Turbulence and noise

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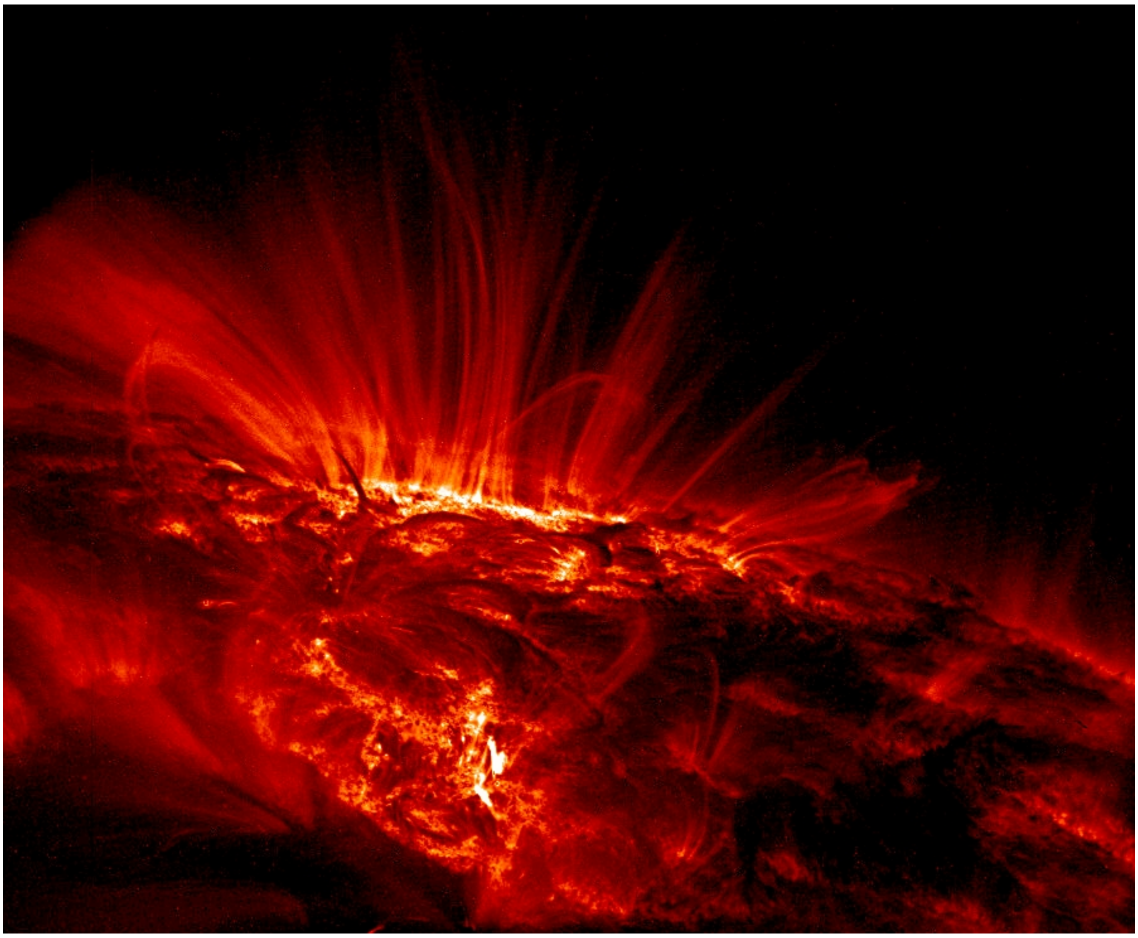


Being to treat of the Doctrine of Sounds*, I hold it convenient to premise something in the general concerning this Theory; which may serve at once to engage your attention, and excuse my pains, when I shall have recommended them, as bestow'd on a subject not altogether useless and unfruitful.*

Narcissus Marsh, 1683/4, *Phil. Trans. Roy. Soc. Lond.*, **156**:472–486.

A large fraction of the world's energy consumption is devoted to compensating for turbulent energy loss!

Robert Ecke, 2005, *Los Alamos Science*, **29**:124–141.

Does the wind possess a velocity? This question, at first sight foolish, improves on acquaintance.

Lewis F. Richardson, 1926, *Proc. Roy. Soc. Lond. A*, **110**:709–737.

Some things you should get

These notes are some, but not all, of the course. You should also get a number of papers and other documents. One of these (Tyler & Sofrin) is on the moodle page for the course. The rest are available from the library or online. You will probably not need to print them out in full, so wait until you need them before putting anything on paper.

- TYLER, J. M., & SOFRIN, T. G. 1962, Axial flow compressor noise studies, *Transactions of the Society of Automotive Engineers*, **70**:309–332.
- LILLEY, G. M. 1995, Jet noise classical theory and experiments, in *Aeroacoustics of flight vehicles*, Hubbard, H., ed., Acoustical Society of America (available from the NASA website).
- LIGHTHILL, M. J. 1952, On sound generated aerodynamically: I General theory, *Proceedings of the Royal Society* A, **211**:564–587.
- HUSSAIN, A. K. M. FAZLE 1986, Coherent structures and turbulence, *Journal of Fluid Mechanics*, **173**:309–356.

Contents

Chapter 1

The basics

Turbulence is one of the deepest mysteries in fluid mechanics, and probably the single biggest thing which makes it both especially difficult and especially interesting. One of the side effects of many flows is sound (if you like it) or noise (if you don't). The noise generated by flows typically carries only a tiny fraction of the energy of the flow proper, even in cases where the noise is sufficient to cause hearing damage. The mystery of turbulence has captivated serious thinkers for centuries and some of those people have made progress in understanding the nature of the problem and the physics which gives rise to the intriguing patterns of turbulent flow. Acoustics, on the other hand, is a study of very weak perturbations which propagate over large distances and interact in a way which depends on a very delicate balance of quantities.

A definition of turbulence which will help us to analyze the problem at something higher than a hand-waving level is given by George (2007):

> Turbulence is that state of fluid motion which is characterized by apparently random and chaotic three-dimensional vorticity.

Panton (2005, page 732) gives the definition:

> Turbulent flows contain self-sustaining velocity fluctuations in addition to the main flow.

The study of turbulence and acoustics will require us to analyze the physics using the mathematical theory of fluid flow (the Navier–Stokes equations) and statistical methods appropriate to random problems. Both acoustics and turbulence, although they are very different in many ways, are governed by the same basic equations of fluid motion.

The equations of fluid motion

The basic equations governing the motion of a fluid are usually known as the Navier–Stokes equations. They can be found in standard fluid dynamics texts (Panton, 2005; George, 2007):

$$\rho\left[\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right] + \frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} = 0; \tag{1.1a}$$

$$\left[\frac{\partial \rho}{\partial t} + u_j\frac{\partial \rho}{\partial x_j}\right] + \rho\frac{\partial u_j}{\partial x_j} = 0. \tag{1.1b}$$

These are equations of momentum and continuity, written in *tensor* notation. Density and pressure have symbols ρ and p. The vectors for position and velocity are denoted by x_i and u_i respectively where the index $i = 1, 2$ or 3, corresponding to the three Cartesian axes, so that (1.1a) is actually three equations, found by setting i to each of its values. Tensor notation is often a much more compact way of writing the equations of fluid motion, and you should be able to switch between it and vector notation.

The *Einstein summation convention* is used which says that where an index is repeated in a term, the term is read as a sum over the three values of the index. For example, the quantity $u_iv_i \equiv u_1v_1 + u_2v_2 +$

u_3v_3. Some more definitions which are useful in dealing with tensors are given in Appendix C. The term τ_{ij} is the *viscous stress tensor* and will turn out to be very important in the study of turbulence. In a Newtonian fluid,

$$\tau_{ij} = 2\mu\left[S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right], \tag{1.2}$$

where the viscosity μ is a property of the fluid and:

$$S_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right], \tag{1.3}$$

is the *strain rate tensor*. The *Kronecker delta* is defined:

$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases} \tag{1.4}$$

Many problems in fluid mechanics deal with incompressible flow (not the same thing as incompressible fluid) where the density ρ is constant. In this case:

$$\frac{\partial u_j}{\partial x_j} = 0 \text{ and } S_{kk} = 0, \tag{1.5}$$

which will allow us to study the behaviour of turbulence without the complications introduced by compressibility.

