Group Theory, 2016
Exercise sheet 1

Exercise 1. Considering multiplication tables, show that there can be only one group multiplication on a set $G = \{e, a, b\}$ with three elements where e is the identity element.

Exercise 2. Identify the Galois group of the polynomial $x^3 - 1 \in \mathbb{Q}[x]$ as a subgroup of S_3.

Exercise 3. Consider a square A in the Euclidean plane whose corner points, in anticlockwise order around the centre, are x_1, x_2, x_3 and x_4. Every isometry that preserves A permutes the corner points and can thus be thought naturally as an element of S_4. The symmetry group of A can thus be identified with a subgroup of S_4. Which elements of S_4 belong to the symmetry group of A?

Exercise 4. Let $(G, *)$ be a set with a binary operation that is associative. Suppose that there is an element $e \in G$ such that $a * e = a$ for all $a \in G$ and that for any $a \in G$ there exists $b \in G$ such that $a * b = e$. Show that $(G, *)$ is a group. (Notice that we are only assuming that there is a 'right identity' and that each element has a 'right inverse').

Exercise 5. Let G be a finite group and let $\mathbb{Z}^G$ be the set of all functions $\phi : G \to \mathbb{Z}$. Define an addition and multiplication on $\mathbb{Z}^G$ as follows:

$$\begin{aligned} [\phi + \psi](g) &= \phi(g) + \psi(g) \\ [\phi \cdot \psi](g) &= \sum_{\substack{f,h \in G \\ fh = g}} \phi(f)\psi(h). \end{aligned}$$

Let $\epsilon \in \mathbb{Z}^G$ be the function such that $\epsilon(1_G) = 1$ but $\epsilon(g) = 0$ if $g \neq 1_G$. Show that $(\mathbb{Z}^G, +, \cdot)$ is a ring with ϵ as a multiplicative identity.

Course webpage: http://people.bath.ac.uk/gt223/MA30237.html