M216: Exercise sheet 8

Warmup questions

1. Let $\alpha_1, \dots, \alpha_k$ span $E \leq V^*$. Show that

$$\operatorname{sol} E = \bigcap_{i=1}^{k} \ker \alpha_i.$$

2. Define $\alpha, \beta \in (\mathbb{R}^3)^*$ be given by

$$\begin{aligned}\alpha(x) &= 2x_1 + x_2 - x_3 \\ \beta(x) &= x_1 - x_2 + x_3,\end{aligned}$$

for $x \in \mathbb{R}^3$.
Let $E = \operatorname{span}\{\alpha, \beta\}$ and compute $\operatorname{sol} E$.

Homework

3. Let $A, B \in M_4(\mathbb{C})$ be given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Compute the Jordan normal forms of A and B.
Are A and B similar?

4. Let $U \leq V$ and $v \in V$ with $v \notin U$. Show that there is $\alpha \in V^*$ such that α is zero on U but $\alpha(v) \neq 0$.
Hint: Apply theorem 5.3 to V/U.

Extra questions

5. Let V be a vector space over a field $\mathbb{F}$ and let $\alpha, \beta \in V^*$ be non-zero linear functionals. Prove that $\ker \alpha = \ker \beta$ if and only there is non-zero $\lambda \in \mathbb{F}$ such that $\alpha = \lambda\beta$.
Hint: If $v_0 \notin \ker \alpha$, show that $V = \operatorname{span}\{v_0\} + \ker \alpha$.

6. Let V be a vector space over $\mathbb{F}$. For $v \in V$, define $\operatorname{ev}(v) : V^* \to \mathbb{F}$ by

$$\operatorname{ev}(v)(\alpha) = \alpha(v).$$

(a) Show that $\operatorname{ev}(v)$ is linear so that $\operatorname{ev}(v) \in V^{**}$.
(b) We therefore have a map $\operatorname{ev} : V \to V^{**}$. Show that ev is linear.
(c) Show that ev is injective.
(d) Deduce that if V is finite-dimensional then $\operatorname{ev} : V \to V^{**}$ is an isomorphism.

