Group Theory, 2016
Exercise sheet 4

Exercise 1. Let H, K be subgroups of a group G. Show that HK is a subgroup of G if and only if $HK = KH$.

Exercise 2. Let $H_1, \ldots, H_n$ be normal subgroups of a group G such that $G = H_1 \cdots H_n$ is an internal direct product. Show that the external direct product $H_1 \times \cdots \times H_n$ is isomorphic to G.

Exercise 3. Suppose that $H_1, \ldots, H_n$ are normal subgroups of a group G where $H_1, \ldots, H_n$ are all finite and of pairwise coprime orders.

(a) Show that $|H_1 H_2 \cdots H_n| = |H_1| \cdot |H_2| \cdots |H_n|$.
(b) Show that $H_1 H_2 \cdots H_n$ is an internal direct product.

Exercise 4. Let G be a finite group and let $\phi : G \to G$ be an automorphism that only fixes 1 and where $\phi \circ \phi = \mathrm{id}$. Show that $\phi(x) = x^{-1}$ for all $x \in G$ and that G is abelian. (**Hint**. consider the map $\theta : G \to G$, $g \mapsto g\phi(g)^{-1}$. Show first that θ is a bijection).

Exercise 5. We say that a group G is *simple* if $G \neq \{1\}$ and the only normal subgroups are $\{1\}$ and G.

Let G be a finite group that is an (internal) direct product of finite <u>non-abelian</u> simple groups

$$G = H_1 H_2 \cdots H_n$$

Suppose we have another factorization

$$G = K_1 K_2 \cdots K_m$$

into an (internal) direct product of non-abelian simple factors. Show that the factors are unique up to order, i.e. show $\{H_1, \ldots, H_n\} = \{K_1, \ldots, K_m\}$.

Give an example that shows that the result does not hold without the assumption that the simple factors are non-abelian.

Course webpage: http://people.bath.ac.uk/gt223/MA30237.html