M216: Exercise sheet 9

Warmup questions

1. Let $U \leq V$. Show that $\operatorname{ann} U \leq V^*$.
2. Let V be finite-dimensional and $U \leq V$. Show that

$$\dim \operatorname{ann} U + \dim U = \dim V.$$

Homework

3. Prove at least one of the following assertions:
   (a) Let $E, F \leq V^*$. Then

$$\begin{aligned} \operatorname{sol}(E+F) &= (\operatorname{sol} E) \cap (\operatorname{sol} F) \\ (\operatorname{sol} E) + (\operatorname{sol} F) &\leq \operatorname{sol}(E \cap F) \end{aligned}$$

   with equality if V is finite-dimensional.

   (b) Let $U, W \leq V$. Then

$$\begin{aligned} \operatorname{ann}(U+W) &= (\operatorname{ann} U) \cap (\operatorname{ann} W) \\ (\operatorname{ann} U) + (\operatorname{ann} W) &\leq \operatorname{ann}(U \cap W) \end{aligned}$$

   with equality if V is finite-dimensional.

4. Let $\phi \in L(V, W)$ be a linear map of vector spaces. Show that

$$\begin{aligned} \ker \phi^T &= \operatorname{ann}(\operatorname{im} \phi) \\ \operatorname{im} \phi^T &\leq \operatorname{ann}(\ker \phi) \end{aligned}$$

   with equality if V, W are finite-dimensional.

Extra questions

5. Let $U \leq V$ and let $\iota : U \to V$ be the inclusion map (so that $\iota(u) = u$, for all $u \in U$) and $q : V \to V/U$ the quotient map.
   (a) Show that $\iota^T : V^* \to U^*$ is the restriction map: thus $\iota^T(\alpha) = \alpha_{|U}$ with kernel $\operatorname{ann} U$.
   If V is finite-dimensional, show that ι^T is surjective and deduce that $V^*/\operatorname{ann} U \cong U^*$.
   (b) Show that $q^T : (V/U)^* \to V^*$ is injective with $\operatorname{im} q^T \leq \operatorname{ann} U$. If V is finite-dimensional, show that q^T is an isomorphism $(V/U)^* \to \operatorname{ann} U$.
6. Recall the linear injection $\operatorname{ev} : V \to V^{**}$. For $U \leq V$, show that $\operatorname{ev}(U) \leq \operatorname{ann}(\operatorname{ann} U)$ with equality if V is finite-dimensional.

Please hand in at 4W level 1 by NOON on Friday December 8th

M216: Exercise sheet 9—Solutions

1. Firstly, $0 \in \operatorname{ann} U$ so $\operatorname{ann} U \neq \emptyset$. So we just check that $\operatorname{ann} U$ is closed under addition and scalar multiplication. Let $\alpha_1, \alpha_2 \in \operatorname{ann} U$ and $u \in U$. Then, $\alpha_1(u) = \alpha_2(u) = 0$ so that $(\alpha_1 + \alpha_2)(u) = 0 + 0 = 0$ whence $\alpha_1 + \alpha_2 \in \operatorname{ann} U$ also. Similarly, for $\alpha \in \operatorname{ann} U$ and $\lambda \in \mathbb{F}$, $(\lambda\alpha)(u) = \lambda\alpha(u) = \lambda 0 = 0$ so that $\lambda\alpha \in \operatorname{ann} U$.
Alternatively, note that restriction to U, $\alpha \mapsto \alpha_{|U}$ is a linear map $V^* \to U^*$ with kernel $\operatorname{ann} U$.

2. Let $v_1, \dots, v_k$ be a basis of U and extend to a basis $v_1, \dots, v_n$ of V. Let $v_1^*, \dots, v_n^*$ be the dual basis. Now observe that $\alpha \in V^*$ is in $\operatorname{ann} U$ if and only if $\alpha(v_j) = 0$, for $1 \leq j \leq k$. Thus, writing $\alpha = \sum_{i=1}^n \alpha(v_i)v_i^*$, we see that $\alpha \in \operatorname{ann} U$ if and only if $\alpha \in \operatorname{span}\{v_i^* \mid k+1 \leq i \leq n\}$. Thus $\operatorname{ann} U = \operatorname{span}\{v_i^* \mid k+1 \leq i \leq n\}$ so that
$$\dim \operatorname{ann} U = n - k = \dim V - \dim U.$$

