Group Theory, 2016

Exercise sheet 5

Exercsie 1. Let $(G, +)$ be an abelian group of prime exponent p. We introduce a scalar multiplication from the field $\mathbb{Z}_p$ by letting $[m] \cdot x = mx$ for $[m] \in \mathbb{Z}_p$ and $x \in G$.

(a) Check that G becomes a vector space over $\mathbb{Z}_p$ by checking that all the vector space axioms hold.

(b) Let H be a subset of G. Show that H is a subgroup of the group G if and only if H is a subspace of the vector space G.

Exercise 2. We say that an abelian group G is finitely generated, if there exist finitely many elements $x_1, \ldots, x_n \in G$ such that $G = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n$.

(a) Show that $\mathbb{Q}$, the additive group of the rationals, is not finitely generated.

(b) Show that any for any two element $x, y \in \mathbb{Q}$, there exist integers r, s (not both zero) such that $rx + sy = 0$. (One then says that x and y are linearly dependent).

Exercise 3. (a) List all abelian groups of order 144 and all abelian groups of order up to 15.

(b) Let A and B be finite abelian groups. Show that if $A \oplus A$ and $B \oplus B$ are isomorphic then A and B are isomorphic.

Exercise 4. Let G be a finite abelian group that is an internal direct sum of cyclic groups, say $G = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n$. Show that G is cyclic if and only if $o(x_1), \ldots, o(x_n)$ are pairwise coprime.

Exercise 5. Let F be a finite field and let $F^* = F \setminus \{0\}$ be the group of units. Show that F^* is a cyclic group using the Fundamental Theorem for finite abelian groups. (**Hint**. Argue by contradiction and suppose F^* is not cyclic. Use the Fundamental Theorem and Exercise 4 to find for some prime p, p^2 roots to the polynomial $x^p - 1$).

Course webpage: http://people.bath.ac.uk/gt223/MA30237.html