Group Theory, 2016

Exercise sheet 10

Exercise 1. Suppose that $G = HN$ is the internal semidirect product of N by H. For each $h \in H$ let ϕ_h be the automorphism in $\mathrm{Aut}\,(N)$ that maps n to n^h and consider the homomorphism

$$\Psi : H \to \mathrm{Aut}\,(N)$$

with $\Psi(h) = \phi_h$. Show that $G \cong H \ltimes_{\Psi} N$.

Exercise 2. (Groups of order 8). Let G be a non-abelian group of order 8.

(a) Show that there is an $a \in G$ of order 4. Conclude that $H = \langle a \rangle$ is normal in G.
(b) Let $b \in G \setminus H$. Show that $a^b = a^{-1}$.
(c) Show that b^2 is either 1 or a^2 and conclude that there are 2 non-abelian groups of order 8.

Exercise 3. Let G be a non-abelian simple group of order 60

(a) Show that G has no subgroup of index 3.
(b) If G has a subgroup H of index 5, show that G must be isomorphic to A_5.

Exercise 4 (Groups of order 12). Let G be a non-abelian group of order 12. Let P and Q be the Sylow subgroups of order 3 and 4 respectively. Suppose that $P \trianglelefteq G$. Thus $G = PQ$ is the semidirect product of P by Q. Find all these. (**Hint**. There are two cases according to whether $Q \cong \mathbb{Z}_4$ or $Q \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$).

Exercise 5. Show that there is no non-abelian simple group with order less than 60.

Course webpage: http://people.bath.ac.uk/gt223/MA30237.html