3. (a) $E, F \leq E + F$ so $\operatorname{sol}(E+F) \leq \operatorname{sol} E, \operatorname{sol} F$ whence $\operatorname{sol}(E+F) \leq (\operatorname{sol} E) \cap (\operatorname{sol} F)$. Conversely, if $v \in (\operatorname{sol} E) \cap (\operatorname{sol} F)$ then $\alpha(v) = \beta(v) = 0$, for all $\alpha \in E$ and $\beta \in F$. Thus, for $\alpha + \beta \in E + F$, $(\alpha + \beta)(v) = 0 + 0 = 0$ so that $v \in \operatorname{sol}(E+F)$. We conclude that $(\operatorname{sol} E) \cap (\operatorname{sol} F) \leq \operatorname{sol}(E+F)$ and so $(\operatorname{sol} E) \cap (\operatorname{sol} F) = \operatorname{sol}(E+F)$.
For the second statement, $E \cap F \leq E, F$ so that $\operatorname{sol} E, \operatorname{sol} F \leq \operatorname{sol}(E \cap F)$ whence $(\operatorname{sol} E) + (\operatorname{sol} F) \leq \operatorname{sol}(E \cap F)$ by Proposition 2.1(2) of the notes. For equality when V is finite-dimensional, we show that both subspaces have the same dimension using the first part, the formula for $\operatorname{sol} E$ and the dimension formula[1]. The dimension formula gives
$$\begin{aligned}\dim((\operatorname{sol} E) + (\operatorname{sol} F)) &= \dim \operatorname{sol} E + \dim \operatorname{sol} F - \dim((\operatorname{sol} E) \cap (\operatorname{sol} F)) \\ &= \dim \operatorname{sol} E + \dim \operatorname{sol} F - \dim \operatorname{sol}(E+F),\end{aligned}$$
using the first part,
$$\begin{aligned}&= \dim V - \dim E + \dim V - \dim F - (\dim V - \dim(E+F)) \\ &= \dim V - \dim(E \cap F),\end{aligned}$$
by the dimension formula again,
$$= \dim \operatorname{sol}(E \cap F).$$
(b) First we note that if $X \leq Y \leq V$ then $\operatorname{ann} Y \leq \operatorname{ann} X$: if $\alpha \in \operatorname{ann} Y$, then $\alpha_{|Y} = 0$ and so, in particular, $\alpha_{|X} = 0$, that is $\alpha \in \operatorname{ann} X$.
We now put this to work: $U, W \leq U + W$ so $\operatorname{ann}(U+W) \leq \operatorname{ann} U, \operatorname{ann} W$ whence $\operatorname{ann}(U+W) \leq (\operatorname{ann} U) \cap (\operatorname{ann} W)$. For the converse, if $\alpha \in (\operatorname{ann} U) \cap (\operatorname{ann} W)$ we have $\alpha_{|U} = 0$ and $\alpha_{|W} = 0$. So if $v = u + w \in U + W$ then $\alpha(v) = \alpha(u) + \alpha(w) = 0 + 0 = 0$ so that $v \in \operatorname{ann}(U+W)$. Thus $\operatorname{ann}(U+W) = (\operatorname{ann} U) \cap (\operatorname{ann} W)$.
For the second statement, $U \cap W \leq U, W$ so that $\operatorname{ann} U, \operatorname{ann} W \leq \operatorname{ann}(U \cap W)$ and then $(\operatorname{ann} U) + (\operatorname{ann} W) \leq \operatorname{ann}(U \cap W)$ by Proposition 2.1(2). For equality when V is finite-dimensional, we argue as in part (a). The dimension formula says
$$\begin{aligned}\dim((\operatorname{ann} U) + (\operatorname{ann} W)) &= \dim \operatorname{ann} U + \dim \operatorname{ann} W - \dim((\operatorname{ann} U) \cap (\operatorname{ann} W)) \\ &= \dim \operatorname{ann} U + \dim \operatorname{ann} W - \dim \operatorname{ann}(U+W),\end{aligned}$$
using the first part,
$$\begin{aligned}&= \dim V - \dim U + \dim V - \dim W - (\dim V - \dim(U+W)) \\ &= \dim V - \dim(U \cap W),\end{aligned}$$

[1] If $X, Y \leq W$ then $\dim(X+Y) + \dim(X \cap Y) = \dim X + \dim Y$.

by the dimension formula again,

$$= \dim \operatorname{ann}(U \cap W).$$

Notice that the arguments for part (b) are essentially identical to those for part (a): the key points are that ann and sol reverse inclusions and take subspaces to ones of complementary dimension.

