Chapter 3 : Covering spaces

Plan: we abstract the essential properties of the map $\phi : \mathbb{R} \to S^1$ that played a starring role in our calculation of $\pi_1(S^1)$.

Defn Let $p : E \to X$ be a cts map of top. spaces.
p is a covering map if each $x \in X$ has open nbhd U with $p^{-1}(U)$ a disjoint union of open sets S_i with $p|_{S_i} : S_i \to U$ a homeomorphism for each i.

Terminology: Each such U is said to be evenly covered + the S_i are called the sheets over U.

E is a covering space of X if $\exists$ covering map $p : E \to X$.

Examples (i) $\phi : \mathbb{R} \to S^1$ is a covering map. The open sets $U := S^1 \setminus \{-1\}$ and $V = S^1 \setminus \{1\}$ are evenly covered. The sheets over U are the sets $(n - \frac{1}{2}, n + \frac{1}{2})$, $n \in \mathbb{Z}$.

Ex What are the sheets over V?

(ii) $p : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ $\quad p(n,x) = x$ is a covering map & $\mathbb{R}$ itself is evenly covered.

Basic properties: (i) p surjects

(ii) for $x \in X$, $p^{-1}\{x\}$ is discrete (i.e. induced topy is discrete) — ex!

(iii) p is a local homeomorphism i.e. each $e \in E$ has open nbhd S s.t. $p|S$ is homeo $S \to p(S)$

(iv) p is open (ex!) whence X has quot. topy induced by p (ex!).

Thus E & X have the same local properties eg E locally path-connected iff X locally path-connected etc...

Lifts The main property of ϕ we used was the possibility of lifting paths + homotopies uniquely. The same thing works in our more general setting.

First the def^n:

Def^n: $p: E \to X$ a covering map, $f: Y \to X$ cts.
A *lift* of f is a cts $f': Y \to E$ s.t. $p \circ f' = f$
i.e.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Y \arrow[ur, "f'"] \arrow[r, "f"] & X \end{tikzcd}$$

Questions: 1/ For what Y, f does ~~a~~ lift f' exist?

2/ How unique is f' if it exists?

Uniqueness first:

Lemma 3.1 (Unique lifting property) Let Y be connected + $p: E \to X$ a covering map. Let $f: Y \to X$ be cts & $f_1, f_2: Y \to E$ two lifts of f. (i.e. $p \circ f_1 = p \circ f_2 = f$). Then

$$\{y \in Y : f_1(y) = f_2(y)\} = \emptyset \text{ or } Y \quad (\text{i.e. } f_1 = f_2 \text{ if they agree at a single point})$$

Pf Set $A = \{y \in Y : f_1(y) = f_2(y)\}$ & set $D = Y \setminus A$.
We prove A, D both open (whence one is $\emptyset$ by connectedness).
Fix $y \in Y$ + let U be evenly covered nbhd of $f(y)$. For $i = 1, 2$, let S_i be the sheet over U containing $f_i(y)$.
Then $V := f_1^{-1}(S_1) \cap f_2^{-1}(S_2)$ is open nbhd of y.
Let $z \in V$ so that $f_i(z) \in S_i$.

1. $y \in A \Rightarrow f_1(y) = f_2(y) \Rightarrow S_1 = S_2$ $\therefore$ $f_i(z) \in S_1$ $i = 1, 2$ $\quad V \Rightarrow f_1(z), f_2(z) \in S_1$
+ $pf_1(z) = pf_2(z) = f(z)$ whence $f_1(z) = f_2(z)$ since $p|S_1$ is homeo.
Thus $z \in A$ whence $V \subset A$ & A open.
2. $y \in D \Rightarrow f_1(y) \neq f_2(y) \Rightarrow S_1 \neq S_2$ (since $pf_1(y) = pf_2(y)$) $\Rightarrow S_1 \cap S_2 = \emptyset$
Thus $f_1(z) \neq f_2(z)$ & $z \in D$. So $V \subset D$ & D open

$\square$

Useful notation: Write $f: (X, x) \to (Y, y)$ if $x \in X, y \in Y$ $f: X \to Y$ with $f(x) = y$.

Thm 3.2 (Path-lifting theorem) Let $p: (E, e_0) \to (X, x_0)$ be a covering map and $\sigma: I \to X$ a path with $\sigma(0) = x_0$. Then $\exists!$ lift $\sigma'_{e_0}: I \to E$ of σ with $\sigma'_{e_0}(0) = e_0$ i.e.

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (I, 0) \arrow[ur, "\sigma'_{e_0}"] \arrow[r, "\sigma"] & (X, x_0) \end{tikzcd}$$

$*$

* Uniqueness gives:

- For $\sigma' : I \to E$

$$(p \circ \sigma')'_{\sigma'(0)} = \sigma'$$

- $$(\gamma_{x_0})'_{e_0} = \gamma_{e_0}$$ [Thus constant paths lift to constant paths]

[As usual γ_{x_0} is constant path at x_0 ...]

- If $\sigma(1) = \tau(0)$

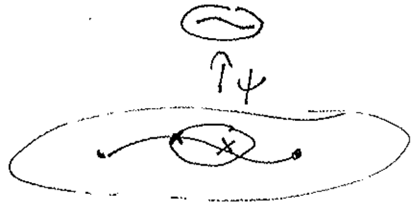
then $(\sigma \circ \tau)'_{e_0} = \sigma'_{e_0} \circ \tau'_{\sigma'(1)}$

Pf Uniqueness is immediate from (3.1).

X has a cover by evenly covered open sets so by Lebesgue Covering Lemma $\exists$ partition $0=t_0<\dots<t_N=1$ with each $\sigma[t_{i-1},t_i]$ lying in evenly covered set.

Subcase: let $e_i\in p^{-1}\{\sigma(t_i)\}$ then $\exists$ $\tilde{\sigma}_{e_i}:([t_i,t_{i+1}],t_i)\to(E,e_i)$ a lift of $\sigma|_{[t_i,t_{i+1}]}$.

Pf Let $\sigma[t_i,t_{i+1}]\subset U$ — evenly covered + let S be sheet over U containing e_i. $p|_S:S\to U$ is homeo with inverse ψ say. Set $\tilde{\sigma}_{e_i}=\psi\circ\sigma|_{[t_i,t_{i+1}]}$



∎

Now construct σ'_{e_0} by induction: induction hypothesis: $\exists$ $\sigma'_i:([0,t_i],0)\to(E,e_0)$ a lift of $\sigma|[0,t_i]$.

