and suppose that N is a proper non-trivial normal subgroup of A_5. We then have that N is a union of some of the conjugacy classes of A_5 (including id). This implies that the number of elements in N is $1 + a$ where a is a partial proper sum of elements from $\{12, 12, 20, 15\}$ (but not all of these). Inspection shows that no such sum divides $60 = |A_5|$ and so N can't exist by Lagrange's Theorem.

Exercise 4. (a) Suppose the cycle of even length is $a = (i_1\ i_2\ \cdots\ i_m)$. Clearly x commutes with a which is odd. It now follows from Exercise 2(a) on sheet 6 that $x^{S_n} = x^{A_n}$.

(b) Suppose that the two cycles of odd length are $(i_1\ \cdots\ i_r)$ and $(j_1\ \cdots\ j_r)$. Let $a = (i_1\ j_1)\cdots(i_r\ j_r)$. Then a is an odd element that swaps (by conjugataion) the two cycles and fixes the remaining cycles of x. Thus x remains the same permuations and x commutes with a. Again we have by Exercise 2(a) on sheet 6 that $x^{S_n} = x^{A_n}$.

(c) We have seen in (a) and (b) that the condition is necessary. It remains to see that it is sufficient. Let the orbits of x be $O_1, \ldots, O_r$ that we assume all have odd number of elements and that no two orbits have the same number of elements. Let $x_1, \ldots, x_r$ be the corresponding cycles. So $x = x_1 \cdots x_r$. Let $a \in S_n$ be any element that commutes with x. Then $x^{a^{-1}} = x_1^{a^{-1}} \cdots x_r^{a^{-1}} = x$ and as there is only one cycle of each length, we must have that $x_i^{a^{-1}} = x_i$ for $i = 1, \ldots, r$. Suppose that $x_1 = (i_1\ \cdots\ i_m)$. Then

$$(i_1\ i_2\ \cdots\ i_m) = x_1 = x_1^{a^{-1}} = (a(i_1)\ a(i_2)\ \cdots\ a(i_m)).$$

This can only happen if for some $1 \leq k \leq m$ we have $a(i_1) = i_k,\ a(i_2) = i_{k+1},\ \cdots\ a(i_m) = i_{k-1}$ which means that a acts on the orbit $O_1 = \{i_1, \ldots, i_m\}$ like a power of $x_1 = (i_1\ i_2\ \cdots\ i_m)$. Similarly a acts on any orbit O_i like a power of x_i and $a = x_1^{s_1} \cdots x_m^{s_m}$ for some integers $s_1, \ldots, x_m$ and since $x_1, \ldots, x_m$ are all even, we conclude that a must be even. Hence there is no odd element that commutes with x.

Exercise 5. (a) Notice that the conjugacy class containing x is the same as the conjugacy class containing x^a. From lectures we then know that $|C_G(x)| = |G|/|x^G| = |G|/|(x^a)^G| = |C_G(x^a)|$.

(b) Let the conjugacy classes of G be $x_1^G, x_2^G, \ldots, x_r^G$. We have $p = l/(|G| \cdot |G|)$ where $l = \sum_{x\in G} |C_G(x)|$. Now summing over the conjugacy class $C_i = x_i^G$ gives (using part (a))

$$\sum_{y\in C_i} |C_G(y)| = \sum_{y\in C_i} |C_G(x_i)| = |x_i^G| \cdot |C_G(x_i)|$$

that by part 1(b) is equal to $\frac{|G|}{|C_G(x_i)|} \cdot |C_G(x_i)| = |G|$. Summing over all the conjugacy classes gives us

$$\sum_{y\in G} |C_G(y)| = \sum_{y\in C_1} |C_G(y)| + \cdots + \sum_{y\in C_r} |C_G(y)| = r|G|$$

and thus $p = \frac{r|G|}{|G|\cdot|G|} = \frac{r}{|G|}$.