M40: Christmas exercise sheet

Some revision questions

1. (1994 exam) Let $n \in \mathbb{N}$. The map $p : S^1 \to S^1$ given by $p(e^{it}) = e^{int}$ is a covering map with group of deck translations H. Show that

$$H \cong \mathbb{Z}/n\mathbb{Z}.$$

2. (1991 exam) Let $f : S^1 \to S^1$ be continuous and not homotopic to the identity map. Show that, for some $x \in S^1$, $f(x) = -x$

3. (1992 exam) Let X be a topological space and $f : X \to S^n$ be a continuous map. If f is not surjective, show that f is homotopic to a constant.

4. (1992 exam) Let X be a topological space. The *cone over* X, CX is the quotient of $X \times [0, 1]$ under the equivalence relation $(x, t) \sim (y, s)$ if and only if $(x, t) = (y, s)$ or $s = t = 1$.
   Show that CX is contractible.

More substantial questions

5. Let X, Y and Z be topological spaces with Y strongly locally compact. Show that the map $c : \mathscr{C}(X, Y) \times \mathscr{C}(Y, Z) \to \mathscr{C}(X, Z)$ given by

$$c(f, g) = g \circ f$$

   is continuous.

6. First a little group theory: let G be a group with subgroup H. The *normaliser of H in G*, denoted $N_G(H)$, is given by

$$N_G(H) = \{g \in G : gHg^{-1} = H\}.$$

   It is easy to show:
   (a) $N_G(H)$ is a subgroup of G containing H;
   (b) H is a normal subgroup of $N_G(H)$ (so that the coset space $N_G(H)/H$ is a group).
   Back to topology: let $p : (E, e_0) \to (X, x_0)$ be a covering map with E connected and locally path-connected but not necessarily simply connected. Recall that $p_* : \pi_1(E, e_0) \to \pi_1(X, x_0)$ is an injection. Set $G = \pi_1(X, x_0)$ and let $H = p_*(\pi_1(E, e_0)) \subset G$.
   Finally let $\mathscr{G}$ be the group of deck translations of E. Show that

$$\mathscr{G} \cong N_G(H)/H.$$

   (Observe that this reduces to Theorem 3.8 when E is simply-connected.)

7. Let G be a topological group with a simply connected, locally path-connected covering space $p : \widetilde{G} \to G$. Choose $e \in p^{-1}\{1\}$, the fibre over the identity of G. Show that $\widetilde{G}$ can be equipped with the structure of a topological group in such a way that e is the identity element and p is a homomorphism.
   Hint: Use the Ultimate Lifting Theorem to define the multiplication and inversion maps and make repeated use of the Unique Lifting Property to establish the group axioms.

Home page: `http://go.bath.ac.uk/ma40040`

8. An H-space is a topological space Ω with two extra ingredients:
   (i) A continuous map $m : \Omega \times \Omega \to \Omega$ written $m(x, y) = x * y$;
   (ii) A *homotopy identity* for m, that is, an element $e \in \Omega$ such that the maps $L,\, R : \Omega \to \Omega$ given by $L(x) = e * x$, $R(x) = x * e$ are both homotopic to the identity relative to $\{e\}$. (In particular, $e * e = e$.)

   The remaining exercises explore examples and properties of certain H-spaces.

   (*a*) Show that any topological group is an H-space.

   Now let Ω be an H-space and let γ_1, γ_2 and γ_3 be loops in Ω based at e. Define a new loop $\gamma_1 * \gamma_2$ by
$$\gamma_1 * \gamma_2(t) = \gamma_1(t) * \gamma_2(t).$$

   (*b*) Show that if $\gamma_2 \sim \gamma_3$, then $\gamma_1 * \gamma_2 \sim \gamma_1 * \gamma_3$.

   (*c*) Show that $\gamma_1 \sim \gamma_1 * \gamma_e \sim \gamma_e * \gamma_1$ where, as usual γ_e is the constant loop based at e.

   (*d*) Deduce that $\gamma_1 \cdot \gamma_2 \sim \gamma_1 * \gamma_2 \sim \gamma_2 \cdot \gamma_1$ and hence that $\pi_1(\Omega, e)$ is abelian.

   In particular, if G is a topological group, $\pi_1(G, 1)$ is abelian.