For case $i=1$, take $\sigma'_1=\tilde{\sigma}_{e_0}$.

Assume ~~case~~ (i) + define σ'_{i+1} by

$$\sigma'_{i+1}(t)=\begin{cases}\sigma'_i(t) & t\in[0,t_i]\\ \tilde{\sigma}_{\sigma'_i(t_i)} & t\in[t_i,t_{i+1}]\end{cases}$$ — well-defined + so cts since agree at t_i

$\therefore$ by induction $\sigma'_{e_0}=\sigma'_N$. ∎

Thm 3.3. (Homotopy lifting thm) Let $p:(E,e_0)\to(X,x_0)$ be a covering map + $F:(I\times I,0)\to(X,x_0)$. Then $\exists!$ lift $F'_{e_0}:(I\times I,0)\to(E,e_0)$ of F:

$$\begin{tikzcd} & E(e_0) \arrow[d] \\ (I\times I,0) \arrow[ur, "F'_{e_0}"] \arrow[r, "F"'] & (X,x_0) \end{tikzcd}$$

Pf Again we get uniqueness from (3.1).

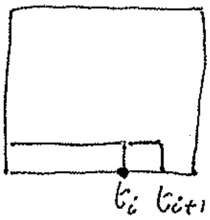
Lebesgue lemma gives partitions $0=t_0<\dots<t_N=1$ $0=s_0<\dots<s_M=1$ s.t. each $F([t_{i-1},t_i]\times[s_{j-1},s_j])$ lies in evenly covered set.

Subcase: For $e_{i,j}\in p^{-1}\{F(t_i,s_j)\}$ $\exists$ $\tilde{F}_{e_{i,j}}:([t_i,t_{i+1}]\times[s_j,s_{j+1}],(t_i,s_j))\to(E,e_{i,j})$ a lift of $F|[t_i,t_{i+1}]\times[s_j,s_{j+1}]$

Proof Just like subcase of (3.2)

Now we construct F'_{e_0} in pieces:

1/ Take $e_{0,0} = e_0$ ~~to~~ in subcase to define $F'_{e_0} = \tilde{F}_{e_{0,0}}$ on $[0, t_1] \times [0, s_1]$

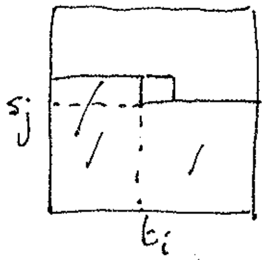
2/ 

Suppose have defined F'_{e_0} on $[0, t_i] \times [0, s_1]$

Use subcase with $e_{i,0} = F'_{e_0}(t_i, 0)$ to define

$$F'_{e_0} = \hat{F}_{e_{i,0}} \text{ on } [t_i, t_{i+1}] \times [0, s_1]$$

To see this is well-defined + cts, we must check our def^s agree on $\{t_i\} \times [0, s_1]$. But ~~both~~ are def^s give lifts of $F|_{\{t_i\} \times [0,s_1]}$ that agree at ~~$\{t_i\}\times$~~ $(t_i, 0)$ + so agree everywhere by (3.1).

3/ 

Suppose have defined F'_{e_0} on shaded area

Use subcase with $e_{i,j} = F'_{e_0}(t_i, s_j)$ to define

$$F'_{e_0} = \tilde{F}_{e_{i,j}} \text{ on } [t_i, t_{i+1}] \times [s_j, s_{j+1}]$$

We must again check this is well-defined + so cts which means checking that the two def^s agree on $J := [t_i, t_{i+1}] \times \{s_j\} \cup \{t_i\} \times [s_j, s_{j+1}]$

Again both ~~lifts~~ def^s give lifts of $F|$ on ~~same in~~ J (a connected set) which agree at (t_i, s_j) + so everywhere.

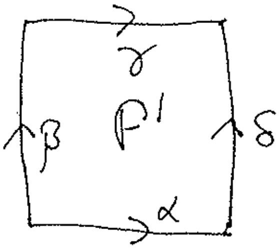
Induction now defines F'_{e_0} on all of $I \times I$ ☒

Applications

Corollary 3.4 If $\sigma, \tau : (I, 0) \to (X, x_0)$ are cts & $\sigma \sim \tau$ then $\sigma'_{e_0} \sim \tau'_{e_0}$ + in particular $\sigma'_{e_0}(1) = \tau'_{e_0}(1)$

Pf Let $\sigma \sim \tau$ via F & let $F' = F'_{e_0}$ — the lift from (3.3).

Define paths as in the diagram:



i.e. $\alpha(t) = F'(t,0)$ etc.

1/ $p \circ \alpha(t) = F(t,0) = \sigma(t)$ i.e. α is lift of σ

$\alpha(0) = e_0$ $\therefore$ $\alpha = \sigma'_{e_0}$

2/ $p \circ \beta(s) = F(0,s) = x_0$ $\forall s$ i.e. β is lift of $s \mapsto x_0$ with $\beta(0) = e_0$. $s \mapsto e_0$ is another such so $\beta(s) \equiv e_0$.

3/ In particular $\gamma(0) = \beta(1) = e_0$ while $p \circ \gamma = \tau$ whence $\gamma = \tau'_{e_0}$.

4/ Finally $p \circ \delta(s) = F(1,s) = \sigma(1)$ $\forall s \in I$ so δ is lift of $s \mapsto \sigma(1)$ with $\delta(0) = \sigma'_{e_0}(1)$. $s \mapsto \sigma'_{e_0}(1)$ is another such so $\delta(s) \equiv \sigma'_{e_0}(1)$.

Thus $\sigma'_{e_0} \sim \tau'_{e_0}$ via F'. ☒

<u>Cor 3.5</u> $p_* : \pi_1(E, e_0) \to \pi_1(X, x_0)$ injects.

<u>Pf</u> Suppose $p_*[\alpha] = 1$ α a loop in E based at e_0.

Thus $p \circ \alpha \sim \gamma_{x_0}$ ← const loop at x_0

Now $(p \circ \alpha)'_{e_0} = \alpha$ while $(\gamma_{x_0})'_{e_0} = \gamma_{e_0}$ so by (3.4)

$\alpha \sim \gamma_{e_0}$. ~~as required~~ $\therefore$ $\mathrm{Ker}\, p_* = \{1\}$ $\therefore$ p_* injects. ☒

<u>Ex</u>: p_* is iso$^{\underline{m}}$ iff p is homeo (i.e. injective).

