M216: Exercise sheet 7

Warmup questions

1. Let $v_1, \dots, v_n$ be a basis for a vector space and $\phi \in L(V)$. Show that the following are equivalent:
   (1) $\phi(v_1) = 0$ and $\phi(v_i) = v_{i-1}$, for $2 \leq i \leq n$.
   (2) $v_i = \phi^{n-i}(v_n)$ and $\phi^n(v_n) = 0$.

2. Let $\phi \in L(V)$ be a nilpotent linear operator on a finite-dimensional vector space V with Jordan normal form $J_{n_1} \oplus \cdots \oplus J_{n_k}$.
   Show that

$$\#\{i \mid n_i = s\} = 2 \dim \ker \phi^s - \dim \ker \phi^{s-1} - \dim \ker \phi^{s+1}.$$

Homework

3. Let $\phi \in L(V)$ be a linear operator on a finite-dimensional complex vector space V with distinct eigenvalues $\lambda_1, \dots, \lambda_k$.
   Show that ϕ is diagonalisable if and only if $m_\phi = \prod_{i=1}^{k}(x - \lambda_i)$.

4. Let $\phi = \phi_A \in L(\mathbb{C}^3)$ where

$$\begin{pmatrix} 0 & 1 & -1 \\ -10 & -2 & 5 \\ -6 & 2 & 1 \end{pmatrix}.$$

   Find the JNF and a Jordan basis for ϕ.
   (You have studied ϕ before in question 3 of sheet 6.)

Extra questions

5. Let $\phi \in L(V)$ be a nilpotent linear operator on a finite-dimensional vector space V with Jordan normal form $J_{n_1} \oplus \cdots \oplus J_{n_k}$.
   Show that $m_\phi = x^s$ where $s = \max\{n_1, \dots, n_k\}$.
   Hint: use question 6 on sheet 5.

6. Let $\phi = \phi_A \in L(\mathbb{C}^3)$ where A is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

   Find the JNF and a Jordan basis for ϕ.
   (You have studied ϕ before in question 4 of sheet 6.)

Please hand in at 4W level 1 by NOON on Friday 24th November

M216: Exercise sheet 7—Solutions

1. Assume (1). Then $v_{n-1} = \phi(v_n)$ and induction gives $v_i = \phi^{n-i}(v_n)$. In particular, $v_1 = \phi^{n-1}(v_n)$ so that $\phi(v_1)$ gives $\phi^n(v_n) = 0$. This establishes (2).
   Assume (2). Then $0 = \phi^n(v_n) = \phi(\phi^{n-1}(v_n)) = \phi(v_1)$. Moreover $\phi(v_i) = \phi(\phi^{n-i}(v_n)) = \phi^{n-(i-1)}(v_n) = v_{i-1}$, for $2 \leq i \leq n$. Thus we have (1).

2. From lectures, we know that, for $s \geq 1$,
$$\#\{i \mid n_i \geq s\} = \dim\ker\phi^s - \dim\ker\phi^{s-1}.$$
   Now
$$\begin{aligned}\#\{i \mid n_i = s\} &= \#\{i \mid n_i \geq s\} - \#\{i \mid n_i \geq s+1\}\\ &= \dim\ker\phi^s - \dim\ker\phi^{s-1} - (\dim\ker\phi^{s+1} - \dim\ker\phi^s)\\ &= 2\dim\ker\phi^s - \dim\ker\phi^{s-1} - \dim\ker\phi^{s+1}.\end{aligned}$$

3. Let $p = \prod_{i=1}^k (x - \lambda_i) \in \mathbb{C}[x]$.
   If ϕ is diagonalisable then $p(\phi) = 0$ since $\phi - \lambda_i \operatorname{id}_V = 0$ on $E_\phi(\lambda_i)$ and V is the direct sum of these eigenspaces.
   Conversely, if $m_\phi = \prod_{i=1}^k (x - \lambda_i)$, a result from lectures tells us that all Jordan blocks in the Jordan normal form of ϕ have size 1. Otherwise said, the JNF is diagonal and the Jordan basis is an eigenbasis. So ϕ is diagonalisable.

4. From question 3 of sheet 6, we know that $m_\phi = (x-3)(x+2)^2$ so that the JNF of ϕ must be $J(3,1) \oplus J(-2,2)$. A Jordan basis of $G_\phi(3) = E_\phi(3)$ is an arbitrary basis and one is given by $(0,1,1)$ as we found out in question 3 of sheet 6.
   For $G_\phi(-2)$, we want $(\phi + 2\operatorname{id}_{\mathbb{C}^3})(v), v$ with $(\phi + 2\operatorname{id}_{\mathbb{C}^3})^2(v) = 0$ so work backwards from an eigenvector w with eigenvalue -2 and then solve $(A + 2I_3)\mathbf{v} = \mathbf{w}$ to get v. We know from sheet 6 that we can take $w = (1,0,2)$ and then
$$(A + 2I_3)\mathbf{v} = \begin{pmatrix} 2 & 1 & -1 \\ -10 & 0 & 5 \\ -6 & 2 & 3 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$
   is clearly solved by $(0,1,0)$. Our Jordan basis is therefore $(0,1,1), (1,0,2), (0,1,0)$.

5. We follow the hint which tells us that when $A = A_1 \oplus \cdots \oplus A_k$, m_A is the smallest monic polynomial divided by each m_{A_i}.
   In the case at hand, with $A = J_{n_1} \oplus \cdots \oplus J_{n_k}$, we know from sheet 6 that $m_{J_{n_i}} = x^{n_i}$. Thus m_A is the monic polynomial of smallest degree divided by each x^{n_i} which is x^s, for $s = \max\{n_1, \dots, n_k\}$.

6. From sheet 6, we have that $m_\phi = x^2(x-5)$ so that the JNF is $J(0,2) \oplus J(5,1)$. A Jordan basis for $G_\phi(5)$ is any non-zero eigenvector with eigenvalue 5. We know from sheet 6 that $(0,0,1)$ is such a vector.
   For $G_\phi(0)$, a Jordan basis is $\phi(v), v$ with $\phi^2(v) = 0$. As usual, work backwards from $w \in \ker\phi$: take $w = (0,1,0)$ and solve
$$A\mathbf{v} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
   to get $v = (1/4, 0, 0)$, for example.
   Thus, a Jordan basis is given by $(0,1,0), (1/4,0,0), (0,0,1)$.