Group Theory, 2016
Exercise sheet 8

Exercise 1. Consider the quoternions $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. The group of quoternions is $Q = \{\pm i, \pm j \pm k, \pm 1\}$, a group of order 8.

(a) Show that Q is a group with respect to the inherited ring multiplication from $\mathbb{H}$.
(b) Show that all the subgroups of Q are normal. List the subgroups.

Exercise 2. Let X be a G-set where G is a group of order p^n for some prime number p and some postive integer n. Suppose that X is finite of size m where p does not divide m. Show that the action of G on X has a fixed point (i.e. there is some $x \in X$ such that $xg = x$ for all $g \in G$).

Exercise 3. Let p be a prime number and let G be a group of order p^2. Show that G is abelian. Hence deduce that up to isomorphism there are exactly two groups of order p^2, $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \oplus \mathbb{Z}_p$. (**Hint**. Use Theorem 5.1 and Exercise 2 from Sheet 2).

Exercise 4. Let G be a finite p-group and let N be a non-trivial normal subgroup of G. Show that $N \cap Z(G) \neq \{1\}$. Conclude that any normal subgroup of order p is contained in $Z(G)$. (**Hint**. Consider $X = N$ as a G-set with right multiplication $x * g = x^g$. Modify the proof of Theorem 5.1).

Exercise 5. (Another proof of Cauchy's Theorem). Let H be a group with order divisible by a prime number p. Let

$$X = \{(a_1, \ldots, a_p) \in H \times \cdots \times H : \; a_1 \cdots a_p = 1\}.$$

Consider the cyclic subgroup $G = \langle (1\ 2\ \ldots\ p) \rangle \leq S_p$. We define an action from G on X by:

$$(a_1,\ \ldots\ , a_p) \cdot \sigma = (a_{(1)\sigma},\ \ldots\ , a_{(p)\sigma}).$$

(In this exercise it will be convenient to perform a composition from the left to right. So for $\alpha \circ \beta$ one first performs α and then β. We also write $(i)\sigma$ for the value of σ in i).

a) Show that $|X| = |H|^{p-1}$ and that X is a G-set with this action.
b) Show that X must have a G-orbit $\{(a, a, \cdots, a)\}$ of size 1 with $a \neq 1$. Conclude that H has an element of order p.

Course webpage: http://people.bath.ac.uk/gt223/MA30237.html