An action of $\pi_1(X, x_0)$ on $p^{-1}\{x_0\}$:

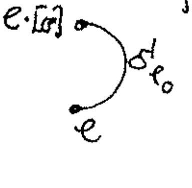
Recall: a <u>right action</u> of a group G on a set $\mathcal{A}$ is a map $\mathcal{A} \times G \to \mathcal{A}$ written $(a, g) \mapsto a g$

$$\left.\begin{array}{l} a \cdot 1 = a \\ a \cdot (g_1 g_2) = (a \cdot g_1) \cdot g_2 \end{array}\right\} \quad a \in \mathcal{A} \quad g_i \in G$$

s.t.

Let $p : E \to X$ be a covering map, $x_0 \in X$ + consider the <u>fibre over x_0</u>: $p^{-1}\{x_0\}$.

For $e \in p^{-1}\{x_0\}$ $[\sigma] \in \pi_1(X, x_0)$ define



$e \cdot [\sigma] = \sigma'_{e_0}(1)$ $\left(\in p^{-1}\{x_0\}\right.$ as $p \circ \sigma'_{e_0}(1) = \sigma(1) = x_0\left.\right)$

N.B 1/ Well-defined: if $\sigma \sim \tau$ then (3.8) $\sigma'_{e_0} \sim \tau'_{e_0}$ so $\sigma'_{e_0}(1) = \tau'_{e_0}(1)$

2/ This defines an action:

(a) $e \cdot 1 = e \cdot [\gamma_{x_0}]$

Now $(\gamma_{x_0})'_e = \gamma_e$ so $e \cdot 1 = \gamma_e(1) = e \quad \forall e \in p^{-1}\{x_0\}$.

(b) If $[\sigma], [\tau] \in \pi_1(X, x_0)$

$(\sigma \cdot \tau)'_e = \sigma'_e \cdot \tau'_{\sigma_e(1)}$ so

$e \cdot ([\sigma][\tau]) = e \cdot [\sigma \cdot \tau] = \sigma'_e \cdot \tau'_{\sigma_e(1)}(1) = \tau'_{\sigma'_e(1)}(1) = \sigma'_e(1) \cdot [\tau]$
$= (e \cdot [\sigma]) \cdot [\tau]$.

<u>Exercises</u> (i) If E path-connected, the action is <u>transitive</u>: if $e_1, e_2 \in p^{-1}\{x_0\}$ $\exists$ $[\sigma] \in \pi_1(X, x_0)$ with $e_1 \cdot [\sigma_1] = [\sigma_2]$.

(ii) the <u>stabiliser</u> of $e \in p^{-1}\{x_0\}$ (i.e. $\{g \in \pi_1(X, x_0) : e \cdot g = e\}$) is given by $p_* \pi_1(E, e)$.

(iii) if E simply connected then $[\sigma] \to e \cdot [\sigma]$, fixed $e \in E$ is a bijection $\pi_1(X, x_0) \to p^{-1}\{x_0\}$. compare $\phi : \mathbb{R} \to S^1$ ....

The ultimate lifting theorem:

<u>Ultimate Lifting Thm</u> Let $p: (E, e_0) \to (X, x_0)$ be covering map, Y connected & path-connected, $f: (Y, y_0) \to (X, x_0)$ cts.
Then $\exists$ lift i.e. cts $f': (Y, y_0) \to (E, e_0)$ with $p \circ f' = f$
iff
$$f_* \pi_1(Y, y_0) \subset p_* \pi_1(E, e_0)$$

<u>Pf</u> ($\Rightarrow$) If f' exists, $p \circ f' = f$ $\therefore$ $p_* \circ f'_* = f_*$

so $f_* \pi_1(Y, y_0) = p_* (f'_* (\pi_1(Y, y_0)))$

($\Leftarrow$) Mission: 1/ define f'

2/ prove our f' cts

How to define f'?

IDEA: <u>if</u> f' exists & τ is path in Y from y_0 to y
then $f' \circ \tau$ starts at e_0 & $p \circ f' \circ \tau = f \circ \tau$
i.e. $f' \circ \tau = (f \circ \tau)'_{e_0}$. In particular,
$$f'(y) = f' \circ \tau(1) = (f \circ \tau)'_{e_0}(1)$$
This gives a recipe for $f'(y)$: we <u>define</u> $f'(y)$ to be $(f \circ \tau)'_{e_0}(1)$ where τ is some path from y_0 to y.

Issues: a/ $f'(y)$ is well-defined (i.e. does not depend on choice of τ)

b/ f' is cts.

First a/.
Note that if $\tau_1, \tau_2^\#$ are paths from y_0 to y with $\tau_1 \sim \tau_2^\#$ then $f \circ \tau_1 \sim f \circ \tau_2^\#$ so $(f \circ \tau_1)'_{e_0} \sim (f \circ \tau_2^\#)'_{e_0}$

+ in particular, $(f\circ\tau_1)'_{e_0}(1) = (f\circ\tau_2)'_{e_0}(1)$.

More generally, for arbitrary τ_1, τ_2 from y_0 to y,

$\tau_1 \sim (\tau_1 \cdot \tau_2^{-1}) \cdot \tau_2 = \sigma \cdot \tau_2$ with σ a loop at y_0

Now $f\circ(\sigma\cdot\tau_2) = (f\circ\sigma)\cdot(f\circ\tau_2)$ so

$(f\circ(\sigma\cdot\tau_2))'_{e_0} = (f\circ\sigma)'_{e_0} \cdot (f\circ\tau_2)'_e$ where $e = (f\circ\sigma)'_{e_0}(1)$.

But by hypothesis $f_*[\sigma] = p_*[\sigma']$ with σ' a loop at e_0

$\therefore$ $f\circ\sigma \sim p\circ\sigma'$ $\therefore$ $(f\circ\sigma)'_{e_0}(1) = \underset{\sigma'}{\underset{\|}{(p\circ\sigma')'_{e_0}}}(1) = \sigma'(1) = e_0$

i.e. $e = e_0$ so that

$$(f\circ(\sigma\cdot\tau_2))'_{e_0}(1) = (f\circ\tau_2)'_{e_0}(1)$$
$$\|$$
$(f\circ\tau_1)'_{e_0}(1)$ by first part.

By construction $p\circ f'(y) = f\circ\tau(1) = f(y)$ so $p\circ f' = f$

while $f(y_0) = e_0$ (take $\tau = \gamma_{y_0}$).

b/ f' is cts

E has base of open sets on which p is a homeo

Let S be one such set — suffices to show that $(f')^{-1}(S)$ open in Y.

Let $U = p(S) \subset X$ & $\phi: U \to S$ be inverse of $p|S$.

