A characterization of the Taylor expansion of $\lambda$-terms

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Recall on quantitative semantics

\[ \lambda \text{-calculus} \]

\[ X \mapsto (F)X \]

\[ \downarrow \beta^* \]

\[ Y \]

\[ x \mapsto f(x) \]

Semantics

is the Taylor expansion of \( f \)
Recall on quantitative semantics

**λ-calculus**

\[ X \mapsto (F)X \]

\[ \downarrow \beta^\ast \]

\[ Y \]

\[ \sum_{n=0}^{\infty} \frac{1}{n!} (\partial_x^n f \cdot x^n)0 \] is the *Taylor Expansion* of \( f \)

**Semantics**

\[ x \mapsto f(x) \]

\[ \sum_{n=0}^{\infty} \frac{1}{n!} (\partial_x^n f \cdot x^n)0 \]
\(\lambda\)-calculus \(\xrightarrow{\text{Taylor Expansion}}\) Resource calculus

**\(\lambda\)-calculus**

Grammar:  \(\Lambda : T, U ::= x | \lambda x. T | (T)U\)

\[(\lambda x. T)U \xrightarrow{\beta} T[U/x]\]

**Resource calculus**

Grammar:  \(\Delta : t, u ::= x | \lambda x. t | \langle t \rangle[u_1, \ldots, u_n]\)

\[\langle \lambda x. t \rangle[u_1, \ldots, u_n] \xrightarrow{r} \sum_{\sigma \in S_n} t\{u_{\sigma(1)}/x_1, \ldots, u_{\sigma(n)}/x_n\}\]

Substitutes each occurrence of \(x\) in \(t\) by only one \(u_i\)

Reduces to 0 otherwise

\(M \xrightarrow{\text{Taylor Expansion}} \sum_{t \in \text{taylor}(M)} \alpha_t t \xrightarrow{\text{NF}} \text{NF}(\sum_{t \in \text{taylor}(M)} \alpha_t t)\)

Goal: Characterize the image of this transformation
Theorem [Characterization]:

\[ \exists M \\lambda \text{-term s.t. } \sum_{t \in \Delta} \alpha_t \cdot t = \text{NF}(\text{taylor}(M)) \text{ iff } \]

0 \ldots
1 \ldots
2 \ldots
3 \ldots
Theorem [Characterization]:

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0 Theorem [Ehrhard - Regnier]:
\[ \forall \alpha_t \in \text{NF}(\text{taylor}(M)), \text{ if } \alpha_t \neq 0 \text{ then } \alpha_t = \frac{1}{m(t)} \]

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Theorem [Characterization]:

∀T ⊆ Δ, ∃M λ-term s.t. T = NF(τ(M)) iff

Theorem [Ehrhard - Regnier]:

∀α_t ∈ NF(taylor(M)), if α_t ≠ 0 then α_t = \frac{1}{m(t)}
Theorem [Characterization]:

\[ \forall \mathcal{T} \subseteq \Delta, \exists M \text{ \(\lambda\)-term s.t.} \ \mathcal{T} = \text{NF}(\tau(M)) \text{ iff} \]

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1 \( \text{FV}(\mathcal{T}) \) is finite
2 \( \mathcal{T} \) is r.e.
3 \( \ldots \)

Conditions 1 and 2: based on Barendregt’s theorem

Theorem [Barendregt]:

Let \( \mathcal{B} \) be a Böhm-like tree. There is a \( \lambda \)-term \( M \) such that \( \text{BT}(M) = \mathcal{B} \) if, and only if, \( \text{FV}(\mathcal{B}) \) is finite and \( \mathcal{B} \) is r.e.
Theorem [Characterization]:

∀\mathcal{T} \subseteq \Delta, \exists M \lambda\text{-term s.t. } \mathcal{T} = \text{NF}(\tau(M)) \text{ iff }

0  Theorem [Ehrhard - Regnier]:
   \forall \alpha_t \in \text{NF}(\text{taylor}(M)), \text{ if } \alpha_t \neq 0 \text{ then } \alpha_t = \frac{1}{m(t)}

1  \text{FV}(\mathcal{T}) \text{ is finite}

2  \mathcal{T} \text{ is r.e.}

3  \mathcal{T} \text{ is an ideal}
**Theorem [Characterization]:**

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1. \( \text{FV}(\mathcal{T}) \) is finite

2. \( \mathcal{T} \) is r.e.