The Navier-Stokes equations are the most general form of the laws governing fluid motion and contain all of the behaviour which we can find in real problems. In practice, we will not try to solve (1.1) (there is a million dollar prize 'simply' for proving that there are reasonable solutions) but will develop appropriate approximations which will let us derive solutions for particular, though still useful, cases and thus find out something about the behaviour of real turbulent systems.

In order to do this, we must first non-dimensionalize the equations, to see which terms are important and which can be neglected. A detailed derivation for the full compressible Navier-Stokes equations can be found in various textbooks (Panton, 2005, page 732, for example) but we will just look at the momentum equation (1.1a):

$$\rho\left[\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right] + \frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} = 0.$$

In order to non-dimensionalize this equation, we introduce a reference length L, velocity U_0, density ρ_0 and viscosity μ_0. The non-dimensional variables are then:

$$x_i^* = x_i/L;\ t^* = tU_0/L;\ u_i^* = u_i/U_0;\ \mu^* = \mu/\mu_0;\ p^* = (p - p_0)/\rho_0 U_0^2$$

where an asterisk indicates that a variable is dimensionless. Pressure appears in these equations as a gradient, so we can subtract off some reference value p_0 before we scale on $\rho_0 U_0^2$.

The first step is to non-dimensionalize the viscous stress term:

$$\begin{aligned}\tau_{ij} &= 2\mu_0\mu^*\left[\frac{U_0}{L}S_{ij}^* - \frac{1}{3}\frac{U_0}{L}S_{kk}^*\delta_{ij}\right], \\ &= \frac{\mu_0 U_0}{L}\tau_{ij}^*\end{aligned}$$

and so:

$$\frac{\partial \tau_{ij}}{\partial x_j} = \frac{\mu_0 U_0}{L^2}\frac{\partial \tau_{ij}^*}{\partial x_j^*}.$$

Going through the same steps for the rest of the momentum equation yields:

$$\rho^* \frac{\rho_0 U_0^2}{L} \left(\frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*} \right) + \frac{\rho_0 U_0^2}{L} \frac{\partial p^*}{\partial x_i^*} - \frac{\mu_0 U_0}{L^2} \frac{\partial \tau_{ij}^*}{\partial x_j^*} = 0,$$

which leads to the dimensionless momentum equation:

$$\rho^* \left(\frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*} \right) + \frac{\partial p^*}{\partial x_i^*} - \frac{1}{\mathrm{Re}} \frac{\partial \tau_{ij}^*}{\partial x_j^*} = 0, \tag{1.6}$$

where the Reynolds number $\mathrm{Re} = \rho_0 U_0 L/\mu_0$. The fundamental importance of the Reynolds number is now obvious. When Re is large, the flow is (almost) inviscid and the inertial terms dominate. When Re is small, the viscous terms dominate. For this reason, large Reynolds numbers flows are also often called 'inviscid', but you must be careful to remember that this does not mean 'flows having zero viscosity' but 'flows in the limit of vanishing viscosity'. The study of turbulence is largely the study of the delicate balance of inertial and viscous forces in a flow.

Dealing with randomness

It is in the nature of turbulent flows that they are random so that we can only talk about them in a statistical sense. In practice, this means using the ideas of averaging and standard deviation, which you have met before, and some extensions of these ideas, which you have probably not.

The basic idea of averaging in turbulence problems is that flows generated under the same conditions will be statistically the same: the instantaneously measured velocities and pressures will be random, but the basic behaviour in terms of statistical properties will not change between repetitions of an experiment. This leads to the idea of *ensemble averaging*: we average across the repetitions of an experiment to get some 'true' underlying mean value. If, for example, we are looking for the ensemble average of the pressure measured in some flow, we compute it, in principle, as:

$$\langle p(\mathbf{x}, t) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} p^{(n)}(\mathbf{x}, t), \tag{1.7}$$

where $p^{(n)}(\mathbf{x}, t)$ is the pressure measured at point $\mathbf{x}$ and time t, during the nth repetition of the experiment. The ensemble average, denoted by $\langle p(\mathbf{x}, t) \rangle$ is found by repeating the experiment to find an estimate of the underlying mean value. The idea is that in a real problem there is some basic mean flow about which there are random fluctuations which we tentatively call turbulence. In practice, we tacitly make the *ergodic assumption*, which can be (roughly) phrased as meaning that time averages and ensemble averages are the same.

To examine the fluctuations, we subtract off the ensemble average. The fluctuation is usually denoted by a prime symbol so:

$$p' = p - \langle p \rangle .$$

Obviously, $\langle p' \rangle = 0$, so it is not a useful measurement to make or quantity to predict. On the other hand the mean of its square will not be zero. This mean is called the variance and defined:

$$\begin{aligned} \mathrm{var}[p'] &= \left\langle (p')^2 \right\rangle = \left\langle (p - \langle p \rangle)^2 \right\rangle , &(1.8)\\ &= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (p^{(n)}(\mathbf{x}, t) - \langle p(\mathbf{x}, t) \rangle)^2. \end{aligned}$$

From the definition of $\mathrm{var}[p]$ and $\langle p \rangle$, you can show that:

$$\mathrm{var}[p] = \left\langle p^2 \right\rangle - (\langle p \rangle)^2. \tag{1.9}$$

Figure 1.1: A two-speaker arrangement

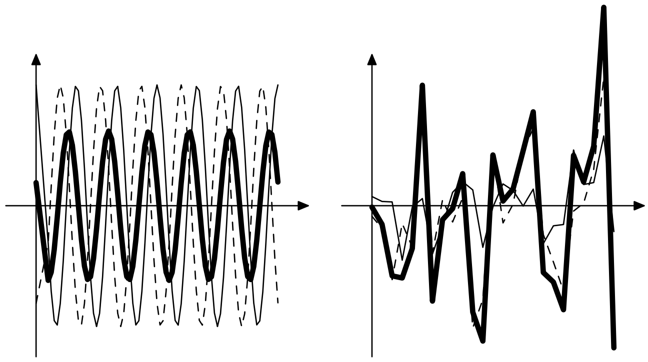


Figure 1.2: Sinusoidal and random signals from the two speakers: first speaker shown solid, second shown dashed, total signal bold.

It is also worth knowing that the square root of the variance is the standard deviation or root mean square of the variable.

Given that we have found averages of p and p^2, we can write down a general formula for the *moments* of a random variable. The mth moment of p is defined:

$$\langle p^m \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (p^{(n)})^m \tag{1.10}$$

and the mth *central* moment is found in the same way, but with the mean subtracted off:

$$\langle (p')^m \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (p^{(n)} - \langle p \rangle)^m, \tag{1.11}$$

so that the variance is the second central moment.

Correlation

An issue which is important in turbulence measurements, and an essential part of estimating the characteristics of noise sources, is the strength of the relationship between the flow at different points in space and time.

Figure 1.1 shows two loudspeakers at positions $\mathbf{x_1}$ and $\mathbf{x_2}$, generating signals $p(t)$ and $q(t)$ respectively. As you will remember, the total signal measured by a microphone is made up of the sum of the two original signals which combine destructively or constructively, depending on the signals and on their distances to the microphone.

