M216: Exercise sheet 4

Warmup questions

1. Let $p, q \in \mathbb{R}[x]$ be given by $p = x^2 - 2x - 3$, $q = x^3 - 2x^2 + 2x - 5$.
   Let $A \in M_2(\mathbb{R})$ and $B \in M_3(\mathbb{R})$ be given by

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

   Compute $p(A), p(B), q(A), q(B)$.

2. Compute the characteristic polynomials of A and B, from question 1.
   What do you notice?

3. Let $\mathbb{F} = \mathbb{Z}_2$, the field of two elements and let $p = x^2 + x \in \mathbb{F}[x]$.
   Show that $p(t) = 0$, for all $t \in \mathbb{F}$.

Homework questions

4. Compute the minimum polynomial of $A \in M_5(\mathbb{R})$ given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

5. Let $\phi \in L(V)$ be an operator on a finite-dimensional vector space over $\mathbb{F}$ and let $p = m_\phi \in \mathbb{F}[x]$.
   Let λ be a root of p.
   (a) Show there is $q \in \mathbb{F}[x]$ with $\deg q < \deg p$ such that

$$p = (x - \lambda)q.$$

   (b) Prove that $q(\phi)$ is non-zero.
   (c) Deduce that λ is an eigenvalue of ϕ.
   This shows that the roots of p are exactly the eigenvalues of ϕ without recourse to the Cayley–Hamilton theorem.
   (d) Deduce that ϕ is invertible if and only if p has non-zero constant term.

Extra questions

6. Let $\phi \in L(V)$ have minimal polynomial $p = 4 + 5x + 6x^2 - 7x^3 - 8x^4 + x^5$, so that ϕ is invertible by question 5(d).
   Compute the minimal polynomial of ϕ^{-1}.
   Hint: Think about multiplying $a_0 \operatorname{id}_V + \cdots + \phi^n$ by ϕ^{-n}.

Please hand in at 4W level 1 by NOON on Friday 3rd November

M216: Exercise sheet 4—Solutions

1. We just compute:

$$A^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}, \qquad A^3 = \begin{pmatrix} 13 & 14 \\ 14 & 13 \end{pmatrix}$$

so that

$$p(A) = A^2 - 2A - 3I_2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$q(A) = A^3 - 2A^2 + 2A - 5I_3 = \begin{pmatrix} 13 & 14 \\ 14 & 13 \end{pmatrix} - 2\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} + 2\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 10 \\ 10 & 0 \end{pmatrix}.$$

Similarly,

$$p(B) = \begin{pmatrix} -6 & -1 & 2 \\ 4 & -6 & -3 \\ -2 & 3 & -1 \end{pmatrix},$$

$$q(B) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. Again, we just compute:

$$\Delta_A = \begin{vmatrix} 1-x & 2 \\ 2 & 1-x \end{vmatrix} = (1-x)^2 - 4 = x^2 - 2x - 3.$$

Similarly,

$$\begin{aligned} \Delta_B = \begin{vmatrix} 1-x & 2 & 1 \\ -2 & -x & 1 \\ 2 & 1 & 1-x \end{vmatrix} &= (1-x)\big(x(x-1)-1\big) - 2\big(2(x-1)-2\big) + (-2+2x) \\ &= (-x^3 + 2x^2 - 1) - 4x + 8 + 2x - 2 = -x^3 + 2x^2 - 2x + 5. \end{aligned}$$

We notice that, with p, q as in question 1, $p = \Delta_A$ and $q = -\Delta_B$ and so, again from question 1,

$$\Delta_A(A) = \Delta_B(B) = 0.$$

As we shall soon see, this is the Cayley–Hamilton theorem in action.

