M216: Exercise sheet 1

Warmup questions

1. Let U be a subset of a vector space V. Show that U is a linear subspace of V if and only if U satisfies the following conditions:
   (i) $0 \in U$;
   (ii) For all $u_1, u_2 \in U$ and $\lambda \in \mathbb{F}$, $u_1 + \lambda u_2 \in U$.

2. Which of the following subsets of $\mathbb{R}^3$ are linear subspaces? In each case, briefly justify your answer.
   (a) $U_1 := \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ (b) $U_2 := \{(x_1, x_2, x_3) \mid x_1 = x_2\}$ (c) $U_3 := \{(x_1, x_2, x_3) \mid x_1 + 2x_2 + 3x_3 = 0\}$

3. Which of the following maps $f : \mathbb{R}^2 \to \mathbb{R}^2$ are linear? In each case, briefly justify your answer.
   (a) $f(x, y) = (5x + y, 3x - 2y)$ (b) $f(x, y) = (5x + 2, 7y)$ (c) $f(x, y) = (\cos y, \sin x)$ (d) $f(x, y) = (3y^2, x^3)$.

Homework

4. Let $\mathcal{I}$ be a set and V a vector space over a field $\mathbb{F}$. Recall that $V^{\mathcal{I}}$ is the set of maps $\mathcal{I} \to V$.
   Show that $V^{\mathcal{I}}$ is a vector space under pointwise addition and scalar multiplication.

5. Let $\mathbb{R}[x]$ be the space of real polynomials. This is a vector space under coefficient-wise addition and scalar multiplication.
   For $d \in \mathbb{N}$, let $P_d \subset \mathbb{R}[x]$ be the set of polynomials of degree no more than d. Show that $P_d \leq \mathbb{R}[x]$ and has basis $1, x, \dots, x^d$
   Define a linear map $D : P_d \to P_d$ by $D(p) = p'$. Compute its matrix with respect to $1, x, \dots, x^d$. What are $\ker D$ and $\operatorname{im} D$?

Additional questions

6. Which of the following subsets of $\mathbb{C}^3$ are linear subspaces over $\mathbb{C}$? In each case, briefly justify your answer.
   (a) $U_1 := \{(z_1, z_2, z_3) \mid z_1 z_2 = 1\}$ (b) $U_2 := \{(z_1, z_2, z_3) \mid z_1 = \bar{z}_2\}$ (c) $U_3 := \{(z_1, z_2, z_3) \mid z_1 + \sqrt{-1} z_2 + 3z_3 = 0\}$

7. Let V be an n-dimensional vector space over $\mathbb{C}$, and let $V_{\mathbb{R}}$ be the underlying vector space over $\mathbb{R}$ (thus $V_{\mathbb{R}}$ has the same set of vectors as V, but scalar multiplication is restricted to real scalars). Prove that $V_{\mathbb{R}}$ has dimension $2n$.
   [**Hint**: let $\mathcal{B} : v_1, v_2, \dots, v_n$ be a basis for V and show that $\mathcal{B}_{\mathbb{R}} : v_1, iv_1, v_2, iv_2, \dots, v_n, iv_n$ is a basis for $V_{\mathbb{R}}$, where $i \in \mathbb{C}$ is $\sqrt{-1}$ rather than an index!]

Please hand in at 4W level 1 by NOON on Friday 13th October 2023

M216: Exercise sheet 1—Solutions

1. First suppose that $U \leq V$. The U is non-empty so there is some $u \in U$ and then, since U is closed under addition and scalar multiplication, $0 = u + (-1)u \in U$ also and condition (i) is satisfied. Now if $u_1, u_2 \in U$ and $\lambda \in \mathbb{F}$, then $\lambda u_2 \in U$ (U is closed under scalar multiplication) and so $u_1 + \lambda u_2 \in U$ (U is closed under addition). Thus condition (ii) holds also.
   For the converse, if conditions (i) and (ii) hold, then, first, $0 \in U$ so U is non-empty and, second, U is closed under addition (take $\lambda = 1$ in condition (ii)) and under scalar multiplication (take $u_1 = 0$ in condition (ii)). Thus $U \leq V$.

2. (*a*) U_1 is not a subspace as it does not contain 0!
   (*b*) U_2 is a subspace: in fact, it is $\ker \phi_A$ where $A = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$.
   (*c*) U_3 is a subspace. It is $\ker \phi_A$ for $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$.

3. (*a*) Here f is linear: it is the map ϕ_A corresponding to the matrix
$$A = \begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix}.$$
   (*b*) This is not linear (because of that +2 term). In particular $f(0, 0) = (2, 0) \neq 0$!
   (*c*) Again $f(0, 0) = (1, 0) \neq 0$ so this f cannot be linear. Of course, we already *know* this because it is certainly not true that $\cos(y_1 + y_2) = \cos y_1 + \cos y_2$.
   (*d*) Another non-linear map: for example $f(2x, 2y) \neq 2f(x, y)$.