Figure 1.2 shows how the two signals can combine. If the signals are sinusoidal, the combination is also a sinusoid with its amplitude depending on the phase of the two original signals. What happens, however, if the signals are random, as in the second part of Figure 1.2? We cannot make a definite statement about how the two signals will combine, unless we consider how they are related. If the two signals are identical, then they can be summed quite easily. If they are completely unrelated, in some sense, then we can treat them independently. The problem is in saying what we mean by ‘related’ or ‘unrelated’, in a statistical sense. This idea can be made more precise using a *correlation function*. For the two signals p and q, this is

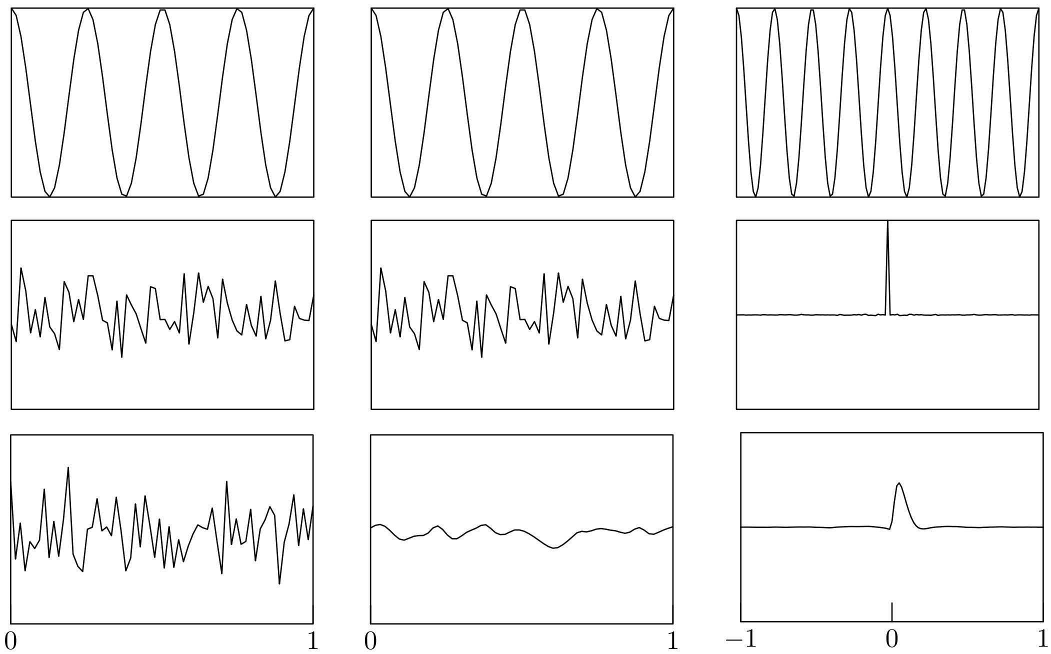


Figure 1.3: The correlation functions of various signals: two sinusoids of the same frequency; a Gaussian random signal autocorrelation; a random signal and its low-pass filtered self. Note that the horizontal axis on the right hand figures runs from -1 to 1.

defined:

$$C(\tau) = \langle p(t)q(t+\tau)\rangle \,, \tag{1.12}$$

which is a statistical quantity which can be measured in a flow or extracted from computations. In turbulent flows, we usually make the assumption that the problem is *statistically stationary*, which means that the statistics (mean, moments, etc.) of the flow do not change with time, so that $C(\tau)$ is a function of time difference τ but not of time t.

An important special case for the correlation is the autocorrelation which is the correlation of a variable with itself:

$$C(\tau) = \langle p(t)p(t+\tau)\rangle \,. \tag{1.13}$$

It is often convenient to remove the effect of the magnitude of the signals by working in terms of the correlation coefficient:

$$\Psi(\tau) = \frac{\langle p(t)q(t+\tau)\rangle}{[\langle p^2\rangle \langle q^2\rangle]^{1/2}} \tag{1.14}$$

which is the correlation function scaled on the r.m.s. values of the variables. The maximum possible magnitude of $\Psi(\tau)$ is one, so that it tells us how well correlated p and q are, independent of their magnitudes.

In order to see how the correlation function can be useful, we can look at some examples.

Figure 1.3 shows a set of basic signals in the first two columns and their correlation coefficients on the right hand side. These were computed numerically using a function in Octave (a program similar to Matlab). The first example is the autocorrelation of a sinusoid, which is itself a sinusoid (see Problem 8).

The second set of figures show the autocorrelation of a random signal. Here the correlation is a spike at time zero: the signal is perfectly correlated with itself at the same time (it is always equal to itself) but because there is no correlation between successive parts of the signal—by definition since the signal is random—the correlation is zero everywhere else. Finally, the last set of figures shows a random signal correlated with a low-pass filtered version of itself. Now, the filter introduces a ‘memory’ into the system and the correlation lasts for a finite length of time, as you can see in the right hand plot. This process is similar to what happens in turbulent flows, where the velocity in the fluid changes as a packet of fluid is swept with the flow, leading to a gradual decay in the correlation function with space and time.

Time and frequency domain analysis

As you might have guessed by now, looking at a picture of a random signal is of little use in characterizing or interpreting turbulence. The basic statistical measures, such as variance, are important but not very informative. In practice, much of our analysis of turbulence will use correlations and frequency domain methods, where we analyze signals in terms of the energy content as a function of frequency, rather than of time. The Fourier transform (Fourier series for a periodic signal) lets us switch between the time and frequency domains:

$$\hat{u}(f) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{j}2\pi f t}\,\mathrm{d}t,\; u(t) = \int_{-\infty}^{\infty} \hat{u}(f)\mathrm{e}^{\mathrm{j}2\pi f t}\,\mathrm{d}f, \tag{1.15}$$

where $\hat{u}(f)$ is the Fourier transform of $u(t)$ and f is frequency. Note that the frequency here is not in radians. There is no single agreed form for the Fourier transform, and different people give slightly different formulations. We use this one, because it is the most symmetric.

The Fourier transform on its own breaks the signal down into different components with a magnitude and phase at each frequency. On its own, this is of little use for a random signal, since it will be random itself. On the other hand, we can look at the Fourier transform of something which is repeatable and well-defined, the autocorrelation. The Fourier transform of the autocorrelation is called the *power spectral density*, or sometimes power spectrum:

$$S(f) = \int_{-\infty}^{\infty} C(\tau)\mathrm{e}^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau. \tag{1.16}$$

The relevance of $S(f)$ is that it gives the mean-square amplitude of $u(t)$ at frequency f. It can be found from experimental measurements by averaging estimates made using the Fast Fourier Transform.

We can define a similar quantity called the *cross spectral density*, which is the Fourier transform of the cross correlation between two signals. This is similar to the transfer function between the input and output of a system.

An important application of the Fourier transform comes when it is applied to spatial data. In this case, we use it to say what *scales* are important in our system, or how “big” particular processes are. An important part of turbulence is the question of how energy is transferred between scales as large scale processes give up their energy to be dissipated by the very small processes governed by viscosity.

Questions

1. Find a tap in a kitchen or bathroom. Open it until the water just starts to flow. Then slowly open it further until it is completely open. Note what you see and describe how the water behaves.

2. Stand on a bridge and look at the water downstream of the bridge as it flows over the pillars of the bridge. How does the water behave? What would the ensemble average of the flow quantities look like? In Bath, a good place to do this is the footbridge at Sainsbury’s.

3. Using the definitions given in the chapter, prove that $S_{kk} = 0$ in an incompressible flow, (1.5).

4. The Navier–Stokes equations (1.1) each contain a term in square brackets. Using the Einstein summation convention, write out this term in full, noting the repeated index j in the derivatives. If u_j is zero, what does this term mean? What does it mean if the derivative with respect to time is zero?

5. You have the job of measuring the pressure fluctuations at a point behind a four-bladed propeller. What would be your first, graphical, guess at a representation of the ensemble averaged pressure in the flow, remembering that the ensemble average is not necessarily constant in time? Given that guess at the form of the ensemble average, what do you think the time average would be? What would the Fourier series look like?

6. Prove (1.9), using (1.7) and (1.8).

7. Prove that the autocorrelation function of a stationary random process is symmetric in time, i.e. $C(\tau) = C(-\tau)$.

8. Derive an expression for the correlation between two sinusoidal signals, $f(t) = \sin(\omega_1 t)$ and $g(t) = \sin(\omega_2 t + \phi)$.

9. If a 'packet' of fluid moves through a point, with its internal velocity changing slowly over time, what do you expect the correlation function for the velocities to look like?

Chapter 2

Simplifications

Short range, intense flows: turbulence

In order to make a start on simplifying our problem, we consider how to derive some averaged form of the Navier-Stokes equations which respects the nonlinearity of the problem, but gives us something we can take a grip on. We start by making the *Reynolds decomposition*, which breaks each quantity into its ensemble average, denoted by a capital letter, and a fluctuating part, denoted by a prime symbol:

$$u_i = U_i + u_i'; \quad p = P + p';$$

and making the assumption of incompressibility, so that $\partial u_j/\partial x_j \equiv 0$. Inserting the Reynolds decomposed velocity into the incompressible continuity equation gives:

$$\frac{\partial U_j}{\partial x_j} = 0; \quad \frac{\partial u_j'}{\partial x_j} = 0; \tag{2.1}$$

so that both the mean flow U_j and the fluctuations are incompressible. Moving to the momentum equation, inserting the Reynolds decomposition gives:

$$\rho\frac{\partial U_i}{\partial t} + \rho U_j\frac{\partial U_i}{\partial x_j} + \rho\frac{\partial}{\partial x_j}\left\langle u_i'u_j'\right\rangle + \frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 U_i}{\partial x_j \partial x_j} = 0. \tag{2.2}$$

We now have an equation for the average flow U_i and P but with one extra term, $\left\langle u_i'u_j'\right\rangle$. This term plays a role similar to that of a shear stress and is called the *Reynolds stress*. It connects the turbulent fluctuations to the underlying base flow. The term 'Reynolds stress' is unfortunate, since the quantity is not really a stress, but it points up the role which is played by $\left\langle u_i'u_j'\right\rangle$, in that it behaves like a stress in the way it affects the underlying flow.