Let $y \in (f')^{-1}(S)$. Suffices to find nbhd V of y contained in $(f')^{-1}(S)$.

Now $f(y) \in U$ i.e. $y \in f^{-1}(U)$ - open $\therefore$ since Y loc. path-connected $\exists$ path-connected nbhd $y \in V \subset f^{-1}(U)$.

Claim $V \subset (f')^{-1}(S)$

Pf Let $y_1 \in V$ + choose path τ_1 in V from y to y_1. Also choose path τ from y_0 to y. Then $\tau \cdot \tau_1$ is path y_0 to y_1 so

$$f'(y_1) = (f \circ (\tau \cdot \tau_1))'_{e_0}$$
$$= (f \circ \tau)'_{e_0} \cdot (f \circ \tau_1)'_{f'(y)} \leftarrow = (f \circ \tau)'_{e_0}(1)$$

But $\psi \circ f \circ \tau_1(0) = \psi \circ f(y) = f'(y)$

$p \circ \psi \circ f \circ \tau_1 = f \circ \tau_1$

$\therefore \quad \psi \circ f \circ \tau_1 = (f \circ \tau_1)'_{f'(y)}$ whence

$$f'(y_1) = (f \circ \tau_1)'_{f'(y)}(1) = \psi \circ f \circ \tau_1(1) \in S$$

$\therefore \; f'(y_1) \in S$ + we're done

$\boxtimes$

Cor 3.7 If Y simply-connected also, any $f : (Y, y_0) \to (X, x_0)$ has a lift $f' : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ ...

§3.2. π_1 and deck translations

Recall: a right action of a group G on a set X is a map $X \times G \to X$ written $(x,g) \mapsto x g$

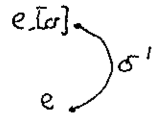
s.t. $x \cdot 1 = x \quad \forall x \in X$

$x \cdot (g_1 g_2) = (x \cdot g_1) \cdot g_2 \quad \forall x \in X,\ g_1, g_2 \in G$.

Let $p : E \to X$ be a covering map at $x_0 \in X$. Consider $p^{-1}\{x_0\}$ – the fibre over x

Define a right action of $\pi_1(X, x_0)$ ~~over~~ on $p^{-1}\{x_0\}$ by

$$e \cdot [\sigma] = \sigma_e'(1)$$



NOTE: 1) This is well-defined: if $\sigma \sim \tau$ then $\sigma_e'(1) = \tau_e'(1)$

and $p\sigma_e'(1) = \sigma(1) = x_0$ so $\sigma_e'(1) \in p^{-1}\{x_0\}$.

2) It is an action:

$e \cdot 1 = e \cdot [\gamma_{x_0}]$ and $(\gamma_{x_0})_e' = \gamma_e$ giving $e \cdot 1 = \gamma_e(1) = e$.

Further, if $[\sigma], [\tau] \in \pi_1(X, x_0)$ $\quad (\sigma\tau)_e' = \sigma_e' \tau'_{\sigma_e'(1)}$ so

$$e \cdot [\sigma\tau] = \sigma_e' \cdot \tau'_{\sigma_e'(1)} = \tau'_{\sigma_e'(1)} = (e \cdot \sigma) \cdot \tau$$

Exercises: (i) if E is path-connected, this action is transitive

(ii) the stabiliser of $e \in p^{-1}\{x\}$ is $p_* \pi_1(E, e) \subset \pi_1(X, x_0)$

(iii) E simply-connected then $[\sigma] \mapsto e_0 \cdot [\sigma]$ bijection $\pi_1(X, x_0) \to p^{-1}\{x\}$

Def$^{\underline{n}}$ A homeo $\phi : E \to E$ is a deck translation of p iff $p \circ \phi = \phi$

i.e.

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Ex (i) What are the deck translations of $\phi : \mathbb{R} \to \mathbb{Z}$

(ii) Prove that deck translations are a group under composition

(iii) If H is discrete normal subgrp of connected G a top. grp – show that the grp of deck translations are isomorphic to H.

The usefulness of this concept is that we can recover $\pi_1(X)$ from the grp of deck translations of a simply connected covering space.

<u>Thm 3.8</u> Let E be <u>simply</u> <u>connected</u> + locally path-connected & $p:(E,e_0)\to(X,x_0)$ a covering map. Let G be the grp of deck translations then

$$G \cong \pi_1(X,x_0)$$

Pf. First recall that since E is simply connected then any $\sigma_1,\sigma_2\in \mathrm{Path}(E,e_1,e_2)$ have

$$\sigma_1 \sim \sigma_2$$

Use this to define $\chi: G\to \pi_1(X,x_0)$ as follows.

For $\varphi\in G$ let σ' be path from e_0 to $\varphi(e_0)$ + set

$$\chi(\varphi) = [p\circ\sigma']$$

Well-defined since $p\circ\sigma'(0) = p(e_0) = p\circ\varphi(e_0) = p\circ\sigma'(1) = x_0$ and if τ' is another path then $\sigma'\sim\tau'$ whence $p\circ\sigma'\sim p\circ\tau'$.

χ will be our isomorphism:

1. χ is homo: let $\varphi_1,\varphi_2\in G$ and σ',τ' paths from e_0 to $\varphi_1(e_0),\varphi_2(e_0)$

then $\varphi_1\circ\tau'$ is path from $\varphi_1(e_0)$ to $\varphi_1\circ\varphi_2(e_0)$ whence

$$\sigma'\cdot\varphi_1\circ\tau' \text{ is path from } e_0 \text{ to } \varphi_1\circ\varphi_2(e_0) \text{ so that}$$

$$\chi(\varphi_1\circ\varphi_2) = [p\circ\sigma']\cdot[p\circ\varphi_1\circ\tau'] = [p\circ\sigma'][p\circ\tau'] = \chi(\varphi_1)\chi(\varphi_2).$$

2. χ is injective: show $\ker\chi=\{id\}$. Suppose $\chi(\varphi)=1$ and let σ' be path from e_0 to $\varphi(e_0)$. Then

$$p\circ\sigma'\sim\gamma_{x_0} \text{ so by (3.1) } \sigma'\sim\gamma_{e_0} \text{ whence } \sigma'(1)=e_0 \text{ i.e. }$$

$$\varphi(e_0)=e_0.$$

But
$$\begin{tikzcd} & E \arrow[d, "p"] \\ E \arrow[ur, "{\varphi, id}"] \arrow[r, "p"] & X \end{tikzcd}$$
both lift p and agree at e_0 so $\varphi = id_E$.