Please hand in at 4W level 1 by NOON on Friday December 1st

M216: Exercise sheet 8—Solutions

1. Let $v \in \operatorname{sol} E$ so that $\alpha(v) = 0$, for all $\alpha \in E$. Then, in particular, each $\alpha_i(v) = 0$ so that $v \in \ker \alpha_i$, for $1 \leq i \leq k$. That is, $v \in \bigcap_{i=1}^k \ker \alpha_i$ and $\operatorname{sol} E \leq \bigcap_{i=1}^k \ker \alpha_i$.
   Conversely, let $v \in \bigcap_{i=1}^k \ker \alpha_i$ so that $\alpha_i(v) = 0$, for $1 \leq i \leq k$. Let $\alpha \in E$. Then $\alpha = \sum_{i=1}^k \lambda_i \alpha_i$, for some $\lambda_1, \dots, \lambda_k \in \mathbb{F}$, since the α_i span E, and
   $$\alpha(v) = \sum_{i=1}^k \lambda_i \alpha_i(v) = 0$$
   so that $v \in \operatorname{sol} E$. Thus $\bigcap_{i=1}^k \ker \alpha_i \leq \operatorname{sol} E$ and we are done.
2. According to question 1, $\operatorname{sol} E$ consists of those $x \in \mathbb{R}^3$ such that $\alpha(x) = \beta(x) = 0$, that is, such that
   $$\begin{aligned} 2x_1 + x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0. \end{aligned}$$
   Adding these gives $3x_1 = 0$ and then the first gives $x_2 = x_3$ so that $\operatorname{sol} E = \operatorname{span}\{(0, 1, 1)\}$.
3. Both being upper triangular, we see that $\Delta_A = \Delta_B = x^4$ so that the only eigenvalue of A or B is 0. Moreover, we compute to see that $A^2 = B^2 = 0$ so that $m_A = x^2$. Thus both A and B have at least one 2×2 Jordan block J_2. Thus the possibilities for the Jordan normal form of either are $J_2 \oplus J_2$ or $J_2 \oplus J_1 \oplus J_1$. To distinguish these, recall that the number of Jordan blocks with eigenvalue 0 is the dimension of the kernel. Now A has clearly has row rank 1 and so 3-dimensional kernel. Thus A has Jordan normal form $J_2 \oplus J_1 \oplus J_1$.
   Meanwhile B has row rank 2, thus nullity 2 so that it has JNF $J_2 \oplus J_2$.
   Since they have different JNF, A and B are not similar.
4. Let $q : V \to V/U$ be the quotient map so that q is a linear surjection with kernel U (this is all we need to know about the quotient construction). Since $v \notin U$, $q(v) \neq 0$ so that, by the Sufficiency Principle (Theorem 5.3), there is $\beta \in (V/U)^*$ such that $\beta(q(v)) \neq 0$.
   Let $\alpha = \beta \circ q : V \to \mathbb{F}$. This is linear, being a composition of linear maps, so $\alpha \in V^*$. Moreover, $\alpha(v) = \beta(q(v)) \neq 0$ while, if $u \in U$, $q(u) = 0$ so that $\alpha(u) = \beta(0) = 0$.
5. The reverse implication is clear: if $\lambda \neq 0$ and $\alpha = \lambda\beta$ then $\alpha(v) = 0$ if and only if $\lambda\alpha(v) = \beta(v) = 0$.
   Now suppose that $\ker \alpha = \ker \beta$ with $\alpha \neq 0$. Thus there is $v_0 \in V$ such that $\alpha(v_0) \neq 0$. Following the hint, let $v \in V$ and observe that $v - (\alpha(v)/\alpha(v_0))v_0 \in \ker \alpha$ so that $V = \operatorname{span}\{v_0\} + \ker \alpha$.
   Now, since $v_0 \notin \ker \alpha = \ker \beta$, $\beta(v_0) \neq 0$ also. Set $\lambda = \alpha(v_0)/\beta(v_0)$ so that
   $$\alpha(v_0) = \lambda\beta(v_0).$$
   Further $\alpha(v) = \lambda\beta(v)$, for all $v \in \ker \alpha$, since both sides are zero. It follows that $\alpha = \lambda\beta$ on $\operatorname{span}\{v_0\} + \ker \alpha = V$.
6. This is a case of thinking carefully what each statement means after which it will be very easy to prove.
   (a) To see that $\operatorname{ev}(v) : V^* \to \mathbb{F}$ is linear, we must show that
   $$\operatorname{ev}(v)(\alpha + \lambda\beta) = \operatorname{ev}(v)(\alpha) + \lambda \operatorname{ev}(v)(\beta),$$
   for all $\alpha, \beta \in V^*$ and $\lambda \in \mathbb{F}$. Using the definition of $\operatorname{ev}(v)$, this reads
   $$(\alpha + \lambda\beta)(v) = \alpha(v) + \lambda\beta(v)$$
   which is exactly the definition of the (pointwise) addition and scalar multiplication in V^*.

(b) Linearity of $\mathrm{ev} : V \to V^{**}$ means that for $v, w \in V$ and $\lambda \in \mathbb{F}$, we have

$$\mathrm{ev}(v + \lambda w) = \mathrm{ev}(v) + \lambda\, \mathrm{ev}(w).$$

This is supposed to be equality of elements of V^{**}, that is to say, equality of two functions on V^*. This holds when the two functions give the same answers on any $\alpha \in V^*$ so we need

$$\mathrm{ev}(v + \lambda w)(\alpha) = \mathrm{ev}(v)(\alpha) + \lambda\, \mathrm{ev}(w)(\alpha).$$

However, using the definition of ev, this reads

$$\alpha(v + \lambda w) = \alpha(v) + \lambda\alpha(w)$$

which is true since α is linear!

(c) ev is injective if and only if $\ker \mathrm{ev} = \{0\}$. Let $v \in \ker \mathrm{ev}$. Thus $\mathrm{ev}(v) = 0 \in V^{**}$, the zero functional on V^*. Otherwise said, $\mathrm{ev}(v)(\alpha) = 0$, for all $\alpha \in V^*$, or equivalently, $\alpha(v) = 0$, for all $\alpha \in V^*$. But the Sufficiency Principle now forces $v = 0$ so that ev injects.

(d) If v is finite-dimensional, $\dim V = \dim V^* = \dim V^{**}$ so that ev is an isomorphism by rank-nullity since we have just seen that it injects.