3. \( \mathcal{T} \) is an ideal

**Resource calculus and Taylor expansion**

**Ideal**

**Two corollaries and further works**
Plan

Resource calculus and Taylor expansion

Ideal

Two corollaries and further works
Resource calculus

Grammar: \( \Delta : t, u ::= x | \lambda x.t | \langle t \rangle[u_1, \ldots, u_n] \)

Relation \( \rightarrow^r \) (strongly normalizing, confluent):

\[
\langle \lambda x.t \rangle[s_1, \ldots, s_n] \rightarrow^r \begin{cases} 
\{t[s_{\sigma(1)}/x_1, \ldots, s_{\sigma(n)}/x_n] \mid \sigma \in S_n \} \\
\emptyset \text{ if } \deg_x(t) \neq n
\end{cases}
\]

Unique normal form: \( \text{NF}(t) \)

\[
\text{NF}(\mathcal{T}) \triangleq \bigcup_{t \in \mathcal{T}} \text{NF}(t)
\]

Taylor expansion: \( \Lambda \longrightarrow \mathcal{P}(\Delta) \)

\[
\tau(x) \triangleq \{x\}
\]

\[
\tau(\lambda x. T) \triangleq \{\lambda x.t \mid t \in \tau(T)\}
\]

\[
\tau((T)U) \triangleq \{\langle t \rangle[u_1, \ldots, u_k] \mid t \in \tau(T); k \in \mathbb{N}; u_1, \ldots, u_k \in \tau(U)\}
\]
A first example : \( S \)

\[
S := \lambda xyz.((x)z)(y)z
\]

Böhm tree of \( S \):

```
    \lambda xyz.x
   / \  
  z   y
   \ /  
  y   z
```

Taylor expansion of \( S \):

\[
\tau(S) = \{ \lambda xyz.\langle x\rangle 11, \lambda xyz.\langle x\rangle [z, \ldots, z] [\langle y\rangle 1, \ldots, \langle y\rangle 1], \ldots \}
\]

\[
= \{ \lambda xyz.\langle x\rangle [z^n][\langle y\rangle [z^{n_1}] \ldots, \langle y\rangle [z^{n_k}]] ; k, n, n_1, \ldots, n_k \in \mathbb{N} \}
= \text{NF}\left(\tau(S)\right)
\]
Two other examples

\((S)II = ((\lambda xyz.((x)z)(y)z)\lambda x.x)\lambda x.x \overset{\beta^*}{\rightarrow} \lambda x.(x)x = \delta\)

\(\tau((S)II) = \{\langle\lambda xyz.<x>11\rangle11, \langle\lambda xz.<x>[z,\ldots,z][<y>1,\ldots,<y>1][I,\ldots,I][I,\ldots,I],\ldots\}\}

\(\text{NF}(\tau((S)II)) = \{\langle\lambda x.\lambda yz.<x>11\rangle11, \ldots\}\)
Two other examples

\[(S)II = ((\lambda xyz.((x)z)(y)z)\lambda x.x)\lambda x.x \xrightarrow{\beta^*} \lambda x. (x)x = \delta\]

\[\tau((S)II) = \{\langle \lambda xyz.\langle x\rangle 11\rangle 11, \langle \lambda xyz.\langle x\rangle [z, \ldots, z][\langle y\rangle 1, \ldots, \langle y\rangle 1]\langle I, \ldots, I\rangle[I, \ldots, I][I, \ldots, I], \ldots\}\]