The problem of dealing with turbulence arises from the nature of (2.2). Firstly, the Reynolds stress is not a property of the fluid, like viscosity, but a property of the flow and one which affects the flow. The problem we have is in determining what happens on the scales typical of turbulence, without knowing in advance how the fluid behaves on these scales, since we need to know how the flow behaves in order to determine its behaviour. Secondly, we need extra equations to close the system. The variables in (2.2) are U_i, P and $u_i'u_j'$, making ten in all. If we include the continuity constraint, we only have four equations in total: we are short by six. There are methods of introducing extra equations to close the problem, but in practice they do not work very well. The attempt to find these extra equations is called the *closure problem*, and is a major area of research in CFD.

The first attempt to develop a closure for the Reynolds stress equations was the mixing length hypothesis, leading to the eddy viscosity model (Tennekes and Lumley, 1972). The idea is shown in Figure 2.1. Consider a particle of fluid which is moved around in a mean shear flow $U_1(x_2)$. The momentum in the x_1 direction per unit volume of the particle is ρu_1. If the particle starts at a position 0 at time 0 and moves to a

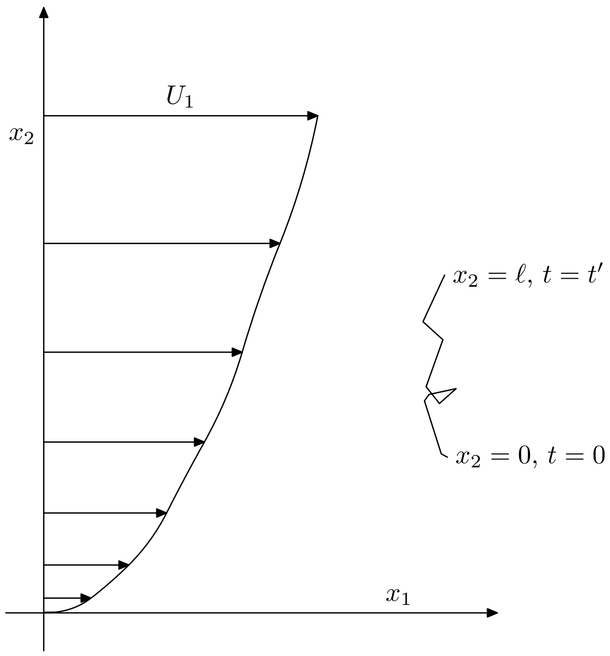


Figure 2.1: A particle of fluid moving in a shear flow

position x_2 at time t', the difference in momentum between start and finish is:

$$\begin{aligned}\Delta M &= \rho(u_1(x_2,t) - u_1(0,0)),\\ &= \rho[U_1(x_2) - U_1(0)] + \rho[u_1'(x_2,t) - u_1'(0,0)],\end{aligned}$$

if we use the Reynolds decomposition. If we can use a linear approximation for the velocity gradient, and assume that the turbulent velocity contribution is negligible, this can be written:

$$\Delta M \approx \rho x_2 \frac{\partial U_1}{\partial x_2}.$$

The rate of momentum transport in the x_2 direction is $u_2 \Delta M$. Now, the particle velocity $u_2 = \partial x_2/\partial t$ so:

$$\begin{aligned}u_2 \Delta M &= \rho \frac{\partial U_1}{\partial x_2} x_2 \frac{\partial x_2}{\partial t},\\ &= \frac{1}{2}\rho \frac{\partial U_1}{\partial x_2}\frac{\partial x_2^2}{\partial t}.\end{aligned}$$

The Reynolds stress is the rate of momentum transfer so we can time average:

$$\overline{\rho u_1' u_2'} = \frac{1}{2}\rho \frac{\partial U_1}{\partial x_2}\frac{\overline{\partial x_2^2}}{\partial t}.$$

Since $\partial x_2^2/\partial t = 2x_2 \partial x_2/\partial t = 2x_2 u_2$,

$$\overline{\rho u_1' u_2'} = \rho \frac{\partial U_1}{\partial x_2}\overline{u_2' x_2}.$$

This equation relates the Reynolds stress to the velocity gradient, similarly to the way viscosity relates the shear stress to the velocity gradient. The relationship depends on the correlation between particle displacement and velocity. This correlation must fall off to zero at some distance, which we call the *mixing*

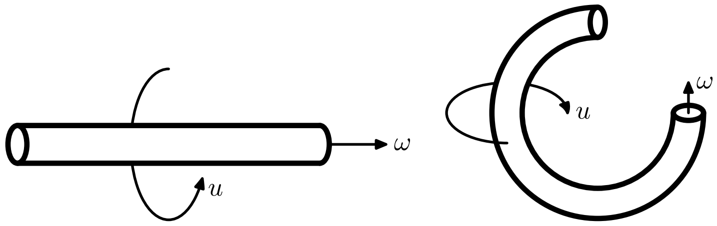


Figure 2.2: A line and a ring vortex

length. On average when a particle has travelled a mixing length, it has lost its identity and given up its momentum to the fluid around it. If we assume that $\overline{x_2 u_2'} = c_1 \ell u_2^{(\mathrm{rms})}$, where ℓ is the mixing length, $u_2^{(\mathrm{rms})}$ is the r.m.s. velocity in the x_2 direction and c_1 is some constant to be determined, we can write the Reynolds stress equation using a constant ν_T:

$$\overline{\rho u_1' u_2'} = \rho \nu_T \frac{\partial U_1}{\partial x_2}, \tag{2.3}$$

where:

$$\nu_T = c_1 \ell u_2^{(\mathrm{rms})} \tag{2.4}$$

is the *eddy viscosity*. It is not really a viscosity, since it is a property of the flow rather than of the fluid, but it relates a 'shear stress' to a velocity gradient so we call it a viscosity for convenience.

This is the simplest turbulence closure we can imagine and in practice works about as well as we might expect. The constant c_1 needs to be determined by experiment or (very) elaborate computation and c_1 determined for one flow will be valid for flows of that type only. It also assumes that the flow has a velocity gradient in one direction. If the velocity varies in two or three directions, the 'medium' is anisotropic and the 'viscosity' becomes a function of direction.

Potential and vorticity

Another way of looking at a flow is to break it into a potential and a vorticity. These are two quantities which are slightly abstract to begin with, but can often give us physical insights into a problem. The velocity field can (always) be written as follows:

$$\mathbf{u}(\mathbf{x}) = \nabla\phi + \nabla \times \mathbf{B}, \tag{2.5}$$

where the scalar ϕ is called the *potential*, and the vector $\mathbf{B}$ is called the *vector potential*. The vector potential itself is rarely used directly, but is expressed in terms of the *vorticity*, $\boldsymbol{\omega}$:

$$\mathbf{B} = \frac{1}{4\pi} \int_V \frac{\boldsymbol{\omega}_1}{|\mathbf{r}|} \,\mathrm{d}V, \tag{2.6}$$

$$\mathbf{r} = |\mathbf{x} - \mathbf{x}_1|, \tag{2.7}$$

and V is the volume of non-zero vorticity. The velocity which results is given by the *Biot–Savart* law:

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi} \int_V \frac{\mathbf{r} \times \boldsymbol{\omega}_1}{|\mathbf{r}|^3} \,\mathrm{d}V. \tag{2.8}$$

The vorticity is a measure of the solid body rotation in the flow and is defined:

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}. \tag{2.9}$$

The easiest way to think physically about vorticity is to look at line and ring vortices, Figure 2.2. A line vortex (a point in two dimensions) generates a flow which rotates about the axis of the vortex. A ring vortex is closed on itself and wraps the velocity field into one which has an axial component parallel to the ring axis.