3. We recall that $\mathbb{Z}_2 = \{\mathbf{0}, \mathbf{1}\}$ with addition and multiplication given by

$$\mathbf{0} = \mathbf{0} + \mathbf{0} = \mathbf{1} + \mathbf{1} \qquad\qquad \mathbf{1} = \mathbf{0} + \mathbf{1} = \mathbf{1} + \mathbf{0}$$

$$\mathbf{0} = \mathbf{00} = \mathbf{01} = \mathbf{10} \qquad\qquad \mathbf{1} = \mathbf{11}.$$

We immediately conclude that $\mathbf{1}^2 + \mathbf{1} = \mathbf{0} = \mathbf{0}^2 + \mathbf{0}$ so that $p(t) = \mathbf{1}$, for both $t \in \mathbb{F}$.

4. Let us compute the first few powers of A:

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 6 & -3 \\ 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad A^3 = \begin{pmatrix} 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 6 & -3 & 0 \\ 0 & 0 & 0 & 6 & -3 \\ 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad A^4 = \begin{pmatrix} 0 & -3 & 0 & 0 & 0 \\ 0 & 6 & -3 & 0 & 0 \\ 0 & 0 & 6 & -3 & 0 \\ 0 & 0 & 0 & 6 & -3 \\ 1 & 0 & 0 & 0 & 6 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} -3 & 0 & 0 & 0 & -18 \\ 6 & -3 & 0 & 0 & 36 \\ 0 & 6 & -3 & 0 & 0 \\ 0 & 0 & 6 & -3 & 0 \\ 0 & 0 & 0 & 6 & -3 \end{pmatrix}$$

Stare at the top row to see that there can be no monic polynomial $p = a_0 + \cdots + x^k$ with $k \leq 4$ with $p(A) = 0$: the -3 on the top row of the leading term would give $a_0 0 + \cdots + a_{k-1} 0 - 3 = 0$. On the other hand, we readily see that $A^5 - 6A + 3I_5 = 0$ so that $m_A = x^5 - 6x + 3$.

5. (*a*) The remainder theorem says we can write $p = (x - \lambda)q + r$ with $\deg r < \deg(x - \lambda) = 1$ so that r is degree zero and so constant. Evaluating at λ gives $0 = p(\lambda) = 0q + r = r$ and we are done.

   (*b*) $q(\phi)$ cannot be zero unless $q = 0$ since $\deg q < \deg p$ and p is the minimal polynomial of ϕ. But q cannot be zero since p is non-zero.

   (*c*) Since $q(\phi)$ is non-zero, there is $v \in V$ such that $q(\phi)v \neq 0$. Now

$$0 = p(\phi)(v) = (\phi - \lambda \operatorname{id}_V)(q(\phi)(v))$$

   so that $q(\phi)v$ is an eigenvector with eigenvalue λ.

   (*d*) ϕ is invertible if and only if ϕ is injective if and only if zero is not an eigenvalue if and only if (thanks to the previous part) zero is not a root of p if and only if p has non-zero constant term.

6. If $a_0 \operatorname{id}_V + a_1\phi + \cdots + \phi^n = 0$ then, multiplying by ϕ^{-n} gives $a_0\phi^{-n} + a_1\phi^{n-1} + \cdots + a_n \operatorname{id}_V = 0$. In the case at hand, this means that

$$4\phi^{-5} + 5\phi^{-4} + 6\phi^{-3} - 7\phi^{-2} - 8\phi^{-1} + \operatorname{id}_V = 0.$$

   If there was a non-zero polynomial $q = \sum_{k=1}^{4} b_k x^k$ of lower degree with $q(\phi^{-1}) = 0$ gives

$$b_4 \operatorname{id}_V + \cdots + b_0\phi^4 = 0,$$

   contradicting the minimality of p. Thus, dividing by 4 to get a monic polynomial, the minimum polynomial of ϕ^{-1} is $1/4 - 2x - 7/4x^2 + 3/2x^3 + 5/4x^4 + x^5$.
   More generally, the same argument says that if $\sum_{k=0}^{n} a_k x^k$ is the minimal polynomial of invertible ϕ with degree n then $1/a_0 \sum_{k=0}^{n} a_{n-k} x^k$ is the minimal polynomial of ϕ^{-1}.