Group Theory, 2016
Exercise sheet 7

Exercise 1. Let X be a G-set where G is a group of order p^n for some prime number p and some postive integer n. Suppose that X is finite of size m where p does not divide m. Show that the action of G on X has a fixed point (i.e. there is some $x \in X$ such that $xg = x$ for all $g \in G$).

Exercise 2. Let $n \geq 5$. Show that apart from $\{1\}$ and S_n, the only normal subgroup of S_n is A_n. What are the composition factors of S_n? Conclude that S_n is not solvable when $n \geq 5$.

Exercise 3.

(a) Determine all the conjugacy classes of A_5 and calculate the size of each of these.
(b) Show that no proper non-trivial subgroup H of A_5 can be normal. Hence, deduce that A_5 is simple.

Exercise 4. Let $x \in A_n$ and write x as a product of disjoint cycles.

(a) Show that if one of the cycles is of even length then $x^{S_n} = x^{A_n}$.
(b) Show that if two of the cycles have the same odd length, then $x^{S_n} = x^{A_n}$.
(c) Show that x^{A_n} and $x^{(1\,2)A_n}$ are disjoint if and only if all the cycles for x are of odd length and no two cycles have the same length.

Exercise 5. Let G be a finite group. For each $x \in G$ we let $C_G(x) = \{a \in G : xa = ax\}$, the *centralizer* of x in G.

(a) Let $x, a \in G$. Show that $|C_G(x)| = |C_G(x^a)|$.
(b) The probability that two elements in G commute is

$$p = \frac{|\{(x, y) \in G \times G :\, xy = yx\}|}{|G \times G|}.$$

Show that $p = r/|G|$ where r is the number of conjugacy classes in G. (**Hint**. The numerator is equal to $\sum_{x \in G} |C_G(x)|$. Write G as a union of pairwise disjoint conjugacy classes and show that the sum over each conjugacy class gives the value $|G|$).

Course webpage: http://people.bath.ac.uk/gt223/MA30237.html