It is common to say that vorticity 'induces' a velocity field, in the same way that an electrical current induces a magnetic field, since the Biot–Savart law applies to both. Strictly, this is not correct, because the vorticity is a property of the velocity field rather than the thing that 'causes' it. A basic equation of fluid dynamics tells us how vorticity evolves:

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega}.\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}, \tag{2.10}$$

where $\mathrm{D}/\mathrm{D}t$ is the material derivative which gives the rate of change of the vorticity in a reference frame moving with the flow (see Question 4 of Chapter 1). The first term on the right relates to the deformation of vortex lines while the second gives the rate of viscous diffusion of vorticity. In a two-dimensional inviscid flow $\mathrm{D}\boldsymbol{\omega}/\mathrm{D}t \equiv 0$ and vorticity is conserved: vortices move around but they do not change in magnitude.

Long range, weak flows: acoustics

The other simplification which we would like to consider is the linear problem of acoustic propagation, where the perturbations are very small, but they travel over long distances. In this case, we neglect viscosity and write the continuity and momentum equations:

$$\rho\left[\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right] + \frac{\partial p}{\partial x_i} = 0; \tag{2.11a}$$

$$\left[\frac{\partial \rho}{\partial t} + u_j\frac{\partial \rho}{\partial x_j}\right] + \rho\frac{\partial u_j}{\partial x_j} = 0. \tag{2.11b}$$

Since this is a linear problem, we assume that the quantities are given by very small fluctuations about some mean. Writing:

$$\rho = \bar{\rho} + \rho',\ u_i = u_i',\ \text{and}\, p = \bar{p} + p',$$

we can insert these terms into the continuity and momentum equations, and neglect any products of fluctuating quantities:

$$\frac{\partial \rho'}{\partial t} + \bar{\rho}\frac{\partial u_i'}{\partial x_i} = 0, \tag{2.12a}$$

$$\bar{\rho}\frac{\partial u_i'}{\partial t} + \frac{\partial p'}{\partial x_i} = 0. \tag{2.12b}$$

These equations can be combined by differentiating them and subtracting one from the other:

$$\frac{\partial}{\partial t}\left[\frac{\partial \rho'}{\partial t} + \bar{\rho}\frac{\partial u_i'}{\partial x_i}\right] = 0,$$

$$-\frac{\partial}{\partial x_i}\left[\bar{\rho}\frac{\partial u_i'}{\partial t} + \frac{\partial p'}{\partial x_i}\right] = 0,$$

$$\frac{\partial^2 \rho'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i \partial x_i} = 0.$$

Now, we can make some assumption about the relationship between density and pressure fluctuations, by writing:

$$
\begin{aligned}
p &= \bar{p} + \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\bar{\rho}} (\rho - \bar{\rho}) + \frac{1}{2}\left.\frac{\partial^2 p}{\partial \rho^2}\right|_{\rho=\bar{\rho}} (\rho - \bar{\rho})^2 + \ldots, \\
p' &= p - \bar{p} \approx \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\bar{\rho}} (\rho - \bar{\rho}) = c^2 \rho', \\
c^2 &= \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\bar{\rho}}.
\end{aligned}
$$

The constant is written c^2 because it is always positive (why?). If we insert this relationship, we get a wave equation for pressure:

$$\boxed{\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0} \tag{2.13}$$

where $\nabla^2 = \partial^2/\partial x_i \partial x_i$ (Appendix C). This is the most fundamental equation in acoustics. It describes the properties of a sound field in space and time and how those properties evolve. It is quite unlike the incompressible flow equations because it describes very weak processes which happen over large distances. The most fundamental property of the wave equation is that it is *linear*. This means that the sum of two solutions of the wave equation is also itself a solution, which is why we can tell a singer from an instrument.

When we come to solve the wave equation, we will find that c is the speed of sound, the speed at which a small disturbance propagates through a fluid. It depends on the thermodynamical properties of the fluid and is calculated on the assumption that sound propagation is *adiabatic*. For an adiabatic process in a gas:

$$p = k\rho^\gamma,$$

where γ is the ratio of the specific heats. Then

$$
\begin{aligned}
c^2 &= \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\rho_0}, \\
&= \gamma k \rho^{\gamma-1} = \frac{\gamma p}{\rho}, \\
p &= \rho R T
\end{aligned}
$$

so that

$$\boxed{c^2 = \gamma R T.}$$

The speed of sound in air at STP is $343\mathrm{m/s}$. The validity of the adiabatic assumption depends on the frequency of the sound. For low-frequency sound, there is no appreciable heat generation by conduction in the fluid and the assumption is a good one. For air, 'low frequency' means 'less than $1\mathrm{GHz}$'.

Note that if $c \to \infty$, the wave equation becomes $\nabla^2 p = 0$, the equation of incompressible flow. Saying $c \to \infty$ is the same as saying that density is independent of pressure, i.e. that the flow is incompressible. Since c is the speed at which disturbances propagate in a fluid, this is equivalent to the statement that disturbances propagate instantaneously in an incompressible flow.

If we write $p = P\exp[-\mathrm{j}\omega t]$ where ω is the radian frequency, the wave equation becomes the *Helmholtz* equation:

$$\boxed{\nabla^2 P + k^2 P = 0.} \tag{2.14}$$

Note that t has disappeared, reducing the order of the equation by one. The *wavenumber* $k = \omega/c$.

Solutions of the wave equation in one dimension: Plane waves

To illustrate some aspects of the solution of the wave equation, we look first at waves in one dimension. This corresponds to low frequency sound propagating in a pipe, for example. If we take x as the coordinate along the pipe, the wave properties are independent of y and z and the wave equation becomes:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = 0. \tag{2.15}$$

You can show quite easily that solutions of the form $p = f(x \pm ct)$ satisfy (2.15). This means that disturbances propagate as fixed shapes which shift along the x-axis at speed c. Figure 2.3 is a simple example, showing both solutions $x \pm ct$.

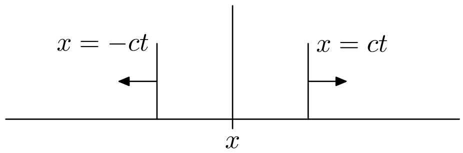


Figure 2.3: Wave propagation: right propagating wave with $x = ct$ and left propagating wave with $x = -ct$.

A pulse starts at a point $x = 0$ at time $t = 0$ so that $x \pm ct = 0$. At a later time, the wave will have moved left to a point $x = -ct$, still satisfying $x + ct = 0$ and right to a point $x = ct$, satisfying $x - ct = 0$. In both cases, the value of p will be the same as at time $t = 0$. As we might expect, the wave travels to the left or right at speed c, which is why c is called the speed of sound.

When waves propagate like this, they are called *plane waves* because their properties are constant over planes of constant x. Waves can be modelled as planar when they propagate at low frequency in pipes or ducts, such as long pipelines or engine exhaust systems. Plane waves also occur in other situations and are very useful in analyzing general problems. If a plane wave propagates in a general direction, we can write it as $f(t - \mathbf{x}.\mathbf{n}/c)$ where $\mathbf{n}$ is the direction of propagation or normal to the wave.

Solutions of the wave equation in three dimensions

Naturally, one-dimensional waves are of little interest to rounded personalities such as ourselves and we must eventually face reality in all of its three dimensions. Solving the wave equation in three dimensions is not much more difficult than doing so in one dimension. The most convenient approach is to work in spherical polar coordinates, Appendix C. In this coordinate system:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2}{\partial \phi^2}.$$

We simplify this by considering the case of sound propagating in free space in a uniform medium. Then, by symmetry, p' is independent of ϕ and θ, so that:

$$\begin{aligned}\nabla^2 p &= \frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} \\ &= \frac{1}{r}\frac{\partial^2}{\partial r^2}(rp) \end{aligned} \tag{2.16}$$

and the wave equation now reads

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(rp) - \frac{\partial^2}{\partial r^2}(rp) = 0, \tag{2.17}$$

which is identical in form to (2.15). Using the solution of that equation, $rp = f(r \pm ct)$, we find

$$p = \frac{f(t - r/c)}{r}. \tag{2.18}$$

For reasons of *causality* (things cannot happen before they have been caused), so the solution $rp = f(r+ct)$ is rejected[1].

