Group Theory, 2016

Exercise sheet 6 (solutions)

Exercise 1.(a) Let e be the identity element in G. Firstly as $x = x^e$, we have that the relation is reflexive. If $y = x^g$ then $x = y^{g^{-1}}$ that shows that $\sim$ is symmetric. Finally if $y = x^a$ and $z = y^b$. Then $z = y^b = (x^a)^b = x^{ab}$ that shows that $\sim$ is transitive. Hence we have an equivalence relation and the equivalence classes form a pairwise disjoint partition of G. Now observe that the equivalence class containing x is the conjugacy class x^G.

(b) Let $a \in N$ then, as $N \trianglelefteq G$, we have $a^G \subseteq N$. This shows that $N = \bigcup_{a\in N} a^G$ and thus N is the union of some conjugacy classes in G.

Exercise 2. As every subgroup of an abelian group is normal, to say that G is simple is the same as saying that the only subgroups are $\{1\}$ and G itself. As $G \neq \{1\}$ there is some $1 \neq a \in G$. As $\{1\} < \langle a\rangle \leq G$ and G is simple, it follows that $G = \langle a\rangle$. Thus G is cyclic. Next we claim that G is of finite order. If not, $\langle a^2\rangle$ would be a non-trivial, proper subgroup of G. Thus $G = \langle a\rangle$ is finite. If $o(a)$ was not a prime, say $o(a) = rs$ for some integers $r, s \leq 2$, then $\langle a^r\rangle$ is again a non-trivial proper subgroup that contradicts simplicity of G. Hence $|G|$ is a prime.

Conversely if $|G|$ is a prime and H a subgroup of G then by Lagrange's Theorem $|H|$ divides $|G| = p$. As p is a prime we then either have $|H| = 1$, and $H = \{1\}$ or $|H| = p = |G|$ and $H = G$. Hence G is simple.

Exercise 3. (a) We have $x^{S_n} = x^{A_n} \cup x^{aA_n} = x^{A_n} \cup (a^{-1}xa)^{A_n} = x^{A_n} \cup x^{A_n} = x^{A_n}$.

(b) For a contradiction, suppose that there is a common element. This means that we have $x^a = x^b$ where a is even and b is odd. Then $x^{ba^{-1}} = x$ and we get the contradiction that x commutes with the odd element ba^{-1}.

Exercise 4. Let $H_n = G$. By Exercise 4 on sheet 3, there exists a subgroup H_{n-1} of G that is of order $p_1 \cdots p_{n-1}$ (as $p_1 \cdots p_{n-1}$ is a divisor of $|G|$). Similarly there exists a subgroup H_{n-2} of H_{n-1} of order $p_1 \cdots p_{n-2}$. continuing in this manner gives us the sequence we want. Notice that

$$|H_{i+1}/H_i| = |H_{i+1}|/|H_i| = \frac{p_1 \cdots p_i p_{i+1}}{p_1 \cdots p_i} = p_{i+1}$$

and it follows that H_{i+1}/H_i is a group of order p_{i+1}. As p_{i+1} is a prime, we know from lecturers that, up to isomorphism, there is only one group of order p_{i+1}, namely $\mathbb{Z}_{p_{i+1}}$.

Notice that it follows from this in particular that for any given positive integer n, all abelian groups of order n have the same composition factors.

Exercise 5. (a) A non-trivial element in S_4 can have the following possible cycle structures

$$(r\ s),\ (r\ s\ t),\ (r\ s\ t\ u),\ (r\ s)(t\ u).$$

From lectures that we know if a is of one of these types, thhen $a^G = \{a^x : x \in G\}$, that consists of all elements in S_4 that have the same cycle structure. In this case the cycle structures give us the following partition of G

	Elements	Size
B_1	id	1
B_2	$(1\,2)^G$	$\binom{4}{2} = 6$
B_3	$(1\,2\,3)^G$	$4 \cdot 2 = 8$
B_4	$(1\,2\,3\,4)^G$	$3! = 6$
B_5	$[(1\,2)(3\,4)]^G$	3

Notice that these add up to $24 = 4! = |S_4|$ elements. Now a subset N of S_4 is a normal subgroup if and only if it is a union of some of the B_i's, including B_1, and this union is a subgroup. By Lagrange's Theorem the number of elements in such a union would then have to divide $24 = |S_4|$. Inspection shows that the only possibilities are the following unions with sizes $1, 4, 12$ and 24

$$\{1\} = B_1, \quad K_4 = B_1 \cup B_5, \quad A_4 = B_1 \cup B_5 \cup B_3, \quad S_4 = B_1 \cup \cdots \cup B_5.$$

One can check that K_4 is closed under multiplication and taking inverses and thus a subgroup so all these are normal subgroups of S_4.

(b) One composition series is

$$\{\mathrm{id}\} < \langle(1\ 2)(3\ 4)\rangle < K_4 < A_4 < S_4$$

with composition factors $\mathbb{Z}_2$, $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_2$.