Group Theory, 2016

Exercise sheet 2 (solutions)

Exercise 1. Let $h \in H$ and $k \in K$ and consider the element

$$h^{-1}k^{-1}hk \in G.$$

Firstly this element is a product $(h^{-1}k^{-1}h)k$ of a conjugate of k^{-1} and k and is thus in K. Secondly this element is also a product $h^{-1}(k^{-1}hk)$ of h^{-1} and a conjugate of h and is therefore in H. So

$$h^{-1}k^{-1}hk \in H \cap K = \{1\}$$

and $h^{-1}k^{-1}hk = 1 \Leftrightarrow hk = kh$.

Exercise 2. (a) To show that $Z(G)$ is a subgroup of G, we apply the usual subgroup criteria. We need to show that the identity element 1 is in $Z(G)$ and that $Z(G)$ is closed under multiplication and taking inverses. Clearly 1 commutes with everything. Now let $a, b \in Z(G)$ and let $g \in G$. Then using the fact that a, b commute with everything, we have $abg = agb = gab$, that shows that $ab \in Z(G)$. Then taking inverses on both sides in $ag^{-1} = g^{-1}a$, we get $ga^{-1} = a^{-1}g$ that shows that $a^{-1} \in Z(G)$. As $aZ(G) = Z(G)a$ for all $a \in G$, we also know that $Z(G)$ is a normal subgroup.

(b) Suppose that $G/Z(G) = \langle aZ(G) \rangle$. Then the cosets of $Z(G)$ in G are

$$a^r Z(G) \quad r \in \mathbb{Z}.$$

So every element in G is of the form $a^r u$ for some $r \in \mathbb{Z}$ and $u \in Z(G)$. Suppose that $a^r u$ and $a^s v$ are any two elements in G, where $u, v \in Z(G)$ and $r, s \in \mathbb{Z}$. Then

$$a^r u \cdot a^s v = a^r a^s vu = a^s a^r vu = a^s v \dot{a}^r u$$

where we have used the fact that u, v commute with everything in G. Hence G is abelian. □

Exercise 3. Let H be a subgroup of G. If H is the trivial subgroup then $H = \langle 1 \rangle$ is clearly cyclic. Suppose now that H is not trivial and suppose $a^m \in H$ for some $m \neq 0$. As H is a subgroup we then also have $a^{-m} \in H$ and thus without loss of generality, we can assume that $m > 0$. Let n be the smallest positive integer such that $a^n \in H$. As H is a subgroup, and thus closed with respect to taking products and inverses, it is clear that $\langle a^n \rangle \leq H$. It now suffices to show that $H \leq \langle a^n \rangle$. To see this, let a^m be an arbitrary element in H. Using division with remainer, we can write $m = sn + r$ for some integer $0 \leq r < n$. As H is a subgroup and $a^n, a^m \in H$ we see that $a^r = a^m \cdot (a^n)^{-s} \in H$. By the minimality of n we must have $r = 0$ and thus $a^m = (a^n)^s \in \langle a^n \rangle$.

Exercise 4. Notice first that

$$a^*b^*(ab) = \sum_{fh=ab} a^*(f)b^*(h) = a^*(a)b^*(b) = 1$$

whereas for $g \neq ab$,

$$a^*b^*(g) = \sum_{fh=g} a^*(f)b^*(h) = 0$$

as for each summand either $f \neq a$, that gives $a^*(f) = 0$, or $h \neq b$, that gives $b^*(h) = 0$. It follows that $a^*b^* = (ab)^*$.

In particular $a^*(a^{-1})^* = 1^* = \epsilon$ and a^* is a unit. Also

$$\phi(ab) = (ab)^* = a^*b^* = \phi(a)\phi(b)$$

that shows that ϕ is a homomorphism. It remains to see that ϕ is injective. However if $b^* = a^*$ then $b^*(a) = a^*(a) = 1$ that can only happen if $b = a$. .

Notice that if $\alpha \in \mathbb{Z}^G$ and $\alpha(a) = n_a$, then

$$\alpha = \sum_{a \in G} n_a a^*.$$

Thus $R = \mathbb{Z}^G = \sum_{a \in G} \mathbb{Z}a^*$ and thus R is generated by G (i.e. the copy of G sitting inside R^*) as a ring.

Exercise 5. (a) Clearly id $\in N$ and as each element is its own inverse, N is closed under taking inverses. Finally, inspection shows that if we multiply two of the elements together, then the result is still in N. (If $N = \{e, a, b, c\}$ then $ab = c$). To see that N is normal notice that $\alpha(i\,j)(r\,s)\alpha^{-1}$ swaps $\alpha(i)$ and $\alpha(j)$ as well as $\alpha(r)$ and $\alpha(s)$. Thus

$$\alpha(i\,j)(r\,s)\alpha^{-1} = (\alpha(i)\,\alpha(j))(\alpha(r)\alpha(s))$$

that is in N. An alternative way (tedious though)of showing that N is normal in S_4, is to show that the left cosest and the right cosets are the same. In part (b) the left cosets are determined and one can similarly determine the right cosets.

(b) The cosets are

$$\begin{aligned}
&\mathrm{id}N = \{\mathrm{id},\ (1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)\},\\
&(1\ 2)N = \{(1\ 2),\ (3\ 4),\ (1\ 3\ 2\ 4),\ (1\ 4\ 2\ 3)\},\\
&(1\ 3)N = \{1\ 3),\ (1\ 2\ 3\ 4),\ (2\ 4),\ (1\ 4\ 3\ 2)\},\\
&(2\ 3)N = \{(2\ 3),\ (1\ 3\ 4\ 2),\ (1\ 2\ 4\ 3),\ (1\ 4)\},\\
&((1\ 2\ 3)N = \{(1\ 2\ 3),\ (1\ 3\ 4),\ (2\ 4\ 3),\ (1\ 4\ 2)\},\\
&(1\ 3\ 2)N = \{(1\ 3\ 2),\ (2\ 3\ 4),\ (1\ 2\ 4),\ (1\ 4\ 3)\}.
\end{aligned}$$

Thus S_4/N has 6 elements.

(c) We have that $S_4/N = \{uN : u \in S_3\}$ and $uN \cdot vN = uvN$. Thus this quotient group is isomorphic to S_3.