This solution contains three useful pieces of information. The first, as in the one-dimensional case, is that the sound at time t depends on what happened at time $t - r/c$, the *emission time* or *retarded time*. The second, again similarly to the one dimensional case, is that the shape of the wave $f(\cdot)$ does not change. The big difference between one and three dimensional waves, however, is that the magnitude of the pressure perturbation (though not its shape) reduces as it propagates.

Acoustic velocity and intensity

When we derived the wave equation, we chose to eliminate velocity and density and concentrated on pressure as our dependent variable. There are two main reasons for doing this: the first is that pressure is a scalar and so is conceptually easier to work with than velocity. In practice, given that we could use a velocity potential, this is not a huge advantage. The second, and more important, reason is that pressure is what we hear and what we measure. Our ears and the microphones we use to measure sound are sensitive to pressure fluctuations, so that is what we choose as our main quantity.

There are times, however, when we will need to use some other quantity. The fundamental theory of aerodynamically generated noise is actually based on density fluctuations (which are usually converted to pressure variations using a linear relationship). A more important relationship is that between pressure and velocity because the acoustic velocity is often used as a boundary condition in calculations involving solid bodies. Remember that acoustics is a branch of fluid dynamics and it is a fluid-dynamical boundary condition that must be satisfied, i.e. usually a velocity.

The linearized momentum equation (2.12b) gives us the relationship we need:

$$\frac{\partial \mathbf{v}'}{\partial t} = -\frac{\nabla p'}{\rho_0},$$

in other words, the acoustic velocity is proportional to the pressure gradient. If we write the solution of the wave equation in terms of a velocity potential $\phi = -f(t - R/c)/R$, the pressure and radial velocity are related via:

$$p = \rho_0 \frac{\partial \phi}{\partial t}, \quad \mathbf{v} = \nabla \phi,$$
$$v = \frac{p}{\rho_0 c} + \frac{f(t - R/c)}{\rho_0 R^2}. \tag{2.19}$$

For a wave of constant frequency, the acoustic velocity amplitude V is related to the acoustic pressure by

$$V = -\mathrm{j}\frac{\nabla P}{\rho_0 \omega}. \tag{2.20}$$

For a plane wave $\nabla \to \partial/\partial x$ and $V = P/\rho_0 c$. For large R, the pressure–velocity relationship for a spherical wave reduces to this form, as seen in (2.19).

A basic characteristic of a source is the rate at which it transfers energy. If we multiply (2.12a) by $c^2\rho'$,

$$c^2\rho'\frac{\partial \rho'}{\partial t} + \rho_0 c^2 \rho' \frac{\partial v}{\partial x} = 0, \tag{2.21}$$

[1] Why did we not do this for one-dimensional waves?

and note that $\rho'\partial\rho'/\partial t = \frac{1}{2}(\partial/\partial t){\rho'}^2$ and that $c^2\rho' = p'$,

$$\frac{c^2}{\rho_0}\frac{1}{2}\frac{\partial}{\partial t}{\rho'}^2 + p'\frac{\partial v}{\partial x} = 0.$$

Multiplying the momentum equation (2.12b) by v gives

$$\rho_0 v\frac{\partial v}{\partial t} + v\frac{\partial p'}{\partial x} = 0,$$

which can be rearranged:

$$\frac{1}{2}\rho_0\frac{\partial}{\partial t}v^2 + v\frac{\partial p'}{\partial x} = 0. \tag{2.22}$$

Adding (2.21) and (2.22) gives a result for the energy transport in the sound field:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 v^2 + \frac{1}{2}\frac{c^2}{\rho_0}{\rho'}^2\right) + \frac{\partial}{\partial x}(p'v) = 0. \tag{2.23}$$

In (2.23), $\rho_0 v^2/2$ is the *kinetic energy* per unit volume, $(c^2/\rho_0){\rho'}^2/2$ is the *potential energy* per unit volume and $p'v$ is the *acoustic intensity* I which is the rate of energy transport across unit area, so that (2.23) is a statement of energy conservation for the system and says that the rate of change of energy in a region is equal to the net rate at which energy is carried into that region.

If we insert the relationship between pressure and velocity (2.19), the acoustic intensity is

$$I = \frac{p^2}{\rho c} + \frac{\partial}{\partial t}\left(\frac{f^2(t-R/c)}{2\rho R^3}\right).$$

If we average I over time for a periodic wave, the second term has a mean value of zero and the resulting mean intensity is:

$$\bar{I} = \frac{\overline{p^2}}{\rho c}. \tag{2.24}$$

Noise from flows

If we repeat the derivation of the wave equation, but using the full continuity and momentum equations, without linearizing or simplifying, we can derive a wave equation which is an exact rearrangement of the Navier-Stokes equations and links the flow to the acoustic source. We start, as before, by differentiating the equations of continuity and momentum:

$$\frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\rho u_i u_j) - \frac{\partial \tau_{ij}}{\partial x_j}\right] = 0; \tag{2.25a}$$

$$-\frac{\partial}{\partial t}\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right] = 0, \tag{2.25b}$$

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 p}{\partial x_i \partial x_i} + \frac{\partial^2}{\partial x_i \partial x_j}(\rho u_i u_j) - \frac{\partial^2 \tau_{ij}}{\partial x_i x_j}. \tag{2.25c}$$

If we now subtract $c^2\partial^2\rho/\partial x_i\partial x_i$ from both sides of this equation, where c is the speed of sound in the undisturbed fluid, we get a wave equation for density:

$$\frac{\partial^2 \rho}{\partial t^2} - c^2\frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{\partial^2}{\partial x_i \partial x_i}(p - c^2\rho) + \frac{\partial^2}{\partial x_i \partial x_j}(\rho u_i u_j) - \frac{\partial^2 \tau_{ij}}{\partial x_i x_j}, \tag{2.26}$$

which can be written:

$$\frac{\partial^2 \rho}{\partial t^2} - c^2 \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}, \tag{2.27}$$

$$T_{ij} = \rho u_i u_j + (p - c^2 \rho)\delta_{ij} - \tau_{ij} \tag{2.28}$$

This is the *Lighthill* wave equation Lighthill (1952, 1954) and forms the basis of modern noise prediction methods. The important point to notice for now is that the source term contains $u_i u_j$: noise is generated by the Reynolds stresses. We have a connection between turbulence and noise.

Questions

1. *Without writing down the equations*, sketch the paths of two point vortices of equal strength interacting in a plane.
2. Repeat Question 1 for the case of two coaxial ring vortices of equal strength and equal radius. What will the velocity and pressure look like? What happens if the vortices have equal and opposite strength?
3. Show that the sum of two solutions of the wave equation is also a solution.
4. Use the chain rule to prove that $f(x \pm ct)$ is a solution of the one-dimensional wave equation.

Chapter 3

Characterizing turbulence

As should be horrifyingly apparent by now, we can say almost nothing about any given instance of turbulent flow. On the other hand, we can discuss a given turbulent flow in terms of correlations and other statistical quantities. In this chapter we consider how to characterize turbulence, and which quantities we will use in describing the system. The physical process which we need to describe is the *turbulent energy cascade*, the name given to the way in which motion on a large scale is dissipated on a small scale. The large scale motion is determined by the geometry of the problem; the small scale motion is determined by the effects of viscosity which dissipates the energy of the motion as heat. On the scales in between, there is a remarkable result that the transfer of energy between scales is independent of the nature of the problem (you will derive this result as a problem at the end of the chapter).

Unless otherwise stated, we deal with the problem of *homogeneous*, *isotropic* turbulence. Homogeneity means that the turbulent properties do not depend on position. Isotropy means that they do not depend on direction.

