Group Theory, 2016

Exercise sheet 2

Exercise 1. Suppose that G is a group with normal subgroups H, K with the property that $H \cap K = \{1\}$. Prove that every element in H commutes with every element in K. (**Hint**. Let $h \in H$ and $k \in K$ consider the element $h^{-1}k^{-1}hk$).

Exercise 2. Let G be a group and let $Z(G) = \{a \in G : \, ag = ga \text{ for all } g \in G\}$.

(a) Show that $Z(G)$ is a normal subgroup of G. (It is called the *center* of G).
(b) Show that if $G/Z(G)$ is cyclic, then G must be abelian.

Exercise 3. Let $G = \langle a \rangle$ be an infinite cyclic group. Show that every subgroup of G is cyclic.

Exercise 4. Let $R = \mathbb{Z}^G$ be the ring from Exercise 5 on sheet 1. For each $a \in G$ let $a^* \in \mathbb{Z}^G$ be the map that maps a to 1 but all other elements in G to 0. Show that each a^* is a unit in $\mathbb{Z}^G$ and that the map

$$\phi : G \to R^*, \;\; g \mapsto g^*$$

is an injective group homomorphism. Conclude that $R = \mathbb{Z}^G$ has a copy of G (namely $\operatorname{im} \phi$) as a subgroup of R^*.

Exercise 5. Consider the group $G = S_4$ and let

$$N = \{(1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3),\ \mathrm{id}\}.$$

(a) Show that N is a normal subgroup of S_4.
(b) List all the cosets of N in S_4. How many elements does S_4/N have?
(c) The quotient group S_4/N is isomorphic to a well known group. Which one?

Course webpage: http://people.bath.ac.uk/gt223/MA30237.html