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Minimal surfaces in quaternionic symmetric spaces

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We describe some birational correspondences between the twistor spaces of quaternionic Kähler compact symmetric spaces obtained by Lie theoretic methods. By means of these correspondences, one may construct minimal surfaces in such symmetric spaces. These results may be viewed as an explanation and a generalisation of some results of Bryant [**1**] concerning minimal surfaces in S^4.

This represents work in progress in collaboration with J.H. Rawnsley and S.M. Salamon.

BACKGROUND

This work has its genesis in our attempt to understand the following result of Bryant [**1**]:

Theorem. *Any compact Riemann surface may be minimally immersed in* S^4.

To prove this, Bryant considers the Penrose fibration $\pi : \mathbf{C}P^3 \to S^4 = \mathbf{H}P^1$. The perpendicular complement to the fibres (with respect to the Fubini-Study metric) furnishes $\mathbf{C}P^3$ with a holomorphic distribution $\mathcal{H} \subset T^{1,0}\mathbf{C}P^3$ and it is well-known that a holomorphic curve in $\mathbf{C}P^3$ tangent to $\mathcal{H}$ (a *horizontal* holomorphic curve) projects onto a minimal surface in S^4. Bryant gave explicit formulae for the horizontality condition on an affine chart which enabled him to integrate it and provide a "Weierstraß formula" for horizontal curves. Indeed, if f, g are meromorphic functions on a Riemann surface M then the curve $\Phi(f,g) : M \to \mathbf{C}P^3$ given on an affine chart by

$$\Phi(f,g) = (f - \tfrac{1}{2}g(dg/df), g, \tfrac{1}{2}(df/dg))$$

is an integral curve of $\mathcal{H}$. For suitable f, g, $\Phi(f,g)$ is an immersion (indeed, an embedding) and the theorem follows.

In [**6**], Lawson gave an interesting interpretation of Bryant's method by introducing the flag manifold $\mathbf{D}^3 = \mathrm{U}(3)/\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$ which may be viewed as the twistor space of $\mathbf{C}P^2$ (the twistor fibring being the non-$\pm$holomorphic homogeneous fibration of $\mathbf{D}^3$ over $\mathbf{C}P^2$). Again we have a holomorphic horizontal distribution $\mathcal{K}$

perpendicular to the fibres of the twistor fibration but, this time, horizontal curves are easy to construct. Indeed, viewing $\mathbf{D}^3$ as $P(T^{1,0}\mathbf{C}P^2)$, $\mathcal{K}$ is just the natural contact distribution and a holomorphic curve in $\mathbf{C}P^2$ has a canonical horizontal lift into $\mathbf{D}^3$ given by its tangent lines.

The remarkable fact, implicit in Bryant's work and brought to the fore by Lawson, is

Theorem. *There is a birational correspondence $\Phi : \mathbf{D}^3 \to \mathbf{C}P^3$ mapping $\mathcal{K}$ into $\mathcal{H}$.*

Recall that a birational correspondence of projective algebraic varieties is a holomorphic map which is defined off a set of co-dimension 2 and biholomorphic off a set of co-dimension 1.

Thus it suffices to produce horizontal curves in $\mathbf{D}^3$ which avoid the singular set of Φ and this may be done by taking the lifts of suitably generic curves in $\mathbf{C}P^2$.

Lawson gave an analytic expression for Φ but a geometrical interpretation of the map seemed quite hard to come by. An algebro-geometric interpretation has been given by Gauduchon [**4**] but it is our purpose here to show how this map arises naturally from Lie theoretic considerations.

QUATERNIONIC SYMMETRIC SPACES

The 4-sphere and $\mathbf{C}P^2$ may be viewed as the 4-dimensional examples of the quaternionic Kähler compact symmetric spaces. These are $4n$-dimensional symmetric spaces N with holonomy contained in $\mathrm{Sp}(1)\mathrm{Sp}(n)$. Geometrically, this means that there is a parallel subbundle E of $\mathrm{End}(TN)$ with each fibre isomorphic to the imaginary quaternions. There is one such symmetric space for each simple Lie group; the classical ones in dimension $4n$ being

$$\mathbf{H}P^n, \quad G_2(\mathbf{C}^{n+2}), \quad G_4(\mathbf{R}^{n+4}).$$

Following [**7**], we consider the *twistor space* Z of N which is the sphere bundle of E or, equivalently,

$$Z = \{j \in E: \quad j^2 = -1\}.$$

This twistor space is a Kähler manifold, indeed a projective variety, and once more the perpendicular complement to the fibres $\mathcal{H}$ is a holomorphic subbundle which is called the *horizontal distribution*. Our main theorem is then

Theorem. *Let N_1, N_2 are compact irreducible quaternionic Kähler symmetric spaces of the same dimension with twistor spaces Z_1, Z_2. Then there is a birational correspondence $Z_1 \to Z_2$ which preserves the horizontal distributions.*

For this we must study the homogeneous geometry of the twistor spaces: if N is the symmetric space G/K then G acts transitively on Z and, moreover, this action extends to a holomorphic action of the complexified Lie group $G^{\mathbf{C}}$. Further, the horizontal distribution is invariant under this $G^{\mathbf{C}}$ action. In fact, Z is a special kind of $G^{\mathbf{C}}$-space: it is a flag manifold, that is, of the form $G^{\mathbf{C}}/P$ where P is a parabolic subgroup.

