M216: Exercise sheet 2

Warmup questions

1. Let $U, W \leq V$ be subspaces of a vector space V.
   When is $U \cup W$ also a subspace of V?

2. Let V, W be vector spaces, $v_1, \ldots, v_n$ a basis of V and $w_1, \ldots, w_n$ a list of vectors in W. Let $\phi : V \to W$ be the unique linear map with

$$\phi(v_i) = w_i,$$

   for all $1 \leq i \leq n$. Show:

   (a) ϕ injects if and only if $w_1, \ldots, w_n$ is linearly independent.

   (b) ϕ surjects if and only if $w_1, \ldots, w_n$ spans W.

   Deduce that ϕ is an isomorphism if and only if $w_1, \ldots, w_n$ is a basis for W.

Homework

3. Let V be a vector space. A linear map $\pi : V \to V$ is called a *projection* if $\pi \circ \pi = \pi$.
   In this case, prove that $\ker \pi \cap \operatorname{im} \pi = \{0\}$ and deduce that $V = \ker \pi \oplus \operatorname{im} \pi$.

4. Let $U_1, U_2, U_3 \leq \mathbb{R}^3$ be the 1-dimensional subspaces spanned by $(1, 2, 0)$, $(1, 1, 1)$ and $(2, 3, 1)$ respectively.
   Which of the following sums are direct?
   (a) $U_i + U_j$, for $1 \leq i < j \leq 3$.
   (b) $U_1 + U_2 + U_3$.

Additional questions

5. Let $V_1, V_2, V_3 \leq V$. Which of the following statements are true? (In each case, give a proof or a counter-example.)

   (a) $V_1 + (V_2 \cap V_3) = (V_1 + V_2) \cap (V_1 + V_3)$.

   (b) $V_1 \cap (V_2 + V_3) = (V_1 \cap V_2) + (V_1 \cap V_3)$.

   (c) $(V_1 \cap V_2) + (V_1 \cap V_3) \subseteq V_1 \cap (V_2 + V_3)$.

6. Let $V_1, V_1', V_2 \leq V$ and suppose that $V = V_1 \oplus V_2$ and $V = V_1' \oplus V_2$.

   (a) Must $V_1 = V_1'$?

   (b) Are V_1 and V_1' isomorphic?

Please hand in at 4W level 1 by NOON on Friday 20th October

M216: Exercise sheet 2—Solutions

1. If $U \subseteq W$ then $U \cup W = W$ is a subspace and similarly if $W \subseteq U$. In any other case, $U \cup W$ is not a subspace: we can find $u \in U \setminus W$ and $w \in W \setminus U$ and then $u + w \notin U$ (else $w = (u + w) - u \in U$) and similarly $u + w \notin W$. Thus $U \cup W$ is not closed under addition.

2. (a) $\lambda_1 w_1 + \cdots + \lambda_n w_n = 0$ if and only if $\lambda_1 v_1 + \cdots + \lambda_n v_n \in \ker \phi$. Thus $w_1, \ldots, w_n$ is linearly independent if and only if ϕ has trivial kernel.

   (b) ϕ surjects if and only if any $w \in W$ can be written $w = \phi(v)$, or equivalently,

   $$w = \phi(\lambda_1 v_1 + \cdots + \lambda_n v_n) = \lambda_1 w_1 + \cdots + \lambda_n w_n,$$

   for some λ_i, $1 \leq i \leq n$.

3. Let $v \in \ker \pi \cap \operatorname{im} \pi$. Then there is $w \in V$ such that $v = \pi(w)$ since $v \in \operatorname{im} \pi$. But $v \in \ker \pi$ also so that

   $$0 = \pi(v) = \pi(\pi(w)) = \pi(w) = v.$$

   Thus $\ker \pi \cap \operatorname{im} \pi = \{0\}$ so it remains to show that $V = \ker \pi + \operatorname{im} \pi$. For this, write $v = (v - \pi(v)) + \pi(v)$. The second summand is certainly in $\operatorname{im} \pi$ while

   $$\pi(v - \pi(v)) = \pi(v) - \pi(\pi(v)) = \pi(v) - \pi(v) = 0$$

   so the first is in $\ker \pi$ and we are done.

4. (a) All these sums are direct as each $U_i \cap U_j = \{0\}$.

   (b) Note that $(2, 3, 1) = (1, 2, 0) + (1, 1, 1)$ and so can be written in two different ways as a sum $u_1 + u_2 + u_3$, with each $u_i \in U_i$:

   $$\begin{gathered}(1, 2, 0) + (1, 1, 1) + (0, 0, 0)\\ (0, 0, 0) + (0, 0, 0) + (2, 3, 1).\end{gathered}$$

   Thus $U_1 + U_2 + U_3$ is not a direct sum.

   This shows us that $U_i \cap U_j = \{0\}$, $i \neq j$, is not enough to force $U_1 + U_2 + U_3$ to be direct.

5. (a) This is false: take $V_1, V_2, V_3 \leq \mathbb{R}^2$ to be the subspaces spanned at $(1, 0)$, $(0, 1)$ and $(1, 1)$ respectively. Then any $V_i + V_j = \mathbb{R}^2$ and $V_i \cap V_j = \{0\}$, for $i \neq j$. Now the left side is $V_1 + \{0\} = V_1$ while the right is $\mathbb{R}^2 \cap \mathbb{R}^2 = \mathbb{R}^2$.

   (b) This is also false. With the same V_i as in part (a), the left side is $V_1 \cap \mathbb{R}^2 = V_1$ while the right is $\{0\} + \{0\} = \{0\}$.

   (c) This is true: $V_2, V_3 \leq V_2 + V_3$ so that $V_1 \cap V_2, V_1 \cap V_3 \leq V_1 \cap (V_2 + V_3)$. It now follows from Proposition 2.1 that $(V_1 \cap V_2) + (V_1 \cap V_3) \subseteq V_1 \cap (V_2 + V_3)$.

6. (a) No: a given V_2 has many complements. For example, take $V = \mathbb{R}^2$, V_2 to be spanned by $(1, 0)$ and then V_1, V_1' to be spanned by $(0, 1)$ and $(1, 1)$ respectively.

   (b) This is true. For example, consider the projection π_1 with image V_1 and kernel V_2 and restrict this to V_1' to get a linear map $V_1' \to V_1$. Then $\ker(\pi_{1|V_1'}) = \ker \pi_1 \cap V_1' = V_2 \cap V_1' = \{0\}$ so that $\pi_{1|V_1'}$ injects. Moreover, for $v_1 \in V_1$, write $v_1 = v_1' + v_2$ with $v_1' \in V_1'$ and $v_2 \in V_2$. Then $v_1 = \pi(v_1) = \pi_1(v_1' + v_2) = \pi_1(v_1')$ so that $\pi_{1|V_1'} : V_1' \to V_1$ surjects also and so is an isomorphism.