Group Theory, 2016

Exercise sheet 6

Exercise 1. Let G be a finite group. For $a \in G$ we refer to the subset $a^G := \{a^g : g \in G\}$ as the *conjugacy class* containing $a \in G$. Define a binary relation on G as follows:

$$x \sim y \text{ iff } y = x^g \text{ for some } g \in G.$$

(a) Show that $\sim$ is an equivalence relation on G and deduce that G is the union of pairwise disjoint conjugacy classes.
(b) Show that if $N \trianglelefteq G$, then N is the union of some conjugacy classes in G.

Exercise 2. Let G be an abelian group. Show that G is simple if and only if G is cyclic of prime order. (Notice that we are not assuming that G is finite to start with).

Exercise 3. Let $x \in A_n$. Notice that if $a \in S_n$ is odd then $x^{S_n} = x^{A_n} \cup (x^a)^{A_n}$.

(a) Suppose x commutes with some odd element $a \in S_n$. Show that $x^{S_n} = x^{A_n}$.
(b) Suppose that x commutes with none of the odd elments of S_n. Show that the conjugacy classes x^{A_n} and $(x^{(1\,2)})^{A_n}$ are disjoint.

Exercise 4. Let G be a finite abelian group of order $p_1 \cdots p_n$ where $p_1, \ldots, p_n$ are the prime factors in some order. Show that there exists a composition series

$$H_0 = \{1\} \leq H_1 \leq \cdots \leq H_n = G$$

where $H_{i+1}/H_i \cong \mathbb{Z}_{p_{i+1}}$ for $0 \leq i \leq n-1$. (**Hint**. Use Exercise 4(b) on Sheet 3).

Exercise 5. Consider the group S_4.

(a) Find all the normal subgroups of S_4.
(b) Write down a composition series for S_4 and deduce that S_4 is solvable.

Course webpage: http://people.bath.ac.uk/gt223/MA30237.html