Group Theory, 2016
Hint sheet 9

Exercise 1. (a) Recall that a group is an internal direct product of P and Q if $P, Q \trianglelefteq G$, $G = PQ$ and $P \cap Q = \{1\}$.

Exercise 2. Theorem 5.10 from the lectures would be useful here.

Exercise 3. (a) Here Poincaré's Lemma is useful.
(b) Try to use a similar argument as the one we used in lectures for dealing with groups of order p^2q.

Exercise 4. (a) To show that $P \cap N$ is a Sylow p-subgroup of N you need to show two things. Firstly that $|P \cap N|$ is a power of p and then that $[N : P \cap N]$ is not divisible by p. For the latter it could be useful to apply the 2nd Isomorphism Theorem.

(b) Again you need to show two things. That $|PN/N|$ is a power of p and that $[G/N : PN/N]$ is not divisible by p. Again the 2nd Isomorphism Theorem is useful.

Exercise 5. The most difficult part of this exercise is to find the Sylow 2-subgroups of S_4. Below in (b) you have a sketch of one method of doing this.

(a) Write up the list of all the elements in A_4. There are $12 = 2^2 \cdot 3$ of these. A Sylow 3-subgroup must then be of order 3 (and thus cyclic) whereas a Sylow 2-subgroup P_2 should have 4 elements. Notice that by Lagrange's Theorem, every element in P_2 must then have an order that divides 4. Which elements could be in P_2?

(b) Write first up the list of all the additional elements (the odd elements that are also 12). Now $|S_4| = 24 = 3 \cdot 8$. Again a Sylow 3-subgroup must be cyclic of order 3 whereas a Sylow 2-subgroup Q should have 8 elements.

To determine all such groups Q argue first that $K_4 \leq Q$ (where $K_4 = \{\text{id}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)$ a normal subgroup of S_4). To see that $K_4 \leq Q$, show that $|K_4Q|$ is a power of 2 (using a formula from lectures) and notice that as $Q \leq K_4Q$ you must have that $|K_4Q| = 8$.

Use the Correspondence Theorem. Finding all subgroups of S_4 of order 8 reduces then to the problem of finding all cyclic groups of order 2 for S_4/K_4.

Course webpage: http://people.bath.ac.uk/gt223/MA30237.html