Kinetic energy

Much of our discussion of turbulence will centre on the process by which energy is generated and dissipated in a flow. When we need reference quantities to develop non-dimensional groups in our analysis, the mean flow velocity might well be of little use. In this case, a good choice is a velocity based on the kinetic energy per unit mass of the flow:

$$k = \frac{\overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2}}{2}. \tag{3.1}$$

Given the kinetic energy k, we can define a reference velocity:

$$u = \left(\frac{2k}{3}\right)^{1/2}. \tag{3.2}$$

We can also look at the rate of energy production and dissipation by considering the *turbulent energy budget*. This is given by Tennekes and Lumley (1972), who note, not unfairly, that its derivation 'is a fairly tedious exercise'. The result of the tedium is:

$$U_j \frac{\partial k}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u_j' p} + \frac{1}{2}\overline{u_i' u_i' u_j'} - 2\nu \overline{u_i' s_{ij}'}\right) - \overline{u_i' u_j'} S_{ij} - 2\nu \overline{s_{ij}' s_{ij}'}, \tag{3.3}$$

where s_{ij}' is the fluctuating strain rate. The left hand side gives us the gradient of kinetic energy k. The right hand side contains the various mechanisms of generation and dissipation of that energy.

There is a corresponding energy equation for the mean flow:

$$U_j \frac{\partial (U_i U_i/2)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(-\frac{\overline{p}}{\rho} U_j + 2\nu U_i S_{ij} - \overline{u_i' u_j'} U_i\right) + 2\nu S_{ij} S_{ij} + \overline{u_i' u_j'} S_{ij}. \tag{3.4}$$

In both equations, we can see the term $\overline{u_i' u_j'} S_{ij}$, but with an opposite sign in each. This means that it represents energy being transferred between the mean flow and the turbulence, usually from the mean to the fluctuating. We can also see, in (3.3), the term $2\nu \overline{s_{ij}' s_{ij}'}$ which represents the work done by viscous stresses in dissipating energy. This always removes energy from the flow and is the mechanism by which energy transferred from the mean flow, which passes into the turbulent flow, is finally dissipated as heat.

Length and time scales

One question we can ask about any physical process is its scale: how 'big' is the process and/or what range of scales does it cover? Turbulence is characterized by activity over a wide range of scales with energy passing from large scales to small scales until it reaches very small scales and is dissipated as heat, via viscosity. This is called the *energy cascade*, since energy 'cascades' down the scales. Different length scales can be used depending on the level at which we study the flow.

The first obvious length scale is some characteristic dimension of the flow itself, the width of a channel, or the chord of a wing for example. If nothing else, we know that there will be no length scale in the flow larger than this one. We call this the *large eddy length scale*, L_t, and take it as characterizing the largest eddies, or identifiable 'sub flows' in the fluid. Given a reference velocity u, there is then a corresponding time scale $T_\mathrm{t} = L_\mathrm{t}/u$. This time is variously interpreted as an eddy turn-over time (a blob of fluid of size L_t turns over in a time T_t); the eddy lifetime (the length of time a blob of fluid maintains its identity); the time over which velocity fluctuations are correlated.

The dissipation of energy also gives us a candidate length scale. The dissipation rate ϵ is of the order of k/T_t so that:

$$\epsilon = \frac{u^3}{L_\mathrm{t}}, \tag{3.5}$$

which leads to a turbulent time and length scale:

$$T_\epsilon = k/\epsilon,\ L_\epsilon = k^{3/2}/\epsilon. \tag{3.6}$$

If you are taking the CFD course, you will see these quantities introduced in the $k - \epsilon$ turbulence model. The term L_ϵ is called the dissipation length scale.

The great Russian mathematician Kolmogorov introduced definitions of the smallest possible scales in turbulence, those scales where viscosity acts and energy is finally dissipated as heat. The length, time and velocity scales are:

$$\eta_K = (\nu^3/\epsilon)^{1/4},\ \tau_K = (\nu/\epsilon)^{1/2},\ u_K = (\nu\epsilon)^{1/4}. \tag{3.7}$$

Thus we have an idea of suitable scales for the 'large' structure of the flow, and for the smallest. The question is what happens in between: how does ϵ depend on the scale of eddies?

It is known that the energy flowing in the cascade is independent of the size of an eddy, so if an eddy has a scale r, with characteristic velocity v:

$$\epsilon = \frac{u^3}{L_\mathrm{t}} = \frac{v^3}{r} = \frac{u_K^3}{\eta_K},$$

when the scale r lies between the largest and smallest scales in the flow.

In practice, we find the distribution of energy as a function of scales by talking about energy at a given *wavenumber* κ, which has the dimensions of inverse length. This comes about by performing a Fourier transform on the energy in space rather than time. The result is an *energy spectrum*, $E(\kappa)$ which gives the energy in each spatial frequency band. It turns out that this is a very powerful way of looking at the problem, since it makes an otherwise intractable problem much simpler.

Scales derived from correlations

If we want to examine the details of a flow by analyzing measured data, we need to extract some scales from the correlation functions. If we consider the velocity correlations, this gives us a correlation tensor:

$$R_{ij}(\mathbf{r}) = \frac{\overline{u_i'(\mathbf{x})u_j'(\mathbf{x}+\mathbf{r})}}{\left(\overline{u_i'^2 u_j'^2}\right)^{1/2}}, \tag{3.8}$$

where $\mathbf{r}$ is the separation between two points. A formal definition of isotropic turbulence is that $\overline{u_1'^2} = \overline{u_2'^2} = \overline{u_3'^2}$ and $\overline{u_i'u_j'} = 0$, for $i \neq j$. In homogeneous turbulence, the spatial derivative is zero for the statistical moments and the value does not depend on $\mathbf{x}$ but only on $\mathbf{r}$. This tells us how well the velocity remains correlated over a distance. We might think that this would give us a means of estimating how 'big' an eddy is. One way to do it is to say that the length of the eddy is the distance, $|\mathbf{r}|$, at which the correlation becomes negligible. The problem with this is deciding what we mean by 'negligible': do we mean 10% or 1% or something smaller still?

An unambiguous length scale is given by integrating the correlation:

$$L_{11} = \int_0^\infty R_{11}(r)\,\mathrm{d}r, \tag{3.9}$$

where L_{11} is called the *integral length scale* and can also be used as a definition of L_t. In anisotropic turbulence, the other integral length scales, L_{22}, for example, will be different, depending on the details of the flow.

A final length scale which is important in characterizing turbulence is the *Taylor microscale*. This can be found by considering the dissipation rate in isotropic turbulence:

$$\epsilon = 2\nu\overline{s_{ij}'s_{ij}'} = 15\nu\overline{\left(\frac{\partial u_1'}{\partial x_1}\right)^2}. \tag{3.10}$$

From this relationship, we define the Taylor microscale λ:

$$\overline{\left(\frac{\partial u_1'}{\partial x_1}\right)^2} = \frac{\overline{u'^2}}{\lambda^2}. \tag{3.11}$$

A similar quantity, based on the time derivative of the velocity, is called the Taylor *temporal* microscale:

$$\overline{\left(\frac{\partial u_1'}{\partial t}\right)^2} = \frac{2\overline{u'^2}}{\lambda_T^2}. \tag{3.12}$$

There is a relationship between λ_T and velocity autocorrelation coefficient. From (1.14):

$$\Psi(\tau) = \frac{\langle u'(t)u'(t+\tau)\rangle}{\langle u'^2\rangle}. \tag{3.13}$$

Since $\left\langle u'^2\right\rangle$ is constant in statistically stationary turbulence, we can find that:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle u'(t)u'(t+\tau)\rangle = 0 = 2\left\langle u'(t+\tau)\frac{\mathrm{d}^2u'}{\mathrm{d}t^2}\right\rangle + 2\left\langle\left(\frac{\mathrm{d}u'}{\mathrm{d}t}\right)^2\right\rangle, \tag{3.14}$$

$$\left\langle u'(t+\tau)\frac{\mathrm{d}^2u'}{\mathrm{d}t^2}\right\rangle = -\left\langle\left(\frac{\mathrm{d}u'}{\mathrm{d}t}\right)^2\right\rangle. \tag{3.15}$$

Near $\tau = 0$, the autocorrelation is well approximated by:

$$\Psi(\tau) \approx 1 + \frac{1}{2}\frac{1}{\langle u'^2 \rangle}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\langle u'(t)u'(t+\tau)\rangle \tau^2, \tag{3.16}$$

$$= 1 - \frac{1}{2}\frac{1}{\langle u'^2 \rangle}\left\langle u'(t+\tau)\frac{\mathrm{d}^2 u'}{\mathrm{d}t^2}\right\rangle \tau^2. \tag{3.17}$$

so that:

$$\Psi(\tau) \approx 1 - \tau^2/\lambda_T^2. \tag{3.18}$$

Questions

1. Use dimensional analysis to show that $\eta_K = (\nu^3/\epsilon)^{1/4}$.
2. Given that energy at a wavenumber κ, $E(\kappa)$, depends only on κ and the dissipation rate ϵ, use dimensional analysis to show that $E = C\kappa^{-5/3}\epsilon^{2/3}$, where C is a constant of proportionality. This is the Kolmogorov spectrum in the inertial range and is one of the deepest results in turbulence.
3. Why can homogeneous and/or isotropic turbulence not exist in nature?