9. Let X be a topological space and fix $x_0 \in X$. Let ΩX be the subspace of $\mathscr{C}(I, X)$ the space of continuous maps $I \to X$ given by
$$\Omega X = \{\alpha \in \mathscr{C}(I, X) \colon \alpha(0) = \alpha(1) = x_0\}$$

   Thus ΩX is the space of loops based at x_0 that we previously called $\text{Path}(X, x_0, x_0)$. We give ΩX the topology induced from the compact open topology on $\mathscr{C}(I, X)$.

   (*a*) Show that $m : \Omega X \times \Omega X \to \Omega X$ given by
$$m(\alpha_0, \alpha_1) = \alpha_0 \cdot \alpha_1$$
   is continuous.

   (*b*) Let $e = \gamma_{x_0} \in \Omega X$ be the constant loop at x_0. Show that e is a homotopy identity for m and thus that ΩX is an H-space.
   Hint: You may find the result of question 5 useful here.

   Since I is strongly locally compact (why?), we learn from a previous exercise sheet that, for α_0, $\alpha_1 \in \Omega X$, $\alpha_0 \sim \alpha_1$ if and only if there is a path in ΩX from α_0 to α_1. Thus we have a bijection between $\pi_0(\Omega X)$ and $\pi_1(X, x_0)$. This motivates the

   Definition. The *second homotopy group* of X based at x_0, denoted $\pi_2(X, x_0)$ is the group $\pi_1(\Omega X, e)$.
   The *n-th homotopy group* of X based at x_0 is then defined inductively by $\pi_n(X, x_0) = \pi_{n-1}(\Omega X, e)$.

   We deduce from question 8 that $\pi_n(X, x_0)$ is abelian for all $n > 1$.

   (*c*) Show that if $f : (X, x_0) \to (Y, y_0)$ is a continuous map then the map $\Omega(f) : \Omega X \to \Omega Y$ given by $\Omega(f)(\alpha_1) = f \circ \alpha_1$ is also continuous.

   (*d*) Use this to define a homomorphism $f_* : \pi_2(X, x_0) \to \pi_2(Y, y_0)$ and show that if $g : (Y, y_0) \to (Z, z_0)$ is also continuous then
$$(g \circ f)_* = g_* \circ f_*.$$

Finally show that id_* is the identity map of π_2 so that we have defined another functor!

These properties extend by induction to all π_n.

10. The groups $\pi_n(X)$ are one of the main families of topological invariants and their study is the vast theory of *Homotopy Theory*. As a final exercise, go to the Library and try and find out about the other main family: the *homology groups*.

November 24, 2011

M40: Christmas exercise sheet—Solutions

1. For $k \in \mathbb{Z}$, define $\phi_k : S^1 \to S^1$ by $\phi_k(z) = e^{2k\pi i/n} z$. Certainly these are homeomorphisms with $\phi_k \circ \phi_l = \phi_{k+l}$ and $p(\phi_k(e^{it})) = e^{2k\pi i} e^{int} = p(e^{it})$ so that each ϕ_k is a deck translation. Thus we have a homomorphism $k \mapsto \phi_k : \mathbb{Z} \to H$ and it is easy to check that its kernel is $n\mathbb{Z}$: $\phi_k = \mathrm{id}_{S^1}$ if and only if $k \in n\mathbb{Z}$. Thus we will be done (via the first isomorphism theorem) if we can see that our homomorphism is onto. But if $\phi \in H$ then $\phi(1) \in p^{-1}\{1\} = \{e^{2k\pi i/n} : k = 1, \dots, n\}$ so that $\phi(1) = e^{2k\pi i/n} = \phi_k(1)$, for some k. Now both ϕ and ϕ_k are lifts of p so that the unique lifting property shows us that $\phi = \phi_k$ and we are done.

2. Suppose that, for all $x \in S^1$, $f(x) \neq -x$. We shall show that f is homotopic to the identity map. Indeed, define $F : I \times S^1 \to S^1$ by
$$F(t,x) = \frac{tf(x) + (1-t)x}{\|tf(x) + (1-t)x\|}.$$
F is continuous as long as it is well-defined and it is well-defined since $tf(x) + (1-t)x = 0$ for some $x \in S^1$ and $t \in I$ if and only if $f(x) = -x$. It is now straight-forward to see that F is the desired homotopy.

