M216: Exercise sheet 5

Warmup questions

1. Write down matrices $A \in M_n(\mathbb{R})$ of the following forms:
   (a) $A_1 \oplus A_2 \oplus A_3$ with each $A_i \in M_2(\mathbb{R})$.
   (b) $A_1 \oplus \cdots \oplus A_5$ with each $A_i \in M_1(\mathbb{R})$.
   (c) $A \in M_3(\mathbb{R})$ such that A is not of the form $A_1 \oplus \cdots \oplus A_k$ with $k > 1$.

2. Let $V_1, \dots, V_k \leq V$ and $\phi_i \in L(V_i)$, $1 \leq i \leq k$. Suppose that $V = V_1 \oplus \cdots \oplus V_k$ and set $\phi = \phi_1 \oplus \cdots \oplus \phi_k$.
   (a) If $U_i \leq V_i$, $1 \leq i \leq k$, show that the sum $U_1 + \cdots + U_k$ is direct.
   (b) Prove that $\operatorname{im} \phi = \operatorname{im} \phi_1 \oplus \cdots \oplus \operatorname{im} \phi_k$.

Homework

3. Let $\phi \in L(V)$ be a linear operator on a vector space V.
   Prove that $\operatorname{im} \phi^k \geq \operatorname{im} \phi^{k+1}$, for all $k \in \mathbb{N}$. Moreover, if $\operatorname{im} \phi^k = \operatorname{im} \phi^{k+1}$ then $\operatorname{im} \phi^k = \operatorname{im} \phi^{k+n}$, for all $n \in \mathbb{N}$.

4. Compute the characteristic and minimum polynomials of

$$A = \begin{pmatrix} 1 & -5 & -7 \\ 1 & 4 & 2 \\ 0 & 1 & 4 \end{pmatrix}.$$

Additional questions

5. Let $\phi \in L(V)$ be a linear operator on a vector space V and $v \in V$, $k \in \mathbb{N}$ such that $\phi^{k+1}(v) = 0$ but $\phi^k(v) \neq 0$.
   Show that $v, \phi(v), \dots, \phi^k(v)$ are linearly independent.
   Hint: Induct on k.

6. In the situation of Question 2, prove:
   (a) m_{ϕ_i} divides m_ϕ, for each $1 \leq i \leq k$.
   (b) If each m_{ϕ_i} divides $p \in \mathbb{F}[x]$, then $p(\phi) = 0$.
   Thus m_ϕ is the monic polynomial of smallest degree divided by each m_{ϕ_i}. Otherwise said, m_ϕ is the *least common multiple* of $m_{\phi_1}, \dots, m_{\phi_k}$.

Please hand in at 4W level 1 by NOON on Friday 10th November

M216: Exercise sheet 5—Solutions

1. There are a gazillion possibilities.

   (*a*)
$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 \\ 0 & 0 & 7 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \oplus \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \oplus \begin{pmatrix} 9 & 0 \\ 1 & 2 \end{pmatrix}.$$

   (*b*) Any 5×5 diagonal matrix will do:
$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix} = (\lambda_1) \oplus \dots \oplus (\lambda_5).$$

   (*c*) Any block matrix with more than one block will have zeros so
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$
   cannot be written $A_1 \oplus \dots \oplus A_k$ with $k > 1$.

2. (*a*) Let $u \in U_1 + \dots + U_k$ so we can write $u = u_1 + \dots + u_k$, with $u_i \in U_i \leq V_i$. However, since the sum of the V_i is direct, there is only one way to write u as a sum of elements of the V_i and so, in particular, as a sum of elements of the U_i. Thus the sum of the U_i is direct.

   (*b*) Let $v \in \operatorname{im}\phi$ so that $v = \phi(w)$, for some $w \in V$. Then, writing $w = w_1 + \dots + w_k$ with each $w_i \in V_i$, we have
$$v = \phi(w) = \phi_1(w_1) + \dots + \phi_k(w_k) \in \operatorname{im}\phi_1 \oplus \dots \oplus \operatorname{im}\phi_k.$$
   Thus $\operatorname{im}\phi \leq \operatorname{im}\phi_1 \oplus \dots \oplus \operatorname{im}\phi_k$.
   For the converse, let $v \in \operatorname{im}\phi_1 \oplus \dots \oplus \operatorname{im}\phi_k$ so that $v = \phi_1(w_1) + \dots + \phi_k(w_k)$ with $w_i \in V_i$, $1 \leq i \leq k$. Since each $\phi_i = \phi_{|V_i}$, this reads
$$v = \phi(w_1) + \dots + \phi(w_k) = \phi(w_1 + \dots + w_k) \in \operatorname{im}\phi.$$
   Thus $\operatorname{im}\phi_1 \oplus \dots \oplus \operatorname{im}\phi_k \leq \operatorname{im}\phi$ and we are done.

