Harmonic tori in Lie groups

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1 Introduction

In recent years, there has been considerable interest in the construction and parametrisation of harmonic maps of Riemann surfaces into a compact Lie group (this is the principal chiral model of the Physicists). After contributions by many workers, the case where the domain is a Riemann sphere is now quite well understood (see [**1**,**9**,**10**, **11**]). Here the key idea is to associate to such harmonic maps, auxiliary holomorphic maps into the infinite dimensional Kähler manifolds of based loops in the group. In as much as this method reduces the problem to the study of (albeit infinite dimensional) complex analytic objects, one may view this approach as a twistorial construction.

These ideas do not seem to generalise to higher genus domains. However, progress has been made in this and related problems when the domain is a torus: Hitchin [**4**] studied harmonic 2-tori in S^3; Pinkall-Sterling [**5**] studied constant mean curvature tori in $\mathbf{R}^3$ (the Gauss maps of which are harmonic tori in S^2); and, very recently, Ferus-Pedit-Pinkall-Sterling have dealt with minimal (i.e. harmonic, conformal) tori in S^4 [**2**]. In all these cases, a complex analytic object arises, although this time it has a rather different flavour: here one has an algebraic curve, the *spectral curve* and the equations of interest reduce to a linear flow on the Jacobian of this curve. This is, of course, a familiar situation in the theory of integrable Hamiltonian systems (c.f. [**3**]).

In this note I wish to show how at least part of the above programme can be carried though for harmonic 2-tori in any compact Lie group.

Much of what follows I learned from Pinkall's lectures in Warwick and Leeds during the spring of 1990. It is a pleasure to acknowledge that debt here.

2 The harmonic map equations

We shall identify harmonic maps of a 2-torus into a compact Lie group G with harmonic maps of $\mathbf{R}^2$ into G which are doubly periodic with respect to some lattice.

So let $\phi : \mathbf{R}^2 \to G$ be a map and let θ be the (left) Maurer-Cartan form of G. Thus θ is a 1-form with values in $\mathbf{g}$, the Lie algebra of G. As is well known, ϕ is harmonic if and only if

$$d^* \phi^* \theta = 0 \tag{1}$$

while we always have the Maurer-Cartan equations pulled back by ϕ:

$$d\,\phi^*\theta + \frac{1}{2}[\phi^*\theta \wedge \phi^*\theta] = 0. \tag{2}$$

Conversely, given a $\mathbf{g}$-valued 1-form α satisfying (1) and (2), we may integrate to obtain a harmonic map $\phi : \mathbf{R}^2 \to G$ such that

$$\alpha = \phi^*\theta.$$

Thus it suffices to consider such $\mathbf{g}$-valued 1-forms α. Now introduce a complex coordinate $z = s + it$ on $\mathbf{R}^2$ and write

$$\alpha = \delta\, dz + \overline{\delta}\, d\overline{z}$$

where $\delta : \mathbf{R}^2 \to \mathbf{g}^{\mathbf{C}}$. The fundamental observation of Uhlenbeck [**9**] (c.f. also [**12**,**13**]) is that if we define a loop of $\mathbf{g}$-valued 1-forms by

$$D_\lambda = \frac{1-\lambda}{2}\delta\, dz + \frac{1-\lambda^{-1}}{2}\overline{\delta}\, d\overline{z}, \quad \lambda \in S^1 \subset \mathbf{C} \tag{3}$$

then $\alpha = D_{-1}$ is $\phi^*\theta$ for ϕ harmonic if and only if D_λ satisfies the Maurer-Cartan equations (2) for all $\lambda \in S^1$. Conversely, any loop D_λ of the form (3) which satisfies the Maurer-Cartan equations for each λ gives rise to a harmonic map ϕ with $D_{-1} = \phi^*\theta$.

Warning: My λ is the reciprocal of Uhlenbeck's.