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Two other examples

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Two other examples

\[(S)II = ((\lambda yz.((x)z)(y)z)\lambda x.x)\lambda x.x \xrightarrow{\beta^*} \lambda x. (x)x = \delta\]

\[\tau((S)II) = \{\langle \lambda yz. \langle x \rangle 11 \rangle 11, \langle \lambda yz. \langle x \rangle [z, \ldots, z] [\langle y \rangle 1, \ldots, \langle y \rangle 1] \rangle [I, \ldots, I][I, \ldots, I], \ldots\} \]

\[\text{NF}(\tau((S)II)) = \{\langle \lambda yz. \langle x \rangle [z, \ldots, z] [\langle y \rangle 1, \ldots, \langle y \rangle 1] \rangle [I, \ldots, I][I, \ldots, I], \ldots = \{\lambda x. \langle x \rangle [x^n], n \in \mathbb{N}\} = \tau(\delta)\]

\[\Omega = (\delta)\delta\]

\[\tau(\Omega) = \{\langle \lambda x. \langle x \rangle [x^{n_0}] \rangle [\lambda x. \langle x \rangle [x^{n_1}], \ldots, \lambda x. \langle x \rangle [x^{n_k}]]; k, n_0, \ldots, n_k \in \mathbb{N}\} \]

\[\text{NF}(\tau(\Omega)) = \emptyset\]
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Terms in normal form: $\Delta^{\text{NF}} : t ::= \lambda x_0 \ldots x_{m-1} \langle y \rangle \mu_0 \ldots \mu_{n-1}$

$\mu_i$: finite multisets of simple terms in normal form

Uniform approximation $\preceq$

\[
\lambda x_0 \ldots x_{m-1} \langle y \rangle \mu_0 \ldots \mu_{n-1} \preceq t \text{ iff }
\]

(i) $t = \lambda x_0 \ldots x_{m-1} \langle y \rangle \nu_0 \ldots \nu_{n-1}$

(ii) $\forall i < n, |\mu_i| \neq \emptyset \implies \exists v \in |\nu_i|, \forall u \in |\mu_i|, u \preceq v$

$\preceq$-ideal

$\mathcal{T} \in \mathcal{P}(\Delta^{\text{NF}})$ ideal: downward closed, directed

- $\tau(S) = \{ \lambda x y z. \langle x \rangle [z^n][\langle y \rangle [z^{n_1}], \ldots, \langle y \rangle [z^{n_k}]]; k, n, n_1, \ldots, n_k \in \mathbb{N} \}$
- $\{ \langle x \rangle [y, z] \}$
- $\{ x[x] \}$
Ideal

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- $\{ \langle x \rangle [y, z] \}$
- $\{ x[x] \}, \{ x1, x[x] \}$
Ideal

Terms in normal form: $\Delta^{\text{NF}} : t ::= \lambda x_0 \ldots x_{m-1}. \langle y \rangle \mu_0 \ldots \mu_{n-1}$

$\mu_i$: finite multisets of simple terms in normal form

Uniform approximation $\preceq$

$\lambda x_0 \ldots x_{m-1}. \langle y \rangle \mu_0 \ldots \mu_{n-1} \preceq t$ iff

(i) $t = \lambda x_0 \ldots x_{m-1}. \langle y \rangle \nu_0 \ldots \nu_{n-1}$

(ii) $\forall i < n, |\mu_i| \neq \emptyset \implies \exists \nu \in |\nu_i|, \forall u \in |\mu_i|, u \preceq \nu$

$\preceq$-ideal

$T \in \mathcal{P}(\Delta^{\text{NF}})$ ideal: downward closed, directed

- $\tau(S) = \{ \lambda x y z. \langle x \rangle [z^n][\langle y \rangle [z^{n_1}], \ldots, \langle y \rangle [z^{n_k}]] ; k, n, n_1, \ldots, n_k \in \mathbb{N} \}$
- $\{ \langle x \rangle [y, z] \}$
- $\{ x[x], \{ x1, x[x] \} \}$ $x[x, x] \preceq x[x]$
Ideal