3. Conceptual answer: if f is not surjective then f has image in $S^n \backslash \{x_0\}$, for some x_0, and this is homeomorphic to $\mathbb{R}^n$ which is contractible. Any continuous map into a contractible space is easily seen to homotopic to a constant (why?).
Low-tech answer: with $x_0 \notin f(X)$ as above, we show that f is homotopic to be constant map with value $-x_0$. Indeed, it is easy to check that $F : I \times S^n \to S^n$ given by
$$F(t,x) = \frac{tf(x) - (1-t)x_0}{\|tf(x) - (1-t)x_0\|}$$
is well-defined (and hence continuous) and thus provides the desired homotopy.

4. Let $[x,s]$ denote the equivalence class of (x,s). We will show that $\{[x,1]\}$ is a deformation retract of CX. We must therefore define a homotopy relative to $\{[x,1]\}$ between id_{CX} and the constant map with value $[x,1]$. We define $F : I \times CX \to CX$ by $F(t,[x,s]) = [x,(1-t)s+t]$ which is well-defined and is our desired homotopy so long as it is continuous. To see the continuity, observe that we have a commuting diagram
$$\begin{array}{ccc} I \times (I \times X) & \xrightarrow{\widehat{F}} & I \times X \\ {\scriptstyle \mathrm{id} \times p}\downarrow & & \downarrow{\scriptstyle p} \\ I \times CX & \xrightarrow{F} & CX \end{array}$$
where p is the quotient map and $\widehat{F}(t,s,x) = ((1-t)s+t,x)$ is clearly continuous. Let $G \subset CX$ be open. Then $F^{-1}(G)$ is open if and only if $(\mathrm{id} \times p)^{-1}(F^{-1}(G))$ is open. But this last is $(p \circ \widehat{F})^{-1}(G)$ which is indeed open since both p and $\widehat{F}$ are continuous.

5. Let $K \subset X$ be compact and $G \subset Z$ be continuous. Let $(f,g) \in c^{-1}(\mathscr{C}_{K,G})$ so that $g \circ f(K) \subset G$. Thus $f(K) \subset g^{-1}(G) \subset Y$ and $f(K)$ is compact and $g^{-1}(G)$ is open.
We would be done if we could find open sets $f\mathscr{C}_{K,N^\circ} \subset \mathscr{C}(X,Y)$ and $g \in \mathscr{C}_{N,G} \subset \mathscr{C}(Y,Z)$ with
$$\mathscr{C}_{K,N^\circ} \times \mathscr{C}_{N,G} \subset c^{-1}(\mathscr{C}_{K,G}).$$
(Here $N^\circ, N \subset Y$ with N° open and N compact). For this to have a chance of working we need $f(K) \subset N^\circ$ (to ensure $f \in \mathscr{C}_{K,N^\circ}$), $g(N) \subset G$ (to ensure $g \in \mathscr{C}_{N,G}$ and finally $N^\circ \subset N$ (to ensure that compositions send K to G). We therefore need to prove the following lemma:

Lemma 0.1. *Let Y be strongly locally compact and $K \subset G \subset Y$ with K compact and G open. Then there is a compact set $N \subset Y$ such that*

$$K \subset N^\circ \subset N \subset G,$$

where N° denotes the interior of N.

Proof. Each $k \in K$ has a compact neighbourhood $U_k \subset G$. The interiors of all these comprise an open cover of K and so passing to a finite subcover gives $N = \bigcup_{i=1}^{n} U_{k_i} \subset G$ with $K \subset N^\circ$. Moreover, N is compact being a finite union of compact sets. □

The lemma now gives us a compact set $N \subset Y$ with $f(K) \subset N^\circ \subset N \subset g^{-1}(G)$. Thus, if $f' \in \mathscr{C}_{K,N^\circ}$ and $g' \in \mathscr{C}_{N,G}$, then $g' \circ f'(K) \subset G$ and we have proved that

$$(f, g) \in \mathscr{C}_{K,N^\circ} \times \mathscr{C}_{N,G} \subset c^{-1}(\mathscr{C}_{K,G}).$$

It now follows that $c^{-1}(\mathscr{C}_{K,G})$ is open so that c is continuous.