4. The basic idea is that the vector space axioms for $V^{\mathcal{I}}$ will follow from those of V applied to the values of elements of $V^{\mathcal{I}}$. Since those elements are completely determined by their values, this will bake the cake.
   In more detail: let $u, v, w \in V^{\mathcal{I}}$, then, for $i \in \mathcal{I}$,
$$(u + v)(i) = u(i) + v(i) = v(i) + u(i) = (v + u)(i),$$
   whence $u + v = v + u$. Here the first and last equalities are just the definition of pointwise addition and the middle one of commutativity of addition in V.
   Similarly,
$$((u+v)+w)(i) = (u+v)(i)+w(i) = (u(i)+v(i))+w(i) = u(i)+(v(i)+w(i)) = (u+(v+w))(i)$$
   so that $(u + v) + w = u + (v + w)$.
   The zero element is the zero map defined by $0(i) := 0$, for all $i \in \mathcal{I}$, while the additive inverse $-v$ of $v \in V^{\mathcal{I}}$ is defined by $(-v)(i) := -(v(i))$. Now
$$\begin{aligned}(v + 0)(i) &= v(i) + 0(i) = v(i) + 0 = v(i)\\ (v + (-v))(i) &= v(i) + (-v)(i) = v(i) - v(i) = 0 = 0(i)\end{aligned}$$
   so that $v + 0 = v$ and $v + (-v) = 0$ as required.
   The axioms around scalar multiplication are verified in the same way. For example, for $\lambda, \mu \in \mathbb{F}$,
$$((\lambda + \mu)v)(i) = (\lambda + \mu)(v(i)) = \lambda(v(i)) + \mu(v(i)) = (\lambda v)(i) + (\mu v)(i) = (\lambda v + \mu v)(i)$$
   so that $(\lambda + \mu)v = \lambda v + \mu v$.
   Again, for $u, v \in V^{\mathcal{I}}$ and $\lambda \in \mathbb{F}$,
$$\begin{aligned}(\lambda(u + v))(i) = \lambda(u + v)(i) = \lambda(u(i) + v(i)) &= \lambda u(i) + \lambda v(i)\\ &= (\lambda u)(i) + (\lambda v)(i) = (\lambda u + \lambda v)(i)\end{aligned}$$
   so that $\lambda(u + v) = \lambda u + \lambda v$.

For $\lambda, \mu \in F$ and $v \in V^{\mathcal{I}}$,

$$((\lambda\mu)v)(i) = (\lambda\mu)v(i) = \lambda(\mu v(i)) = (\lambda(\mu v))(i)$$

so that $(\lambda\mu)v = \lambda(\mu v)$.
Finally, $(1v)(i) = 1v(i) = v(i)$ so that $1v = v$ and we are (at last!) done.

5. Clearly P_d is non-empty as it contains the zero polynomial. Moreover, for any polynomials p, q and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} \deg(p+q) &\leq \max\{\deg p, \deg q\} \\ \deg(\lambda p) &\leq \deg p, \end{aligned}$$

from which it easily follows that P_d is closed under addition and scalar multiplication.
Any polynomial $p \in P_d$ has a unique expression of the form

$$p = a_0 + a_1 x + \cdots + a_d x^d.$$

It now follows from Proposition 1.1 that $1, x, \dots, x^d$ is a basis for P_d.
Set $v_j = x^{j-1}$, for $1 \leq j \leq d+1$, and compute Dv_j in terms of the v_i:

$$Dv_j = (j-1)v_{j-1}$$

so that the matrix A of D with respect to this basis has all entries 0 except just above the diagonal where $A_{(j-1)j} = j - 1$. For example, if $d = 3$, we have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The kernel of D is the constant polynomials P_0 and the image is P_{d-1}.

6. (a) $0 \notin U_1$ so U_1 is not a subspace.
   (b) U_2 is not a subspace because it is not closed under complex scalar multiplication: $(1,1,0) \in U_2$ but $i(1,1,0) = (i,i,0)$ is not (here $i = \sqrt{-1}$). In general, any time you see complex conjugation in the definition of a subset, it is unlikely to be a complex subspace.
   (c) $U_3 = \ker \phi_A$ for $A = \begin{pmatrix} 1 & \sqrt{-1} & 3 \end{pmatrix}$ and so is a subspace.

7. Following the hint we need to show that any $v \in V_{\mathbb{R}}$ can be written uniquely as a real linear combination of vectors in the list $\mathcal{B}_{\mathbb{R}}$. Since $v \in V$, we may write $v = \sum_{j=1}^{n} \lambda_j v_j$ for unique $\lambda_j \in \mathbb{C}$. Write $\lambda_j = a_j + ib_j$ with $a_j, b_j \in \mathbb{R}$. Then $v = \sum_{j=1}^{n} (a_j v_j + b_j i v_j)$ and this expression is unique: it suffices to observe that for $v = 0$, $\lambda_j = 0$ for all j, and hence $a_j = b_j = 0$ for all j.