We interpret this result by introducing the following loop algebra. Let $\Omega\mathbf{g}$ be the algebra of based loops in $\mathbf{g}$, that is,

$$\Omega\mathbf{g} = \{\gamma : S^1 \to \mathbf{g} : \quad \gamma(1) = 0\}.$$

This is a Lie algebra under pointwise Lie bracket. We equip $\mathbf{g}$ with an invariant inner product and, using this, give $\Omega\mathbf{g}$ the H^1 inner product:

$$(\xi_1, \xi_2)_{H^1} = \int_{S^1} (\xi_1', \xi_2').$$

With respect to this inner product, an orthogonal (not orthonormal) basis of $\Omega\mathbf{g}^{\mathbf{C}}$ is given by

$$\{\lambda \mapsto (1-\lambda^n)\eta_i\}_{\substack{1\le i\le \dim \mathbf{g}\\ n\in\mathbf{Z}\setminus\{0\}}}$$

where the $\{\eta_i\}$ are an orthonormal basis of $\mathbf{g}$. Then any $\xi \in \Omega\mathbf{g}$ may be written as

$$\xi_\lambda = \sum_{n\neq 0}(1-\lambda^n)\xi_n$$

with $\xi_n \in \mathbf{g}^{\mathbf{C}}$ and $\xi_n = \overline{\xi_{-n}}$.

Using these definitions, we introduce a filtration of $\Omega\mathbf{g}$ by finite-dimensional subspaces: for $d \in \mathbf{N}$, set

$$\Omega_d = \{\xi \in \Omega\mathbf{g} : \xi_n = 0 \text{ for } |n| > d\}$$

so that

$$\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega\mathbf{g}.$$

Also put

$$\begin{aligned}\Omega^+ &= \{\xi \in \Omega\mathbf{g}^{\mathbf{C}} : \xi_n = 0 \text{ for } n < 0\}\\ \Omega^- &= \{\xi \in \Omega\mathbf{g}^{\mathbf{C}} : \xi_n = 0 \text{ for } n > 0\}.\end{aligned}$$

We note that $\Omega^\pm$ are mutually conjugate subalgebras and that

$$\Omega\mathbf{g}^{\mathbf{C}} = \Omega^+ \oplus \Omega^-.$$

Our above development may now be succinctly summarised in the following

Proposition 1 *Let $A = A_\lambda$ be an $\Omega\mathbf{g}^{\mathbf{C}}$-valued $(1,0)$-form on $\mathbf{R}^2$ such that*

(i) $A + \overline{A}$ satisfies the Maurer-Cartan equations;

(ii) A has values in $\Omega^+ \cap \Omega_1^{\mathbf{C}}$.

Then $A_{-1} = \phi^*\theta^{(1,0)}$ for a harmonic map $\phi : \mathbf{R}^2 \to G$ and all harmonic maps $\mathbf{R}^2 \to G$ arise this way.

Thus our problem is reduced to solving the Maurer-Cartan equations for $\Omega\mathbf{g}$-valued 1-forms subject to be algebraic constraint given by (ii). In the next section, we shall see how to do this by integrating a pair of compatible ordinary differential equations.

3 The ordinary differential equations

On Ω_d, consider the vector fields X_1, X_2 given by

$$\frac{1}{2}(X_1 - iX_2)(\xi) = [\xi, 2i(1-\lambda)\xi_d].$$

We observe that the X_i are tangent to Ω_d since $[\xi_d, \xi_d]$ vanishes. Our main result is

Theorem 2 *(i)* $[X_1, X_2] = 0$.

(ii) If $\xi : \mathbf{R}^2 \to \Omega_d$ *is an integral submanifold of the distribution defined by the* X_i *with*

$$\frac{\partial \xi}{\partial s} = X_1(\xi), \quad \frac{\partial \xi}{\partial t} = X_2(\xi),$$

then, putting $z = s + it$ *and* $A = 2i(1-\lambda)\xi_d\, dz$, *we have that* $A + \overline{A}$ *satisfies the Maurer-Cartan equations (and so produces a harmonic map by proposition* 1*).*

Otherwise put: if $\xi : \mathbf{R}^2 \to \Omega_d$ satisfies

$$\frac{\partial \xi}{\partial z} = [\xi, 2i(1-\lambda)\xi_d] \tag{4}$$

then $4i\xi_d\, dz = \phi^*\theta^{(1,0)}$ for a harmonic map $\phi : \mathbf{R}^2 \to G$.

