Group Theory, 2016

Exercise sheet 4 (hints)

Exercise 1. You can use the fact that a subset U is a subgroup of G if and only if $1 \in U$, $U \cdot U = U$ and $U^{-1} = U$.

Exercise 2. There is a natural candidate for the isomorphism $f : H_1 \times \cdots \times H_n \to G$. Now show that this map is bijective and a homomorphism, using Proposition 2.2 from lectures.

Exercise 3. (a) For the inclusion $\supseteq$ you can use Lagrange's Theorem.

(b) Show that $H_i \cap \prod_{j \neq i} H_j$ has order 1 using Lagrange's Theorem.

Exercise 4. First show that if $g\phi(g)^{-1} = h\phi(h)^{-1}$ then we must have $g = h$. Use here the fact that 1 is the only element fixed by ϕ. This gives you that θ is a bijection. Thus any element x in G can be written of the form

$$x = g\phi(g)^{-1}.$$

Now calculate $\phi(x)$ using this formula.

Exercise 5. Let H_i be any of the groups in the first set. We want to show that $H_i = K_j$ for some K_j in the latter set. There exists some $a \in H_i$ that is not in $Z(H)$ and thus there has to be some K_j and some $b \in K_j$ such that a does not commute with b. Then $a^{-1}b^{-1}ab \in H_i \cap K_j$. What can you conclude about $H_i \cap K_j$ using the fact that both H_i and K_j are simple?