3. χ surjective

Let $[\sigma] \in \pi_1(X, x_0)$. Seek $\varphi \in \mathcal{G}$ s.t. $\varphi(e_0) = \sigma'_{e_0}(1)$

But E simply connected so by 3.7 $\exists$ cts $\varphi : E \to E$ s.t.

$$\begin{tikzcd} & (E, \sigma'_{e_0}(1)) \arrow[d, "p"] \\ (E, e_0) \arrow[ur, "\varphi"] \arrow[r, "p"'] & (X, x_0) \end{tikzcd}$$

(thus $p \circ \varphi = \varphi$)

We must show φ has cts inverse, but $\exists$ cts $\bar{\varphi} : E \to E$ s.t.

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (E, \sigma'_{e_0}(1)) \arrow[ur, "\bar{\varphi}"] \arrow[r, "p"'] & (X, x_0) \end{tikzcd}$$

~~+ then~~ $\cancel{\varphi \circ \bar{\varphi} : (E, \sigma'_{e_0}(1)) \to (E, \sigma'_{e_0}(1))}$ $\cancel{\bar{\varphi} \circ \varphi : (E, e_0) \to (E, e_0)}$

+ have

$$\begin{tikzcd} & (E, \sigma'_e(1)) \arrow[d] \\ (E, \sigma'_e(1)) \arrow[ur, "{\varphi \circ \bar{\varphi},\, id}"] \arrow[r] & (X, x_0) \end{tikzcd} \qquad \begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (E, e_0) \arrow[ur, "{\bar{\varphi} \circ \varphi,\, id}"] \arrow[r, "p"'] & (X, x_0) \end{tikzcd}$$

so by (3.1) $\varphi \circ \bar{\varphi} = \bar{\varphi} \circ \varphi = id$ so $\bar{\varphi} = \varphi^{-1}$ & φ is homeo. ∎

<u>Remark</u> Observe that

$$e_0 \cdot \chi(\varphi) = \varphi(e_0)$$

Ex (if time) $N(H)$ - normaliser
$\mathcal{G} \cong N(p_* \pi_1(X, x_0)) / p_* \pi_1(X, x_0)$

* INSERT BORSUK-ULAM HERE.

§3.3. Universal covering spaces [Henceforth all spaces connected + locally path-connected]

Last result shows that simply connected covering spaces are a good thing.

So:

Def^n E is a universal cover of X if E simply connected + there is a covering map $p : E \to X$

Prop^n 3.9 $8\frac{1}{2}$ (Uniqueness of universal covers) If $p : E \to X$, $\hat{p} : \hat{E} \to X$ are universal covers then $\exists$ homeo $\hat{\varphi} : E \to \hat{E}$ s.t.

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & \hat{E} \arrow[dl, "\hat{p}"] \\ & X & \end{tikzcd}$$

Proof Fix $x_0 \in X$, $e_0 \in p^{-1}\{x\}$, $\tilde{e}_0 \in \tilde{p}^{-1}\{e_0\}$.

By (3.7) have cts φ s.t. and $\bar{\varphi}$

$$\begin{tikzcd} & (\tilde{E}, \tilde{e}_0) \arrow[d, "\tilde{p}"] \\ (E, e_0) \arrow[ur, "\varphi"] \arrow[r, "p"'] & (X, x_0) \end{tikzcd} \qquad \begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (\tilde{E}, \tilde{e}_0) \arrow[ur, "\bar{\varphi}"] \arrow[r, "\tilde{p}"'] & (X, x_0) \end{tikzcd}$$

and, again

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (E, e_0) \arrow[ur, "\bar{\varphi}\circ\varphi, id"] \arrow[r, "p"] & (X, x_0) \end{tikzcd}$$

so $\bar{\varphi} \circ \varphi = id_E$ by (3.1) + simly $\varphi \circ \bar{\varphi} = id_{\tilde{E}}$
so that φ is homeo with inverse $\bar{\varphi}$.

∎

Finally, we show that any 'nice' space has a universal cover:

Def^n X is semi locally simply connected, if each $x \in X$ has nbhd U_x s.t. any loop in U_x is homotopic rel $\{0,1\}$ to a constant loop (in X)

Remark If U_x has this property so does any open sset of U_x so that X has base of open sets with this property.

Exercise If X has a universal cover then X is semi locally simply connected.

<u>Thm 3.10</u> If X is semi-locally simply connected then X has a universal cover.

<u>Pf</u> Fix $x_0 \in X$ and let $\Omega(X, x_0) = \{\gamma : I \to X : \gamma \text{ cts } \& \ \gamma(0) = x_0\}$

As usual, say $\gamma_1 \sim \gamma_2$ iff $\gamma_1 \simeq \gamma_2 \text{ rel}\{0,1\}$ + let $E = \Omega(X, x_0)/\sim$ be set of equiv. classes.

Define $p : E \to X$ by $p[\gamma] = \gamma(1)$ - well-defined + onto since X path-connected.

<u>1. Topologise E</u>

First note that we have base $\mathcal{U}$ for topology on X s.t each $U \in \mathcal{U}$ has

1. path-connected
2. any loop in U is $\sim$ constant loop.

For $[\gamma] \in E$ & $V \in \mathcal{U}$ s.t $\gamma(1) \in V$ set

$$U([\gamma], V) = \{[\gamma \cdot \beta] : \beta \text{ path in } V \text{ with } \beta(0) = \gamma(1)\}$$

N.B For $[\alpha] = [\gamma \cdot \beta] \in U([\gamma], V)$, $p[\alpha] = \beta(1) \in V$ whence (V path-connected)

$$p(U([\gamma], V)) = V \qquad (*).$$

Claim: The set of all such $U([\gamma], V)$ is base for topology on E

For this let $[\gamma''] \in U([\gamma], V) \cap U([\gamma'], V')$ then $\gamma''(1) \in V \cap V'$ so $\exists V'' \in \mathcal{U}$ s.t

$$\gamma''(1) \in V'' \subset V \cap V'$$

It suffices to have $U([\gamma''], V'') \subset U([\gamma], V) \cap U([\gamma'], V')$

So let $[\alpha] \in U([\gamma''], V'')$ then $\alpha \sim \gamma'' \cdot \beta''$ with β'' path in V''

But $\gamma'' \sim \gamma \cdot \beta$, β path in V and $\gamma'' \sim \gamma' \cdot \beta'$, β' a path in V'

so that 1). $\alpha \sim \gamma \cdot \underbrace{(\beta \cdot \beta'')}_{\text{path in } V}$ so $[\alpha] \in U([\gamma], V)$

2) $\alpha \sim \gamma' \cdot \underbrace{(\beta' \cdot \beta'')}_{\text{path in } V'}$ so $[\alpha] \in U([\gamma'], V')$

Thus $U([\gamma''], V'') \subset U([\gamma], V) \cap U([\gamma'], V')$ whence

$$\{U([\gamma], V) : [\gamma] \in E \ \ V \in \mathcal{U} \ \ \gamma(1) \in V\} \text{ is base for a topology by (0.1).}$$

2. p is a covering map

We shall see that each $\mathcal{U}([\gamma],V)$ is a sheet over V.

