Group Theory, 2016

Exercise sheet 4 (solutions)

Exercise 1. Suppose first that HK is a subgroup of G. Then as HK is invariant under taking inverses (and every element in HK is an inverse $a = (a^{-1})^{-1}$), we see that

$$HK = (HK)^{-1} = K^{-1}H^{-1} = KH.$$

Conversely, suppose $HK = KH$. We check that the subgroup criteria hold. Firstly, $1 = 1 \cdot 1 \in HK$. Also if $a = hk \in HK$ with $h \in H$ and $k \in K$, then

$$a^{-1} = k^{-1}h^{-1} \in KH = HK.$$

Finally if $a = h_1k_1$ and $b = h_2k_2$ are in HK with $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Then

$$ab = h_1k_1h_2k_2 \in HKHK = HHKK \subseteq HK.$$

Thus all the subgroup criteria hold and $HK \leq G$.

Exercise 2. Consider the map

$$\phi : H_1 \times \cdots \times H_n \to G,\ (x_1, \ldots, x_n) \mapsto x_1 \cdots x_n.$$

By Proposition 2.2 from lectures, we know that this map is bijective. Using part (b) of Proposition 2.2, we also have

$$\begin{aligned}
\phi((x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n)) &= \phi(x_1y_1, \ldots, x_ny_n) \\
&= (x_1y_1) \cdots (x_ny_n) \\
&= x_1 \cdots x_n y_1 \cdot y_n \\
&= \phi(x_1, \ldots, x_n) \cdot \phi(y_1, \ldots, y_n).
\end{aligned}$$

Hence ϕ is a homomorphism and thus an isomorphism.

Exercise 3.(a) As there are at most $|H_i|$ ways of picking x_i in $x_1 \cdots x_n \in H_1 \cdots H_n$, there is a limit of $|H_1| \cdots |H_n|$ of elements in $H_1 \cdots H_n$. Thus $|H_1 \cdots H_n| \leq |H_1| \cdots |H_n|$. For the reverse inequality notice that, by Lagrange, each $|H_i|$ divides $|H_1 \cdots H_n|$ as $H_i \leq H_1 \cdots H_n$. As the numbers $|H_1|, \ldots, |H_n|$ are pairwise coprime it follows that their product divides $|H_1 \cdots H_n|$. But then it follows in particular that $|H_1| \cdots |H_n| \leq |H_1 \cdots H_n|$.

(b) We need to show that $H_i \cap \prod_{j \neq i} H_j$ is trivial. We do this by showing that its order is 1. By Lagrange's Theorem this order must divide both $|H_i|$ and $|\prod_{j \neq i} H_j|$. By part (a) we know that latter number is equal to $\prod_{j \neq i} |H_j|$ that is coprime to $|H_i|$. As $|H_i \cap \prod_{j \neq i} H_j|$ divides both, it follows that it must be 1.

Exercise 4. First consider the map $\theta : G \to G,\ g \mapsto g \cdot \phi(g)^{-1}$. We first show that θ is injective. If $\theta(a) = \theta(b)$, then $a\phi(a)^{-1} = b\phi(b)^{-1}$ and thus $b^{-1}a = \phi(b)^{-1}\phi(a) = \phi(b^{-1}a)$. But 1 is the only element fixed by ϕ and thus $b^{-1}a = 1$, i.e. $a = b$. This shows that θ is injective and as G is finite it follows that θ is bijective.

Now let $x \in G$. As θ is bijective we have that $x = \theta(a) = a\phi(a)^{-1}$ for some $a \in G$. Then

$$\phi(x) = \phi(a \cdot \phi(a)^{-1}) = \phi(a) \cdot \phi^2(a)^{-1} = \phi(a)a^{-1} = x^{-1}.$$

As ϕ is a homomorphism, we have $ab = (b^{-1}a^{-1})^{-1} = \phi(b^{-1}a^{-1}) = \phi(b^{-1})\phi(a^{-1}) = ba$ for all $a, b \in G$. Hence G is abelian.

Exercise 5. Consider some H_i, $i = 1, \ldots, n$. Now H_i is not abelian so in particular there is some $a \in H_i$ that is not contained in $Z(G)$. So there must be some $j = 1, \ldots, m$ such that a does not commute with all the elements of K_j. Let $b \in K_j$ such that a and b do not commute. Then the element $1 \neq a^{-1}b^{-1}ab \in H_i \cap K_j$. Then $H_i \cap K_j$ is a normal subgroup of G that is a subgroup of both H_i and K_j. In particular $H_i \cap K_j$ is a non-trivial normal subgroup of both H_i and K_j and as H_i, K_j are simple, it follows that $H_i = H_i \cap K_j = K_j$. This argument shows that

$$\{H_1, \ldots, H_n\} \subseteq \{K_1, \ldots, K_m\}.$$

By symmetry we have that a similar argument shows that

$$\{K_1, \ldots, K_m\} \subseteq \{H_1, \ldots, H_n\}.$$

Hence $n = m$ and we have the same set of subgroups.

This result does not hold without the extra assumption that the simple factors are non-abelian. Take for example the group $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Here G can be written as direct sum of two abelian sugroups of order 2 in three different ways:

$$\begin{aligned} G &= \mathbb{Z}(1,0) + \mathbb{Z}(0,1) \\ &= \mathbb{Z}(1,0) + \mathbb{Z}(1,1) \\ &= \mathbb{Z}(0,1) + \mathbb{Z}(1,1). \end{aligned}$$