6. The assertions about normalisers are utterly straight-forward so let's get down to the nitty-gritty.
The idea is to mimic the proof of Theorem 3.8 and define a map from $\mathscr{G}$ to $N_G(H)/H$ as follows: let $\phi \in \mathscr{G}$ and choose a path σ' from e_0 to $\phi(e_0)$. Then $\sigma = p \circ \sigma'$ is a loop at x_0 and hopefully its homotopy class will give us an element of $N_G(H)/H$. Let us see how this works out:
First we show that $[\sigma] \in N_G(H)$. So let $[\tau] \in H$. Then $[\tau] = [p \circ \tau']$ for some loop τ' in E based at e_0 and we must show that $[\sigma \cdot p \circ \tau' \cdot \sigma^{-1}] \in H$. But $\phi \circ \tau'$ is a loop in E based at $\phi(e_0)$ and covering $p \circ \tau'$ so that $\sigma' \cdot (\phi \circ \tau') \cdot \sigma'^{-1}$ is a well-defined loop at e_0 covering $\sigma \cdot p \circ \tau' \cdot \sigma^{-1}$. This means that $[\sigma \cdot p \circ \tau' \cdot \sigma^{-1}] \in H$ as required.
So far, the element of $N_G(H)$ that we have constructed depends on the path σ' from e_0 to $\phi(e_0)$. So let ρ' be another such path with $\rho = p \circ \rho'$. Then $\sigma' \sim (\sigma' \cdot \rho'^{-1}) \cdot \rho'$ so that, setting $\tau' = \sigma' \cdot \rho'^{-1}$ which is a loop at e_0 we have

$$[\sigma] = [p \circ \tau'][\rho].$$

Now $[p \circ \tau'] \in H$ so $H[\sigma] = H[\rho]$ or, since H is normal in $N_G(H)$, $[\sigma]H = [\rho]H$. To conclude then, we have found that the cosets of $[\sigma]$ and $[\rho]$ are the same element of $N_G(H)/H$ and so have a well-defined map $\chi : \mathscr{G} \to N_G(H)/H$ given by

$$\chi(\phi) = [p \circ \sigma']H,$$

where σ' is any path from e_0 to $\phi(e_0)$.
We now show that χ is an isomorphism. First we show that it is a homomorphism: let ϕ_1, $\phi_2 \in \mathscr{G}$ and let σ' and τ' be paths from e_0 to $\phi_1(e_0)$ and $\phi_2(e_0)$ respectively. Let $\sigma = p \circ \sigma'$ and $\tau = p \circ \tau'$. Now $\phi_1 \circ \tau'$ is a path from $\phi_1(e_0)$ to

$$\phi_1(\tau'(1)) = \phi_1(\phi_2(e_0)).$$

Thus $\sigma' \cdot (\phi_1 \circ \tau')$ is a path from e_0 to $\phi_1 \circ \phi_2(e_0)$ so that

$$\chi(\phi_1 \circ \phi_2) = [\sigma \cdot (p \circ \phi_1 \circ \tau')]H = [\sigma][\tau]H = \chi(\phi_1)\chi(\phi_2),$$

as required.
As for injectivity of χ: let $\phi \in \ker \chi$. We must show that $\phi = \mathrm{id}_E$. So let σ' be a path from e_0 to $\phi(e_0)$ with $\sigma = p \circ \sigma'$. By hypothesis, $[\sigma] \in H$ so that $\sigma \sim p \circ \tau'$ with τ' a loop in E based at e_0. The homotopy lifting property then gives that $\sigma' \sim \tau'$ and, in particular, $\phi(e_0) = \sigma'(1) = \tau'(1) = e_0$ since τ' is a loop. Thus ϕ is a lift of p which agrees with id_E at e_0 and so, by the unique lifting property, $\phi = \mathrm{id}_E$ as required.

Finally we show surjectivity: so let $[\sigma]H \in N_G(H)$ and let $e_1 = \sigma'_{e_0}(1)$. We must find $\phi \in \mathscr{G}$ for which $\phi(e_0) = e_1$. We will have done this if we can get continuous maps ϕ, $\overline{\phi} : E \to E$ which lift p such that $\phi(e_0) = e_1$ and $\overline{\phi}(e_1) = e_0$. For then the usual unique lifting argument shows that ϕ and $\overline{\phi}$ are mutually inverse deck translations. Now the ultimate lifting theorem guarantees the existence of ϕ as soon as we know that $p_*(\pi_1(E, e_0) \subset p_*(E, e_1)$ while $\overline{\phi}$ exists when $p_*(\pi_1(E, e_1) \subset p_*(E, e_0)$. So it remains to show that these two groups are equal.
For this, let $[\tau] \in p_*(\pi_1(E, e_0)$ so that we have a loop τ' based at e_0 with $p \circ \tau' = \tau$. Now $(\sigma'_{e_0})^{-1} \cdot \tau' \cdot \sigma'_{e_0}$ is a loop at e_1 and taking p of this, we see that $[\sigma]^{-1}[\tau][\sigma] \in p_*\pi_1(E, e_1)$. Thus