First, by (*) each $p(\mathcal{U}[\gamma],V)) = V$ - open whence p open.

Second: fix $x \in X$ and pick $V \in \mathcal{U}$ with $x \in V$

I claim that $p^{-1}(V) = \bigcup_{[\gamma] \in \pi_1(X,x_0,x)} \mathcal{U}([\gamma],V)$

Proof of claim: Certainly each $p(\mathcal{U}[\gamma],V) = V$ so r.h.s $\subset p^{-1}(V)$

Conversely, let $[\alpha] \in p^{-1}(V)$ i.e. $\alpha(1) \in V$. Fix path β in V from x to $\alpha(1)$ (V path-connected)

then $\alpha \sim (\alpha.\beta^{-1}).\beta$ then $[\alpha.\beta^{-1}] \in \pi_1(X,x_0,x)$ and

$[\alpha] \in \mathcal{U}([\alpha.\beta^{-1}],V)$ so claim follows.

As a consequence each $p^{-1}(V)$ is open so that p cts but more is true:

Claim each $V \in \mathcal{U}$ is evenly covered with sheets $\{\mathcal{U}([\gamma],V)\}_{\gamma \in \pi_1(X,x_0,x)}$

Pf First show sheets are disjt: let $[\alpha] \in \mathcal{U}([\gamma],V) \cap \mathcal{U}([\gamma'],V)$ $[\gamma],[\gamma'] \in \pi_1(X,x_0,x)$

Want $[\gamma]=[\gamma']$. But $\left.\begin{array}{l}\alpha \sim \gamma.\beta \\ \alpha \sim \gamma'.\beta'\end{array}\right\}$ β, β' paths from x to $\alpha(1)$

So $\gamma \simeq \alpha.\beta^{-1} \simeq \gamma'.(\beta'.\beta^{-1})$ but $\beta'.\beta^{-1}$ is loop in V based at x + so $\beta'.\beta^{-1} \sim \gamma_x$ - constant loop

so $\gamma \sim \gamma'.\gamma_x \sim \gamma'$ as required.

Finally want $p|\mathcal{U}([\gamma],V)$ homeo onto V.

Already have
1. p open
2. p cts
3. p onto

so only want p injective: so suppose $[\alpha],[\alpha'] \in \mathcal{U}([\gamma],V)$ have $p[\alpha]=p[\alpha']$ i.e.

$\alpha(1) = \alpha'(1)$.

Now $\left.\begin{array}{l}\alpha \sim \gamma.\beta \\ \alpha' \sim \gamma.\beta'\end{array}\right\}$ β, β' paths from x to $\alpha(1) = \alpha'(1)$

then $\alpha \sim \gamma.(\beta.\beta'^{-1}).\beta'$ but $\beta.\beta'^{-1}$ is loop at x in V so $\beta.\beta'^{-1} \simeq$

whence $\alpha \sim \gamma \cdot \beta' \sim \alpha'$ so that $[\alpha] = [\alpha']$ and we are done.

3. E path-connected

Let $e_0 = [\gamma_{x_0}] \in E$ and $[\alpha] \in E$ – it suffices to find path from e_0 to $[\alpha]$.

Let $\alpha_s : I \to X$ be given by $\alpha_s(t) = \alpha(st)$ and define

$$\alpha' : I \to E \quad \text{by} \quad \alpha'(s) = [\alpha_s]$$

N.B $\alpha_0 \equiv x_0$ i.e. $[\alpha'(0)] = e_0$ while $\alpha_1 = \alpha$ so $\alpha'(1) = [\alpha]$

So we must show α' is cts. Let $\mathcal{U}([\gamma], V)$ be basic open set : it suffices to show that $(\alpha')^{-1}\mathcal{U}([\gamma], V)$ is open.

Let $s \in (\alpha')^{-1}\mathcal{U}([\gamma], V)$ then $p\alpha'(s) = \alpha_s(1) = \alpha(s) \in V$ so $s \in \alpha^{-1}(V)$ – open and $\exists\, J$ open ~~set~~ interval s.t $s \in J \subset \alpha^{-1}(V)$

Suffices to show $\alpha'(J) \subset \mathcal{U}([\gamma], V)$: for this let $u \in J$.

Now $\alpha_u \sim \alpha_s \cdot \alpha_{s,u}$ where $\alpha_{s,u}(t) = \alpha((1-t)s + tu)$ (ex!) is path in V (since $J \subset \alpha^{-1}(V)$)

while $\alpha_s \sim \gamma \cdot \beta$, β path in V so

$$\alpha_u \sim \gamma \cdot \underbrace{(\beta \cdot \alpha_{s,u})}_{\text{path in } V} \quad \text{so} \quad [\alpha_u] \in \mathcal{U}([\gamma], V)$$

i.e. $\alpha'(u) \in \mathcal{U}([\gamma], V)$ and we are done.

Thus ~~cts~~ α' cts so E path-connected.

In fact we get a little more: $p \circ \alpha'(s) = \alpha_s(1) = \alpha(s)$ i.e. $p \circ \alpha' = \alpha$

so that $\alpha' = \alpha'_{e_0}$ – the lift of α starting at e_0

In particular $\alpha'_{e_0}(1) = [\alpha]$

4. E simply connected

Let τ be loop at e_0 and let $\alpha = p \circ \tau$. By (3.1)

$$\tau = \alpha'_{e_0}$$

and since τ is loop we have $\alpha'_{e_0}(1) = \tau(1) = e_0$ i.e. $e_0 = [\alpha]$

i.e. $\alpha \sim \gamma_{x_0}$

So by homotopy lifting $\alpha'_{e_0} \sim (\gamma_{x_0})'_{e_0}$ i.e. $\tau \sim (\gamma_{e_0})$ as required. $\boxtimes$

§3.3. Topology of $\mathbb{RP}^n$ and the Borsuk-Ulam Theorem

Recall: $\mathbb{RP}^n = S^n/\sim$ where $x \sim y$ iff $x = \pm y$

Exercises (1) $S^n \xrightarrow{p} S^n/\sim$ is a covering map

(2) $\mathbb{RP}^1 \cong S^1$ [Have homeo

$$\begin{tikzcd} S^1 \arrow[d, "p"'] \arrow[dr, "z \mapsto z^2"] & \\ \mathbb{RP}^1 \arrow[r, "\cong"] & S^1 \end{tikzcd}$$

$[z] \mapsto z^2$]

From this we can compute $\pi_1(\mathbb{RP}^n)$:

Thm 3.9 $\pi_1(\mathbb{RP}^1) \cong \mathbb{Z}$

$\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2 \ (\mathbb{Z}/2\mathbb{Z})$ for $n \geq 2$.