4. Let $\alpha \in W^*$. Then $\alpha \in \ker \phi^T$ if and only if $\alpha \circ \phi = 0$ if and only if $\alpha(\operatorname{im} \phi) = \{0\}$, that is $\alpha \in \operatorname{ann}(\operatorname{im} \phi)$. Thus $\ker \phi^T = \operatorname{ann}(\operatorname{im} \phi)$.
For the second statement, suppose that $\beta \in \operatorname{im} \phi^T$ so that $\beta = \phi^T(\alpha) = \alpha \circ \phi$, for some $\alpha \in W^*$. Then if $v \in \ker \phi$, $\beta(v) = \alpha(\phi(v)) = 0$ so that $\beta \in \operatorname{ann}(\ker \phi)$. Thus $\operatorname{im} \phi^T \leq \operatorname{ann}(\ker \phi)$.
For equality when V is finite-dimensional, recall that we already know from lectures that $\operatorname{rank} \phi = \operatorname{rank} \phi^T$ from which we see from rank-nullity that

$$\dim \operatorname{im} \phi^T = \operatorname{rank} \phi = \dim V - \dim \ker \phi = \dim \operatorname{ann}(\ker \phi),$$

where the last equality comes from Question 2.

5. (a) For $\alpha \in V^*$ and $u \in U$, $\iota^T(\alpha)(u) = \alpha(\iota(u)) = \alpha(u) = \alpha_{|U}(u)$. Thus $\iota^T(\alpha) = \alpha_{|U}$ and ι^T is the restriction map. Now $\ker \iota^T = \{\alpha \in V^* \mid \alpha_{|U} = 0\} = \operatorname{ann} U$.
Proposition 2.11 tells us[2] that any $\beta \in U^*$ is the restriction of some $\alpha \in V^*$ so that ι^T surjects: $\operatorname{im} \iota^T = U^*$. Thus, the First Isomorphism Theorem, applied to ι^T, tells us that

$$V^* / \operatorname{ann} U = V^* / \ker \iota^T \cong \operatorname{im} \iota^T = U^*.$$

This gives us another approach to Question 2.

(b) All we need to know about q is that it is a linear surjection with kernel U. Then, by Question 4, $\ker q^T = \operatorname{ann}(\operatorname{im} q) = \operatorname{ann} V/U = \{0\}$ (any $\alpha \in (V/U)^*$ that vanishes on V/U is zero by definition!) so that q^T injects. Moreover, Question 4 tells us that $\operatorname{im} q^T \leq \operatorname{ann}(\ker q) = \operatorname{ann} U$ with equality when V is finite-dimensional. Thus, in that case, q^T is a linear bijection $(V/U)^* \to \operatorname{ann} U$ and so an isomorphism.

6. This is just a matter of not panicking! Let $f \in \operatorname{ev}(U)$ so that $f = \operatorname{ev}(u)$, for some $u \in U$. Let $\alpha \in \operatorname{ann} U$. We need $f(\alpha) = 0$. But

$$f(\alpha) = \operatorname{ev}(u)(\alpha) = \alpha(u) = 0,$$

since $\alpha \in \operatorname{ann} U$.
When V is finite-dimensional, we know that ev is an isomorphism so that $\dim \operatorname{ev}(U) = \dim U$. Meanwhile

$$\dim(\operatorname{ann}(\operatorname{ann} U)) = \dim V^* - \dim \operatorname{ann} U = \dim V - (\dim V - \dim U) = \dim U$$

so that $\operatorname{ev}(U)$ and $\operatorname{ann}(\operatorname{ann} U)$ have the same dimension and so coincide.

[2] This is where we use that V is finite-dimensional.