Before discussing the proof of this result, let us consider what kind of harmonic maps we get from this construction. We observe that equation (4) is of Lax type and so has many conserved quantities. In particular, if P is a homogeneous invariant polynomial on $\mathbf{g}^{\mathbf{C}}$, then $P(\xi_d)$ is constant for any solution ξ of (4). On the other hand, if $\phi : \mathbf{R}^2 \to G$ is harmonic, it is well known that $P(\phi^*\theta^{(1,0)})$ is a holomorphic section of $\otimes^l T^*\mathbf{R}^2$, $l = \deg P$, so that if ϕ factors through a torus then $P(\phi^*\theta^{(1,0)})$ is constant by Liouville's theorem. Again, if ϕ extends to a harmonic map $S^2 \to G$ then, since S^2 has no non-vanishing holomorphic differentials, each $P(\phi^*\theta^{(1,0)})$ must vanish identically so that $\phi^*\theta^{(1,0)}$ is nilpotent (c.f. [**6**]).

In the work of Hitchin and Pinkall et al. described above, it was shown that similar constructions account for all harmonic (respectively minimal) maps of a 2-torus into S^3 (respectively S^4) for which $\phi^*\theta^{(1,0)}$ is regular in the sense that at each point its centraliser in $\mathbf{g}$ is a Cartan subalgebra. It is therefore reasonable to make the following

Conjecture *Let* $\phi : T^2 \to G$ *for which* $\phi^*\theta^{(1,0)}$ *is regular at one (and hence every) point. Then* ϕ *arises by solving* (4) *for some* d.

I shall return to this question elsewhere.

4 r-matrices and all that

The Reader may easily verify that our main theorem is true but this does not explain where it comes from. For this, we briefly sketch a development which is mostly well known in the theory of integrable Hamiltonian systems (c.f. [**8**]).

Let $\mathbf{g}$ be a (possibly infinite-dimensional) Lie algebra. A linear map $R : \mathbf{g} \to \mathbf{g}$ is called an *r-matrix* if the bracket defined by

$$[X, Y]_R \stackrel{\text{def}}{=} [RX, Y] + [X, RY]$$

satisfies the Jacobi identity. This is certainly the case if R satisfies the (modified) classical Yang-Baxter equations

$$R[X, Y]_R - [RX, RY] = \alpha[X, Y]$$

for some fixed $\alpha \in \mathbf{C}$.

An r-matrix endows $\mathbf{g}$ with a second Lie algebra structure and thus $\mathbf{g}^*$ with a second Poisson structure. Indeed, for f, $g \in C^\infty(\mathbf{g}^*)$, setting

$$\{f, g\}_R(x) =< x, [df, dg]_R >$$

defines a Poisson bracket on $\mathbf{g}^*$ and thus a Lie algebra homomorphism $C^\infty(\mathbf{g}^*) \to C^\infty(T\mathbf{g}^*)$ given by

$$f \mapsto X_f$$

where $X_f g = \{f, g\}_R$.

Among the well known properties [**8**] of such brackets are

(*i*) If f, g are invariant functions on $\mathbf{g}^*$ then they Poisson commute so that $[X_f, X_g]$ vanishes.

(*ii*) If f is invariant then, for $x \in \mathbf{g}^*$,

$$X_f(x) = -\mathrm{ad}^*(R\, df)(x).$$

This last equation is essentially a Lax equation. Indeed, if we can identify $\mathbf{g}$ and $\mathbf{g}^*$ via an invariant inner product then the equation becomes

$$X_f(x) = [x, R\nabla f(x)],$$

for $x \in \mathbf{g}$, which is truly a Lax equation. Further, we note that, for any invariant function f, $[x, \nabla f(x)]$ vanishes identically so that we may write our equation as

$$X_f(x) = [x, (R + \mu)\nabla f(x)]$$

for any $\mu \in \mathbf{C}$.

Suppose now we have invariant functions f_1, f_2 on $\mathbf{g}^*$ and corresponding vector fields X_1, X_2 which commute by (ii) above. Let $\xi : \mathbf{R}^2 \to \mathbf{g}$ be an integral submanifold of the corresponding distribution with

$$\frac{\partial \xi}{\partial s} = X_1(\xi), \quad \frac{\partial \xi}{\partial t} = X_2(\xi),$$

and define a $\mathbf{g}$-valued 1-form A by

$$A = (R + \mu_1) df_1(\xi)\, ds + (R + \mu_2) df_2(\xi)\, dt.$$

We now have

Lemma 3 *With ξ, A as above, if R satisfies the modified classical Yang-Baxter equations* (4) *then A satisfies the Maurer-Cartan equations.*

To apply this to the case at hand, we first define an r-matrix on $\Omega\mathbf{g}$ by

$$R = \begin{cases} i & \text{on } \Omega^+ \\ -i & \text{on } \Omega^-. \end{cases}$$