$$[\sigma]^{-1} p_*\pi_1(E, e_0)[\sigma] \subset p_*\pi_1(E, e_1).$$

But $[\sigma]$ is in the normaliser of $p_*\pi_1(E, e_0)$ so that

$$p_*\pi_1(E, e_0) = [\sigma]^{-1} p_*\pi_1(E, e_0)[\sigma] \subset p_*\pi_1(E, e_1).$$

For the converse, let $[\tau] \in p_*\pi_1(E, e_1)$ with $\tau = p \circ \tau'$ for some loop τ' based at e_1. Then $\sigma'_{e_0} \cdot \tau' \cdot (\sigma'_{e_0})^{-1}$ is a loop at e_0 which we take p of to conclude that $[\sigma][\tau][\sigma]^{-1} \in p_*\pi_1(E, e_0)$. Then

$$[\tau] \in [\sigma]^{-1} p_*\pi_1(E, e_0)[\sigma] = p_*\pi_1(E, e_0)$$

and we are done!

7. The first thing to do is define the multiplication $\widetilde{\mu} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ and the inversion $\widetilde{\imath} : \widetilde{G} \to \widetilde{G}$. We define $\widetilde{\imath}$ to be the unique lift of $i \circ p : \widetilde{G} \to G$ such that $\widetilde{\imath}(e) = e$. Such a lift exists by the Ultimate Lifting Theorem since $\widetilde{G}$ is simply connected. Thus

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\widetilde{\imath}} & \widetilde{G} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ G & \xrightarrow{i} & G \end{array}$$

commutes.
Similarly, define $p \times p : \widetilde{G} \times \widetilde{G} \to G \times G$ by $p \times p(x, y) = (p(x), p(y))$ (this is clearly a continuous map) and define $\widetilde{\mu} : \widetilde{G} \times \widetilde{G}$ to be the unique lift of $\mu \circ (p \times p) : \widetilde{G} \times \widetilde{G} \to G$ with $\widetilde{\mu}(e, e) = e$. Again, this lift exists by the Ultimate Lifting Theorem since $\widetilde{G} \times \widetilde{G}$ is simply connected so we get a commuting diagram:

$$\begin{array}{ccc} \widetilde{G} \times \widetilde{G} & \xrightarrow{\widetilde{\mu}} & \widetilde{G} \\ {\scriptstyle p \times p}\downarrow & & \downarrow{\scriptstyle p} \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

Now we must show that the multiplication and inversion so defined actually satisfy the group axioms. First associativity: define maps $\widetilde{A}_1$, $\widetilde{A}_2 : \widetilde{G} \times \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ by

$$\begin{aligned} \widetilde{A}_1(g_1, g_2, g_3) &= \widetilde{\mu}(g_1, \widetilde{\mu}(g_2, g_3)) = g_1(g_2 g_3) \\ \widetilde{A}_2(g_1, g_2, g_3) &= \widetilde{\mu}(\widetilde{\mu}(g_1, g_2), g_3) = (g_1 g_2) g_3 \end{aligned}$$

Both $\widetilde{A}_i$ are continuous and $\widetilde{A}_i(e, e, e) = e$. However,

$$p(\widetilde{A}_1(g_1, g_2, g_3)) = p(g_1)(p(g_2)p(g_3)) = (p(g_1)p(g_2))p(g_3) = p(\widetilde{A}_2(g_1, g_2, g_3)),$$

by the associativity of the multiplication in G, so that by the unique lifting property, $\widetilde{A}_1 = \widetilde{A}_2$ and $\widetilde{\mu}$ is associative.