Terms in normal form: \( \Delta^{\text{NF}} : t ::= \lambda x_0 \ldots x_{m-1}.\langle y \rangle \mu_0 \ldots \mu_{n-1} \)

\( \mu_i \): finite multisets of simple terms in normal form

Uniform approximation \( \preceq \)

\[ \lambda x_0 \ldots x_{m-1}.\langle y \rangle \mu_0 \ldots \mu_{n-1} \preceq t \text{ iff } \]

(i) \( t = \lambda x_0 \ldots x_{m-1}.\langle y \rangle \nu_0 \ldots \nu_{n-1} \)

(ii) \( \forall i < n, |\mu_i| \neq \emptyset \implies \exists v \in |\nu_i|, \forall u \in |\mu_i|, u \preceq v \)

\( \preceq \)-ideal

\( \mathcal{T} \in \mathcal{P}(\Delta^{\text{NF}}) \text{ ideal: downward closed, directed} \)

- \( \tau(S) = \{ \lambda x y z.\langle x \rangle [z^n] [\langle y \rangle [z^{n_1}], \ldots, \langle y \rangle [z^{n_k}]], k, n, n_1, \ldots, n_k \in \mathbb{N} \} \)
- \( \{ \langle x \rangle [y, z] \} \)
- \( \{ x[x], \{ x1, x[x] \}? \text{ } x[x, x] \preceq x[x] \implies \{ x[x^n] | n \in \mathbb{N} \} \)
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∀T ⊆ Δ, ∃M λ-term s.t. T = NF(τ(M)) iff

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∀α_t ∈ NF(taylor(M)), if α_t ≠ 0 then α_t = \frac{1}{m(t)}

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Two corollaries and further works
Corollary 1

Let $\mathcal{T} \in \mathcal{P}(\Delta^{\text{NF}})$.

There is a normalizable $\lambda$-term $M$ such that $\text{NF}(\tau(M)) = \mathcal{T}$ iff

(i) $\text{height}(\mathcal{T})$ is finite

(ii) $\mathcal{T}$ is a maximal clique

$\Delta^{\text{NF}} : t ::= \lambda x_0 \ldots x_{m-1}.\langle y \rangle \mu_0 \ldots \mu_{n-1}$, $\mu_i$ finite multisets of simple terms in normal form.

Coherence $\preceq$ on $\Delta^{\text{NF}}$:

$\lambda x_0 \ldots x_{m-1}.\langle y \rangle \mu_0 \ldots \mu_{n-1} \preceq t$ iff

(i) $t = \lambda x_0 \ldots x_{m-1}.\langle y \rangle \nu_0 \ldots \nu_{n-1}$

(ii) $\forall i < n, \forall u, u' \in |\mu_i \cdot \nu_i|, u \preceq u'$

Clique: subset of a $\preceq$-ideal

$\mathcal{T} \in \mathcal{P}(\Delta^{\text{NF}})$ clique: $\forall t, t' \in \mathcal{T}, t \preceq t'$
Corollary 2

Let $\mathcal{T} \in \mathcal{P}(\Delta^{NF})$.
There is a total $\lambda$-term $M$ such that $NF(\tau(M)) = \mathcal{T}$ iff

1. $FV(\mathcal{T})$ is finite
2. $\mathcal{T}$ is r.e.
3. $\mathcal{T}$ is a maximal clique

Total terms

(i) $M \xrightarrow{h^*} \lambda x_0 \ldots x_{m-1}.(y)M_0 \ldots M_{n-1}$
(ii) $M_0, \ldots, M_{n-1}$ are total
Further works

Bring the results to more expressive calculi:

- $\Lambda\mu$-calculus
  - Cannot use Barendregt’s theorem
- Non-Deterministic settings
  - Cannot use Ehrhard Regnier’s theorem
- ...