Group Theory, 2016

Exercise sheet 3

Exercise 1. (a) Let G be an abelian group and let n be a positive integer. Show that the subsets

$$G^n = \{a^n : a \in G\} \text{ and } G[n] = \{a \in G : a^n = 1\}$$

of G are subgroups, and that $G^n \cong G/G[n]$.

(b) Let $G = \mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$. For which pairs n, m of integers are G^n and G^m isomorphic? What is the answer when $G = \mathbb{C}^*$?

Exercise 2. Let G be a finite group with a normal subgroup N and let $a \in G$.

(a) Show that the order, $o(aN)$, of aN in G/N divides the order, $o(a)$, of a in G.
(b) Show that if $o(a)$ is coprime to $[G : N]$, then $a \in N$.

Exercise 3. Let $\mathbb{Q}^+$ be the set of all positive rational numbers. Are the groups $(\mathbb{Q}^+, \cdot)$ and $(\mathbb{Q}, +)$ isomorphic? Justify your answer.

Exercise 4. Let G be a finite abelian group.

(a) If p dividies $|G|$ for some prime p, show that G has an element of order p. (**Hint**. Prove this by induction on $|G|$. For the induction step take any cyclic subgroup $H = \langle a \rangle$ and consider the groups H and G/H).

(b) Use (a) to deduce that if m is any positive integer that divides $|G|$ then G has a subgroup H where $|H| = m$. (**Hint**. Use induction on $|G|$ and the Correspondence Theorem).

Exercise 5. Let G be a group and let $\mathrm{A}ut\,(G)$ be the set of all isomorphisms from G to itself. These are usually referred to as the *automorphisms* of G. For $a \in G$ consider the map $\phi_a : G \to G$, $x \mapsto x^{a^{-1}} = axa^{-1}$.

(a) Show that the set $\mathrm{Aut}\,(G)$ is a group with respect to composition.
(b) Show that $\phi_a \in \mathrm{Aut}\,(G)$ and that the set $\mathrm{Inn}(G) = \{\phi_a : a \in G\}$ is a normal subgroup of $\mathrm{Aut}\,(G)$. ($\mathrm{Inn}\,(G)$ is called the group of inner automorphisms).
(c) Show that the map $\Psi : G \to \mathrm{Aut}\,(G), a \mapsto \phi_a$ is a homomorphism.
(d) What are $\mathrm{Im}\,\Psi$ and $\mathrm{Ker}\,\Psi$ in this case? What can we derive from the 1st isomorphism theorem?

Course webpage: http://people.bath.ac.uk/gt223/MA30237.html