As for e being the identity, define $L : \widetilde{G} \to \widetilde{G}$ by $L(g) = \widetilde{\mu}(e, g) = eg$ and observe that $p \circ L(g) = \mu(p(e), p(g)) = 1p(g) = p(g)$ while $L(e) = \widetilde{\mu}(e, e) = e$. Thus both L and $\mathrm{id}_{\widetilde{G}}$ lift p and agree at e so that, by the unique lifting property, $L = \mathrm{id}_{\widetilde{G}}$. This means $eg = g$ for all $g \in \widetilde{G}$. Similarly $ge = g$ for all $g \in \widetilde{G}$ whence e is the identity for $\widetilde{\mu}$.
Finally, consider $I : \widetilde{G} \to \widetilde{G}$ given by $I(g) = \widetilde{\mu}(g, \widetilde{\imath}(g))$. Clearly I is continuous and $I(e) = e$. Moreover, $p(I(g)) = \mu(p(g), i(p(g))) = p(g)(p(g))^{-1} = 1$, for all $g \in \widetilde{G}$. Thus I takes values in the discrete set $p^{-1}\{1\}$ and so is constant giving $I(g) = 1$ or $g\widetilde{\imath}(g) = 1$. Similarly, $\widetilde{\imath}(g)g = 1$, for all $g \in \widetilde{G}$ and we conclude that the group axioms are satisfied.
It is straight-forward to show that if a space is Hausdorff so is any covering space thereof so that $\widetilde{G}$ is Hausdorff and therefore we see that $\widetilde{G}$ is a topological group.
Finally, the commuting diagram describing $\widetilde{\mu}$ gives

$$p(g_1 g_2) = p(\widetilde{\mu}(g_1, g_2)) = \mu(p(g_1), p(g_2)) = p(g_1)p(g_2)$$

so that p is a homomorphism.

8. (*a*) This is very easy: let m be the usual multiplication and e the identity element. In this case, the maps L and R are *equal* to the identity map.

   (*b*) Let F be a homotopy from γ_2 to γ_3 and define $G : I \times I \to \Omega$ by

   $$G(t, s) = m(\gamma_1(t), F(t, s)).$$

   G is clearly continuous being a composition of continuous maps and moreover, $G(0, s) = G(1, s) = m(e, e) = e$ while $G(t, 0) = m(\gamma_1(t), \gamma_2(t)) = (\gamma_1 * \gamma_2)(t)$ and, similarly, $G(t, 1) = (\gamma_1 * \gamma_3)(t)$. Thus G is the desired homotopy.

   (*c*) Let $F : I \times \Omega \to \Omega$ be the homotopy from the identity to L relative to $\{e\}$ and define $G : I \times I \to \Omega$ by

   $$G(t, s) = F(s, \gamma_1(t))$$

   which is certainly continuous. Then $G(t, 0) = F(0, \gamma_1(t)) = \gamma_1(t)$ while $G(t, 1) = F(1, \gamma_1(t)) = L(\gamma_1(t)) = e * \gamma_1(t) = (\gamma_e * \gamma_1)(t)$. Moreover $G(s, 0) = G(s, 1) = F(s, e) = e$ since F is a homotopy relative to $\{e\}$. Thus $\gamma_1 \sim \gamma_e * \gamma_1$.
   A similar argument establishes the other identity.

   (*d*) We know that $\gamma_1 \sim \gamma_1 \cdot \gamma_e$ and $\gamma_2 \sim \gamma_e \cdot \gamma_2$. Thus, from the above, we get

   $$\gamma_1 * \gamma_2 \sim (\gamma_1 \cdot \gamma_e) * (\gamma_e \cdot \gamma_2).$$

   However, we see from the definitions that

   $$(\gamma_1 \cdot \gamma_e) * (\gamma_e \cdot \gamma_2) = (\gamma_1 * \gamma_e) \cdot (\gamma_e * \gamma_2),$$

   while the above shows that this last is homotopic to $\gamma_1 \cdot \gamma_2$. Thus $\gamma_1 \cdot \gamma_2 \sim \gamma_1 * \gamma_2$. The other identity is proved in a similar fashion.

9. Recall the the compact open topology is that topology on $\mathscr{C}(Y, X)$ with sub-base consisting of all sets $\mathscr{C}_{K,G}$ of the form

   $$\mathscr{C}_{K,G} = \{f \in \mathscr{C}(Y, X) \colon f(K) \subset G\},$$

   where $K \subset Y$ is compact and $G \subset X$ is open. We denote the induced sub-basic open sets of ΩX by $\Omega_{K,G}$: thus

   $$\Omega_{K,G} = \{\alpha \in \Omega X \colon \alpha(K) \subset G\}$$

   where $K \subset I$ is compact and $G \subset X$ is open.