Pf For $n \geq 2$, S^n simply connected (+ loc. path. connected) &

$p: S^n \to \mathbb{RP}^n$ is covering map.

$\therefore$ by (3.8) $\pi_1(\mathbb{RP}^n) = \{\text{deck translations of } p\}$

However $\{\text{deck translations}\} = \{id_{S^n}, -id_{S^n}\}$:

certainly id_{S^n}, $-id_{S^n}$ are deck translations: $p(x) = p(-x)$

Further if $\phi: S^n \to S^n$ is deck translation + $x_0 \in S^n$ then

$$\begin{tikzcd} & S^n \arrow[d, "p"] \\ S^n \arrow[ur, "\phi"] \arrow[r, "p"'] & \mathbb{RP}^n \end{tikzcd}$$

and $\phi(x_0) = x_0$ or $\phi(x_0) = -x_0$

$\therefore$ by unique lifting prop. $\phi = id_{S^n}$ or $\phi = -id_{S^n}$.

We use this to prove

Borsuk-Ulam Theorem $\exists$ no cts map $\phi: S^n \to S^{n-1}$ which is antipode preserving i.e. $\phi(-x) = -\phi(x)$

Pf Only prove for $n \leq 2$.

$n=1$: S^1 connected S^0 disconnected $\Rightarrow$ any cts $\phi: S^1 \to S^0$ const

+ $\therefore$ not antipode preserving.

$\underline{n=2}$: Suppose such a ϕ exists + define

$\psi: \mathbb{RP}^2 \to \mathbb{RP}^1$ by $\psi([x]) = [\phi(x)]$ — well-defined since $\phi(-x) \sim \phi(x)$.

Have commuting diagram:

$$\begin{tikzcd} S^2 \arrow[r, "\phi"] \arrow[d, "p"'] & S^1 \arrow[d, "p"] \\ \mathbb{RP}^2 \arrow[r, "\psi"] & \mathbb{RP}^1 \end{tikzcd}$$

so ψ cts since $p \circ \phi$ is ...

Fix $x_0 \in S^2$ + contemplate $\psi_* : \pi_1(\mathbb{R}^2\mathbb{P}^2, [x_0]) \to \pi_1(\mathbb{RP}^1, [\phi(x_0)])$

$$\begin{array}{ccc} \pi_1(\mathbb{RP}^2, [x_0]) & \to & \pi_1(\mathbb{RP}^1, [\phi(x_0)]) \\ \shortparallel\wr & & \\ \mathbb{Z}_2 & \to & \mathbb{Z} \end{array}$$

$\underline{\text{FACT}}$ Any homomorphism $\mathbb{Z}_2 \to \mathbb{Z}$ is constant.

$\underline{\text{Pf}}$ If $\chi: \underset{\{\pm 1\}}{\mathbb{Z}_2} \to \mathbb{Z}$ is homo then

$0 = \chi(1) = \chi(-1 \cdot -1) = \chi(-1) + \chi(-1)$

$\therefore \chi(\pm 1) = 0$. ∎

$\therefore$ For any loop σ in $\mathbb{RP}^2$ at $[x_0]$ we have $\psi_*[\sigma] = 1_{\pi_1(\mathbb{RP}^1, \phi[x_0])}$

i.e. $\psi \circ \sigma \sim \gamma_{\phi[x_0]}$.

From this we get a contradiction: Let $\sigma' : I \to S^2$ be any path from x_0 to $-x_0$ + set $\sigma = p \circ \sigma'$ — a loop at $[x_0]$.

$\therefore$ $\psi \circ \sigma \sim \gamma_{\phi[x_0]}$ + taking lifts gives

$$(\psi \circ \sigma)'_{\phi(x_0)} \sim (\gamma_{\psi[x_0]})' = \gamma_{\phi(x_0)}$$

But $\phi \circ \sigma' : I \to S^1$ has $\phi \circ \sigma'(0)) = \phi(x_0)$

$p \circ \phi \circ \sigma' = \psi \circ p \circ \sigma' = \psi \circ \sigma$

$\therefore$ $(\psi \circ \sigma^{\not{*}})'_{\phi(x_0)} = \phi \circ \sigma'$

In particular

$$(\psi \circ \sigma)'_{\phi(x_0)}(1) = \phi(-x_0) = -\phi(x_0) \neq \gamma_{\phi(x_0)}(1)$$

$\times\!\!\!\times$ ∎

Cor 3.10 If $f: S^n \to \mathbb{R}^n$ is cts and $f(-x) = -f(x)$ $\forall x \in S^n$ then $\exists\, x_0$ s.t. $f(x_0) = 0$

Pf If not, let $g: S^n \to S^{n-1}$ be given by $g(x) = f(x)/\|f(x)\|$ — cts + antipode-preserving ※.

Cor 3.11 If $f: S^n \to \mathbb{R}^n$ cts then $\exists\, x_0 \in S^n$ s.t. $f(x_0) = f(-x_0)$

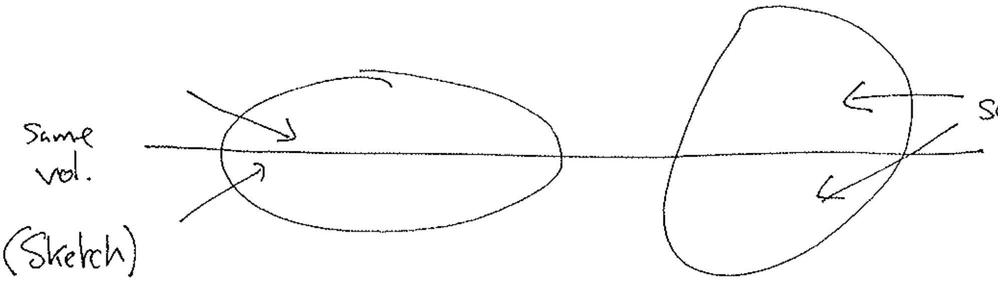
[In particular, f not injective]

Pf ~~Suppose not +~~ Contemplate $g: S^n \to \mathbb{R}^n$ $g(x) = f(x) - f(-x)$

g cts & $g(-x) = -g(x)$ so $\exists\, x_0 \in S^n$ s.t. $g(x_0) = 0$ by (3.10).