3. Let $v \in \operatorname{im}\phi^{k+1}$ so that $v = \phi^{k+1}(w)$, for some $w \in V$. Then $v = \phi^k(\phi(w)) \in \operatorname{im}\phi^k$. Thus $\operatorname{im}\phi^k \geq \operatorname{im}\phi^{k+1}$.
   Suppose now that $\operatorname{im}\phi^k = \operatorname{im}\phi^{k+1}$. We prove that $\operatorname{im}\phi^k = \operatorname{im}\phi^{k+n}$ by induction on n. We are given that this holds for $n = 1$ so we suppose this holds for some n ($\operatorname{im}\phi^k = \operatorname{im}\phi^{k+n}$) and prove it then holds for $n+1$. Thus, let $v \in \operatorname{im}\phi^k = \operatorname{im}\phi^{k+1}$ so that $v = \phi(\phi^k(w))$, for some $w \in V$. Then $\phi^k(w) \in \operatorname{im}\phi^k = \operatorname{im}\phi^{k+n}$, by the induction hypothesis, so that $\phi^k(w) = \phi^{k+n}(u)$, some $u \in V$, whence $v = \phi(\phi^{k+n}(u)) = \phi^{k+n+1}(u) \in \operatorname{im}\phi^{k+n+1}$. We conclude that $\operatorname{im}\phi^k \leq \operatorname{im}\phi^{k+n+1}$. The converse inclusion always holds so we have equality. Induction now bakes the cake.

4. We compute the characteristic polynomial of A to be
$$\Delta_A = -x^3 + 9x^2 - 27x + 27 = -(x-3)^3.$$

We learn from the Cayley–Hamilton theorem that $m_A = (x-3)^k$, for some k with $k \leq 1 \leq 3$. Clearly $k = 1$ is out, since A is not diagonal, so we try $k = 2$:

$$(A - 3I)^2 = \begin{pmatrix} -2 & -5 & -7 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & -5 & -7 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -3 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{pmatrix},$$

which is non-zero. This means we must have $m_A = (x-3)^3$.

5. We follow the hint and induct on k. For $k = 0$, the assertion is that if $v \neq 0$ and $\phi(v) = 0$ then the one element list v is linearly independent which is certainly true (whether or not $\phi(v) = 0$).
   Suppose now that the result holds for $\ell < k$ so that $\phi^{\ell+1}(w) = 0$ and $\phi^{\ell}(w) \neq 0$ forces $w, \dots, \phi^{\ell}(w)$ to be linearly independent. Now suppose that $\phi^{k+1}(v) = 0$ and $\phi^k(v) \neq 0$. If there are $\lambda_i \in \mathbb{F}$ with
   $$\lambda_1 v + \cdots + \lambda_k \phi^k(v) = 0, \tag{1}$$
   then taking ϕ of this gives
   $$\lambda_1 \phi(v) + \cdots + \lambda_{k-1} \phi^{k-1}(\phi(v)) = 0.$$
   But with $w = \phi(v)$, we have $\phi^k(w) = 0$ and $\phi^{k-1}(w) \neq 0$ so the induction hypothesis applies with $\ell = k-1 < k$ to give that $w, \dots, \phi^{k-1}(w)$ are linearly independent. Therefore $\lambda_1, \dots, \lambda_{k-1}$ all vanish. Plugging this back into (1) gives $\lambda_k \phi^k(v) = 0$ whence $\lambda_k = 0$ also. We conclude that $v, \dots, \phi^k(v)$ are linearly independent and so the result holds for all $k \in \mathbb{N}$ by induction.

6. (a) We have that $m_\phi(\phi) = 0$ so that $0 = m_\phi(\phi)_{|V_i} = m_\phi(\phi_i)$. It follows that m_{ϕ_i} divides m_ϕ.
   (b) Since m_{ϕ_i} divides p, $p(\phi_i) = 0$ for each i. But then
   $$p(\phi) = p(\phi_1) \oplus \cdots \oplus p(\phi_k) = 0$$
   so that m_ϕ divides p.