   (a) It suffices to show that the inverse image of any sub-basic open set of ΩX by m is open. So consider $m^{-1}(\Omega_{K,G})$ with $K \subset I$ compact and $G \subset X$ open. Then $(\alpha_0, \alpha_1) \in m^{-1}(\Omega_{K,G})$

if and only if $\alpha_0 \cdot \alpha_1(K) \subset G$ if and only if $\alpha_0(2t) \in G$ for $t \in K \cap [0, \frac{1}{2}]$ and $\alpha_1(2t-1) \in G$ for $t \in K \cap [\frac{1}{2}, 1]$. To say this another way, define K_1, $K_2 \subset I$ by

$$K_1 = \{2t : t \in K \cap [0, \tfrac{1}{2}]\} \qquad K_2 = \{2t - 1 : t \in K \cap [\tfrac{1}{2}, 1]\}.$$

These are easily seen to be compact subsets of I and the above remarks show that

$$\Omega_{K_1,G} \times \Omega_{K_2,G} = m^{-1}(\Omega_{K,G})$$

so that $m^{-1}(\Omega_{K,G})$ is open in $\Omega X \times \Omega X$ and m is continuous.

(b) We must find homotopies relative to $\{e\}$ between the identity map $\mathrm{id} : \Omega X \to \Omega X$ and the maps L, $R : \Omega X \to \Omega X$ given by $L\alpha = e \cdot \alpha$ and $R\alpha = \alpha \cdot e$. We shall just give the argument for L.
It is not so difficult to see what the homotopy must be: any α is homotopic to $e \cdot \alpha$ and this suggests defining $F : I \times \Omega X \to \Omega X$ by

$$F(s, \alpha)(t) = \begin{cases} \alpha(0) = x_0 & \text{for } 0 \le t \le s/2 \\ \alpha((2t - s)/(2 - s)) & \text{for } s/2 \le t \le 1 \end{cases}$$

It is easy to check that F is well-defined, that $F(0, \alpha) = \alpha$, $F(1, \alpha) = e \cdot \alpha$ and $F(s, e) = e$, for all $\alpha \in \Omega X$ and $s \in I$. Thus F is the required homotopy if we can prove that it is continuous.
For this, we argue as follows: on a previous sheet we proved that when Y is strongly locally compact, f,$g \in \mathscr{C}(Y, Z)$ were homotopic if and only if there was a path in $\mathscr{C}(Y, Z)$ from f to g. This amounted to showing that $G : I \times Y \to Z$ is continuous if and only if the map $\widehat{G} : I \to \mathscr{C}(Y, Z)$ given by $\widehat{G}(s)(y) = G(s, y)$ is continuous. We apply this result to the map $G : I \times I \to I$ given by

$$G(s, t) = \begin{cases} 0 = x_0 & \text{for } 0 \le t \le s/2 \\ (2t - s)/(2 - s) & \text{for } s/2 \le t \le 1 \end{cases}$$

which is obviously continuous. The point now is that $F(s, \alpha) = \alpha \circ \widehat{G}(s)$ and so is the composition of the continuous map $(s, \alpha) \mapsto (\widehat{G}(s), \alpha) : I \times \Omega X \to \mathscr{C}(I, I) \times \Omega X$ and the composition map $c : \mathscr{C}(I, I) \times \mathscr{C}(I, X)$ which is continuous by question 5. Thus F is continuous and we are done.

(c) This is a bit easier: let $\Omega_{K,G} \subset \Omega Y$ be a sub-basic open set with $K \subset I$ compact and $G \subset Y$ open. Then $\alpha \in \Omega(f)^{-1}(\Omega_{K,G})$ if and only if $f \circ \alpha(K) \in G$ if and only if $\alpha(K) \subset f^{-1}(G)$ if and only if $\alpha \in \Omega_{K,f^{-1}(G)}$. Thus $\Omega(f)^{-1}(\Omega_{K,G}) = \Omega_{K,f^{-1}(G)}$ which is open. Thus $\Omega(f)$ is continuous.

(d) We therefore define $f_* : \pi_1(\Omega X, e) \to \pi_1(\Omega Y, e)$ by $f_* = \Omega(f)_*$ and the functorial properties are easy to check using $\Omega(g \circ f) = \Omega(g) \circ \Omega(f)$.