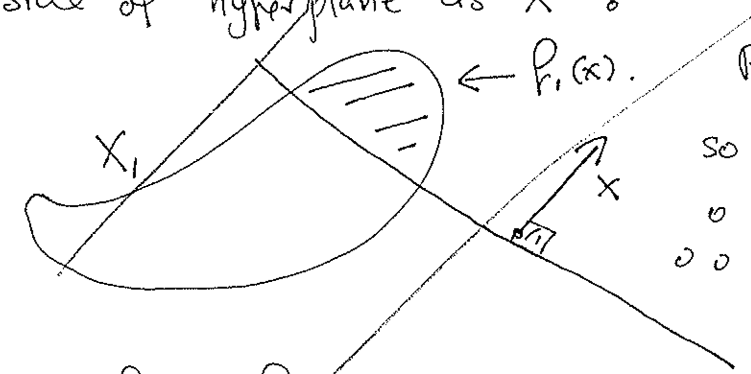
[Chat about pressure + temperature ...]

Ham Sandwich Theorem Let $X_1, \dots X_n$ be bdd measureable (e.g. open or closed) subsets of $\mathbb{R}^n$ then $\exists$ hyperplane bisecting each X_i simultaneously i.e. each X_i has same volume on each side of hyperplane



(Sketch)

Pf For $x \in S^n$, let Π_x be hyperplane orthogonal to x thru origin + let $f_i(x)$ = vol of that part of X_i lying on same side of hyperplane as x ∴

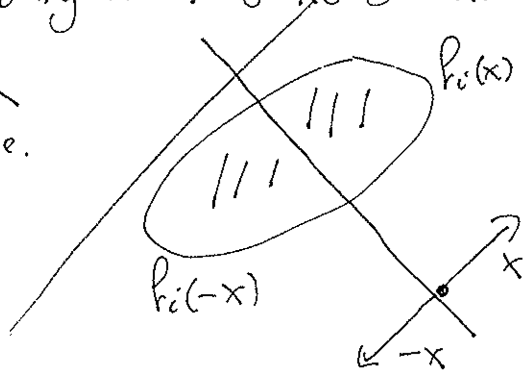


FACT Each $f_i: S^n \to \mathbb{R}$ cts so have cts $f = (f_1, \dots f_n): S^n \to \mathbb{R}^n$

∴ by (3.11) $\exists\, x \in S^n$ s.t. $f(x) = f(-x)$

i.e. $f_i(x) = f_i(-x)$ each i i.e.

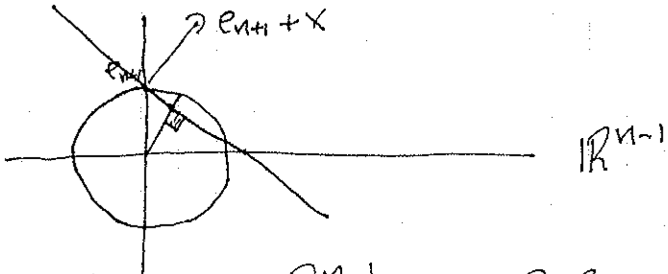
Π_x bisects each X_i.



☒

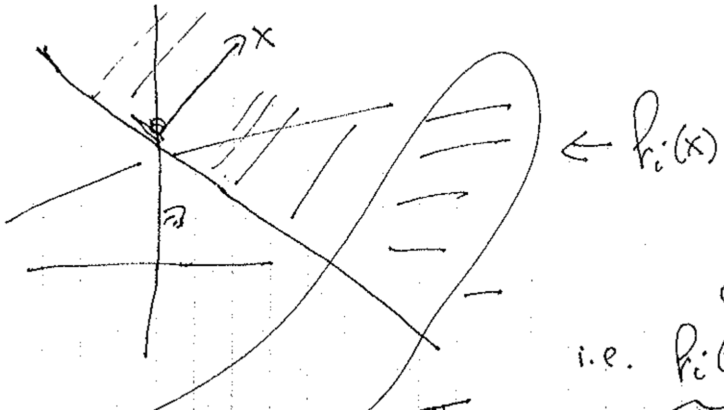
Proof Let $e_{n+1} = \{0, \dots, 1\} \in \mathbb{R}^{n+1}$ & view $\mathbb{R}^n = e_{n+1}^{\perp} \subset \mathbb{R}^{n+1}$

For $x \in S^n$ let Π_x be hyperplane orthogonal to x & passing thru e_{n+1}:



Then $\Pi_x \cap \mathbb{R}^n$ is hyperplane in $\mathbb{R}^{n-1}$ or $\{0\}$ (when $x = \pm e_n$)

Let $f_i(x)$ = vol of that part of X_i lying on same side as $x + e_n$ of Π_x

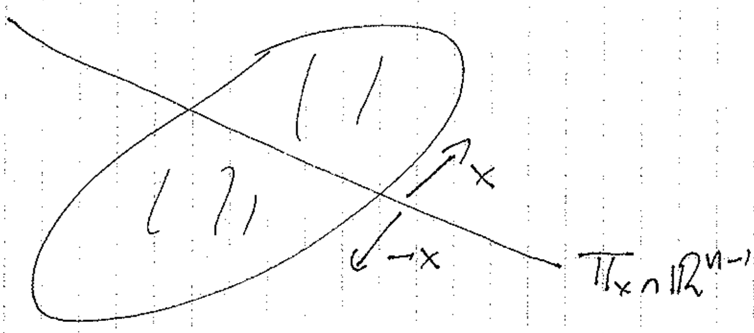


FACT Each $f_i : S^n \to \mathbb{R}$ cts so have cts $f = (f_1, \dots, f_n) : S^n \to \mathbb{R}^n$

$\therefore$ by (8.11) $\exists x \in S^n$ s.t. $f(x) = f(-x)$

i.e. $f_i(x) = f_i(-x)$ each i.

Then since $\Pi_x = \Pi_{-x}$, Π_x bisects each X_i



$\blacksquare$