For any flag manifold $G^{\mathbf{C}}/P$, let $\mathbf{p}$ be the Lie algebra of P. We have a decomposition of the Lie algebra of $G^{\mathbf{C}}$

$$\mathbf{g}^{\mathbf{C}} = \mathbf{p} \oplus \overline{\mathbf{n}}$$

where $\mathbf{n}$ is the nilradical of $\mathbf{p}$ so that $\overline{\mathbf{n}} \cong T^{1,0}_{eP} G^{\mathbf{C}}/P$ is a nilpotent Lie algebra. Let $\overline{N}$ be the corresponding nilpotent Lie group and consider the $\overline{N}$ orbits in $G^{\mathbf{C}}/P$. The orbit Ω of the identity coset is a Zariski dense open subset of $G^{\mathbf{C}}/P$ (it is the "big cell" in the Bruhat decomposition of the flag manifold). In fact, the map $\overline{\mathbf{n}} \to \Omega$ given by

$$\xi \mapsto \exp \xi \cdot P$$

is a biholomorphism with polynomial components (since $\overline{\mathbf{n}}$ is nilpotent) and so extends to give a birational correspondence of $G^{\mathbf{C}}/P$ with $P(\overline{\mathbf{n}} \oplus \mathbf{C})$. Thus $G^{\mathbf{C}}/P$ is a rational variety: a classical result of Goto [**5**]. However, more is true: let $G_1^{\mathbf{C}}/P_1$ and $G_2^{\mathbf{C}}/P_2$ be flag manifolds and suppose that the nilradicals $\mathbf{n}_1$ and $\mathbf{n}_2$ are isomorphic as complex Lie algebras. Then we have an isomorphism $\phi : \overline{\mathbf{n}}_1 \to \overline{\mathbf{n}}_2$ which we may exponentiate to get an isomorphism of Lie groups $\Phi : \overline{N}_1 \to \overline{N}_2$ and thus a biholomorphism $\Omega_1 \to \Omega_2$ which extends to a birational correspondence between the flag manifolds. Moreover, on Ω_1, this biholomorphism is $\overline{N}_1$-equivariant and so will preserve any invariant distribution so long as ϕ does when viewed as a map $T^{1,0}_{eP_1} G_1^{\mathbf{C}}/P_1 \to T^{1,0}_{eP_2} G_2^{\mathbf{C}}/P_2$.

We now specialise to the case at hand: if Z is the twistor space of a quaternionic Kähler compact irreducible symmetric space then Z is a rather special kind of flag manifold. In fact, Wolf [**8**] has shown that here $\overline{\mathbf{n}}$ is two-step nilpotent with 1-dimensional centre and so is precisely the complex Heisenberg algebra. Thus any two of our twistor spaces of the same dimension have isomorphic $\overline{\mathbf{n}}$ and so the main theorem follows.

APPLICATIONS TO MINIMAL SURFACES

The relevance of these constructions to minimal surface comes from the well-known fact that, just as in the 4-dimensional case, horizontal holomorphic curves in Z project onto minimal surfaces in N. Moreover, in some of the classical cases, horizontal holomorphic curves are quite easy to come by. For example, the twistor space of $G_2(\mathbf{C}^{n+2})$ is the flag manifold $Z = \mathrm{U}(n+2)/\mathrm{U}(1) \times \mathrm{U}(n) \times \mathrm{U}(1)$ which we may realise as the set of flags

$$\{\ell \subset \pi \subset \mathbf{C}^{n+2} \colon \dim \ell = 1,\, \dim \pi = n+1\}.$$

Horizontal, holomorphic curves in this setting are just a special kind of ∂'-pair in the sense of Erdem-Wood [**3**] and may be constructed as follows: if $f : M \to \mathbf{C}P^{n+1}$ is a holomorphic curve, we may construct the associated holomorphic curves $f_r : M \to G_{r+1}(\mathbf{C}^{n+2})$ given locally by

$$f_r = f \wedge \frac{\partial f}{\partial z} \wedge \ldots \wedge \frac{\partial^r f}{\partial z^r}.$$

Generically, f is full so that $f_1, \ldots, f_n$ are defined and then the map $\psi : M \to Z$ given by

$$\psi = (f \subset f_n)$$

is horizontal and holomorphic. Note that for $n = 1$, ψ is just the lift of $f : M \to \mathbf{C}P^2$ discussed above. Composing these curves with the birational correspondences of the previous section, we then have horizontal holomorphic curves in all the other twistor spaces of the same dimension so long as we can ensure that the curves avoid the singular sets of the correspondences. Thus, for example, one has the possibility of constructing minimal surfaces in the 8-dimensional exceptional quaternionic symmetric space $\mathrm{G}_2/\mathrm{SO}(4)$ from holomorphic curves in $\mathbf{C}P^3$.

However, to carry out such a programme, a rather more detailed analysis of these singular sets is required so as to ensure that they are avoided for suitably generic f. Work is still in progress on this issue.

EXTENSIONS

Many parts of the above development apply to arbitrary generalised flag manifolds. Burstall-Rawnsley [**2**] have shown that any flag manifold fibres in a canonical way over a Riemannian symmetric space of compact type and, moreover, any such symmetric space with inner involution is the target of such a fibration. In this setting, the perpendicular complement to the fibres is not in general holomorphic but there is a sub-distribution thereof, the *superhorizontal distribution*, which is holomorphic and $G^{\mathbf{C}}$-invariant. Again, holomorphic integral curves of this distribution project onto minimal surfaces in the symmetric space.

The above discussion applies so that isomorphisms of nilradicals exponentiate to give birational correspondences of flag manifolds which preserve the super-horizontal distributions. However, apart from the quaternionic symmetric case, we have not yet found any examples of differing flag manifolds with isomorphic nilradicals.

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