Group Theory, 2015

Hint sheet 10

Exercise 1. There is a natural map from $H \ltimes_{\Psi} N$ to HN, that maps (h, n) to hn. You need to show that this map is bijective and that it is a homomorphism.

Exercise 2. (a) Argue first that there can't be an element in G of order 8. Thus all non-trivial elements must have order (dividing 8), 2 or 4. They can't all have order 2 (why?).

(b) $a^b \in \{1, a, a^2, a^3\}$ has the same order as a.

(c) In order to determine the structure, notice that you have two cosets in G, H and bH where $H = \langle a \rangle$. So there are products of four different types:

$$a^r \cdot a^s = a^{r+s},\ ba^r \cdot a^s = ba^{r+s},\ a^r \cdot ba^s = ba^k,\ ba^r \cdot ba^s = b^2 a^k.$$

(Determine through direct calculations what k is in terms of r and s). In order to determine the structure fully, you now only need to know what the value of b^2 is (this is in $\langle a \rangle$, why?).

Exercise 3. For both parts Theorem 5.10 and Poincaré's Lemma will be useful. For part (b) notice that a subgroup of index 2 must be normal in S_5. We have seen on an earlier sheet what the normal subgroups of S_5 are.

Exercise 4. Let $P = \langle a \rangle$. When Q is cyclic show that the structure of G is like that of the group T from the lectures.

For the case when Q is a direct product of two cyclic groups of order 2, let b, c, d be the elements in Q of order 2. Show that one of a^b, a^c, a^d but be equal to a and thus that we get a cyclic subgroup of G of order 6 (and then index 2). Argue that G has to be isomorphic to D_{12}.

Exercise 5. The groups of order p^n, pq, p^2q and p^2q^2 are dealt with in lectures and on a previous exercise sheet. This leaves only few remaining cases to be dealth with.

Course webpage: http://people.bath.ac.uk/gt223/MA30237.html