This is easily checked to satisfy (4). As invariant inner product on $\Omega\mathbf{g}$ we take the L^2 inner product

$$(\xi_1, \xi_2)_{L^2} = \int_{S^1} (\xi_1, \xi_2),$$

while, on Ω_d, we take f_1, f_2 to be given by

$$(f_1 + i f_2)(\xi) = \int_{S^1} \lambda^{1-d} (\xi, \xi).$$

Then the Hamiltonian equations of motion on Ω_d are

$$\frac{\partial \xi}{\partial z} = [\xi, (R + \mu)\lambda^{1-d}\xi]$$

and, taking $\mu = i$, this becomes

$$\frac{\partial \xi}{\partial z} = [\xi, 2i(1 - \lambda)\xi_d]$$

which is precisely (4). Thus the theorem is a consequence of lemma 3.

5 The spectral curve

Let us conclude by indicating the link between the differential equation (4) and Algebraic Geometry. We begin by reducing our situation to the case where $\mathbf{g} = \mathbf{u}(n)$ by choosing a faithful representation if necessary.

Consider $\xi_\lambda \in \Omega_d$. We define an affine curve in (λ, η)-space by the equation

$$\det(\xi_\lambda - \eta \mathrm{Id}) = 0.$$

Compactifying gives us a curve C_0 which, since (4) is a Lax equation, will remain invariant under the flow of (4). Since $\xi_1 = 0$, C_0 is necessarily singular so we pass to its normalisation C which is our *spectral curve*. Suppose now we choose an initial condition $\xi(0)$ which has simple eigenvalues for generic λ. Then the η-eigenspace of ξ_λ at (λ, η) defines a line bundle on C which changes as ξ evolves under (4). Thus (4) corresponds to a dynamical system on the Jacobian of C.

In fact this flow is *linear*, a result suggested by the work of Reyman and Semenov-Tian-Shansky [**7**]. However, their proof does not work in our setting since they assume that C_0 is smooth. On the other hand, it turns out that the linearity of the flow may be established by using the rather general algebro-geometric methods of Griffiths [**3**].

6 References

1. F. E. Burstall and J. H. Rawnsley, *Twistor theory for Riemannian symmetric spaces with applications to harmonic maps of Riemann surfaces*, Lect. Notes in Math., vol. 1424, Springer, Berlin, Heidelberg, New York, 1990.

2. D. Ferus, F. Pedit, U. Pinkall and I. Sterling, *Minimal tori in* S^4, T.U. Berlin preprint, 1990.

3. P. A. Griffiths, *Linearizing flows and a cohomological interpretation of Lax equations*, Amer. J. Math. **107** (1985), 1445–1483.

4. N. J. Hitchin, *Harmonic maps from a* 2*-torus to the* 3*-sphere*, J. Diff. Geom. **31** (1990), 627–710.

5. U. Pinkall and I. Sterling, *On the classification of constant mean curvature tori*, Ann. of Math. **130** (1989), 407–451.

6. J. Ramanathan, *Harmonic maps from* S^2 *to* $G_{2,4}$, J. Diff. Geom. **19** (1984), 207–219.

7. A. G. Reyman and M. A. Semenov-Tian-Shansky, *Reduction of Hamiltonian systems, affine Lie algebras and Lax equations, II*, Invent. Math. **63** (1981), 423–432.

8. ———, *Compatible Poisson structures for Lax equations: an r-matrix approach*, Phys. Let. A **130** (1988), 456–460.

9. K. Uhlenbeck, *Harmonic maps into Lie groups (classical solutions of the chiral model)*, J. Diff. Geom. **30** (1989), 1–50.

10. G. Valli, *On the energy spectrum of harmonic 2-spheres in unitary groups*, Topology **277** (1988), 129–136.

11. J. C. Wood, *Explicit construction and parametrisation of harmonic two-spheres in the unitary group*, Proc. Lond. Math. Soc. **58** (1989), 608–624.

12. V. E. Zakharov and A. V. Mikhaĭlov, *Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method*, Sov. Phys. JETP **47** (1978), 1017–1027.

13. V. E. Zakharov and A. B. Shabat, *Integration of non-linear equations of mathematical physics by the method of inverse scattering, II*, Funkts. Anal i Prilozhen **13** (1978), 13–22.