Chapter 4

Instability and transition to turbulence

A question which we have to consider is how turbulence is initiated: what happens to a flow to make it turbulent? There are a number of different mechanisms of transition to turbulence, but the essential features are usually the same: a flow responds to a perturbation in an unstable manner, with the instability growing until non-linear mechanisms take over to generate and sustain turbulence. Since we are concerned with the response to a small perturbation, our analysis can be linear, rather than non-linear, and it is possible to say something about the early stages of transition to turbulence.

Kelvin–Helmholtz instability

Figure 4.2 shows the notation for our problem (Panton, 2005). There is a shear layer dividing two streams of fluid, one travelling at velocity U_1 and the other at velocity U_2, in the x direction. We want to know what will happen to the shear layer if it is given a small perturbation. If the flow is incompressible and

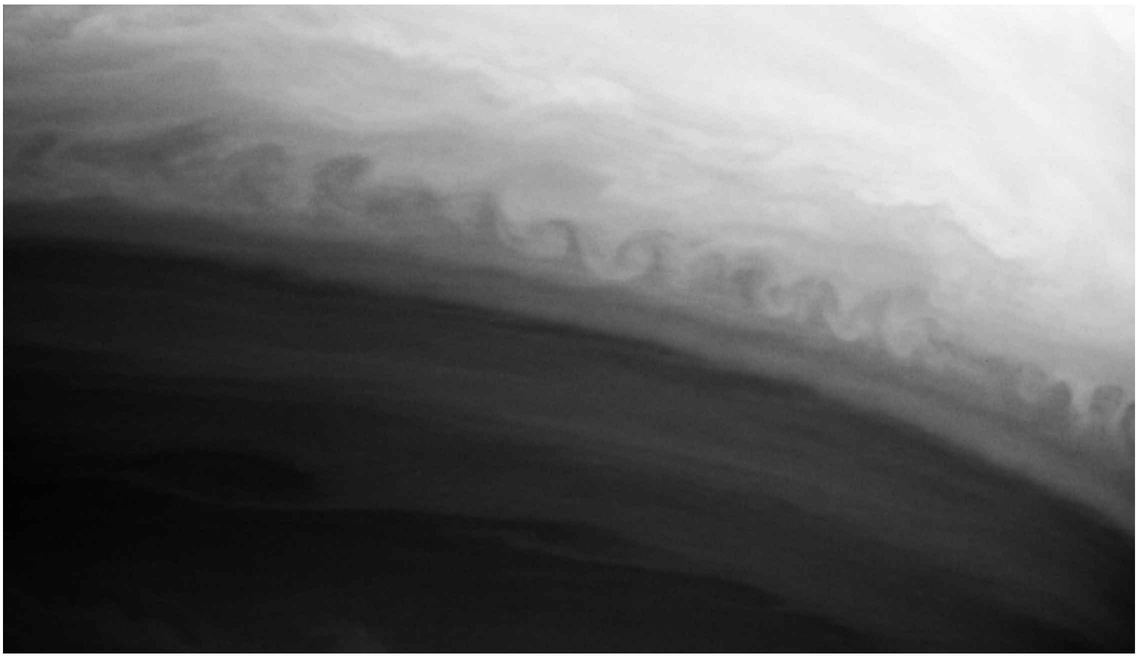


Figure 4.1: The Kelvin–Helmholtz instability in nature: a Cassini orbiter image of the interface between two regions of different density in the atmosphere of Saturn (©NASA).

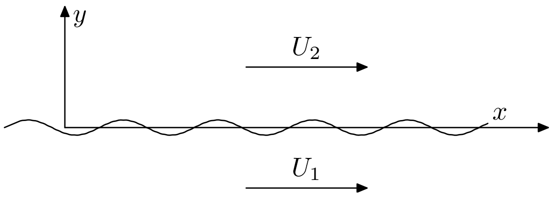


Figure 4.2: Notation for Kelvin–Helmholtz instability analysis

irrotational, then the velocity potential in the two regions of the flow is governed by the Laplace equation:

$$\nabla^2\phi_1 = 0, \quad \nabla^2\phi_2 = 0, \tag{4.1}$$

with the boundary conditions:

$$\nabla\phi_1 = U_1, \quad \text{as} \quad y \to -\infty, \tag{4.2a}$$

$$\nabla\phi_2 = U_2, \quad \text{as} \quad y \to \infty. \tag{4.2b}$$

When it is perturbed, the shear layer moves vertically to a displacement ξ where:

$$0 = F(x, \xi, z, t) = \xi - f(x, z, t). \tag{4.3}$$

There is a condition linking the vertical velocity of the interface with the velocity of the fluid travelling past it:

$$\frac{\partial F}{\partial t} + \mathbf{w}.\nabla F = 0, \tag{4.4}$$

giving:

$$-\frac{\partial f}{\partial t} + w_y - w_x\frac{\partial f}{\partial x} - w_z\frac{\partial f}{\partial z} = 0. \tag{4.5}$$

If fluid does not cross the shear layer, $w = v_1$, the vertical velocity of the fluid, and, at $y = \xi$:

$$\frac{\partial\phi_1}{\partial y} = v_1 = \frac{\partial\xi}{\partial t} + u_1\frac{\partial\xi}{\partial x} + w_1\frac{\partial\xi}{\partial z}. \tag{4.6}$$

Similarly:

$$\frac{\partial\phi_2}{\partial y} = v_2 = \frac{\partial\xi}{\partial t} + u_2\frac{\partial\xi}{\partial x} + w_2\frac{\partial\xi}{\partial z}. \tag{4.7}$$

The Bernoulli equation gives us:

$$\frac{\partial\phi_1}{\partial t} + \frac{\nabla\phi_1.\nabla\phi_1}{2} + c_1 = \frac{\partial\phi_2}{\partial t} + \frac{\nabla\phi_2.\nabla\phi_2}{2} + c_2. \tag{4.8}$$

These equations govern the flow on both sides of the shear layer and its movement. To make some progress in analyzing the problem, we linearize it in the usual way: break each variable into a mean and a (small) fluctuating part and neglect products of small parts.

With ξ small, the basic steady flow satisfies the constraints and the Bernoulli equation (4.8) gives:

$$c_1 - U_1^2/2 = c_2 - U_2^2/2. \tag{4.9}$$

In the unsteady case:

$$\phi_1 = U_1 x + \phi_1', \tag{4.10a}$$

$$\phi_2 = U_2 x + \phi_2', \tag{4.10b}$$

where ϕ'_1 and ϕ'_2 are small perturbations on the base potential. These potentials are also solutions of the incompressible flow equation:

$$\nabla^2\phi'_1 = \nabla^2\phi'_2 = 0, \tag{4.11}$$

and they decay to zero far from the shear layer:

$$\nabla\phi'_1 = 0, \quad \text{as} \quad y \to -\infty, \tag{4.12a}$$
$$\nabla\phi'_2 = 0, \quad \text{as} \quad y \to \infty. \tag{4.12b}$$

Linearizing the Bernoulli equation:

$$\frac{\partial\phi'_1}{\partial t} + U_1\frac{\partial\phi'_1}{\partial x} = \frac{\partial\phi'_2}{\partial t} + U_2\frac{\partial\phi'_2}{\partial x}, \tag{4.13}$$

and the displacement conditions:

$$\frac{\partial\phi'_1}{\partial y} = \frac{\partial\xi}{\partial t} + U_1\frac{\partial\xi}{\partial x}, \tag{4.14a}$$
$$\frac{\partial\phi'_2}{\partial y} = \frac{\partial\xi}{\partial t} + U_2\frac{\partial\xi}{\partial x}. \tag{4.14b}$$

We can solve the problem by assuming that the solution is made up of normal modes with:

$$\left\{\begin{array}{c}\xi\\ \phi'_1\\ \phi'_2\end{array}\right\} = \left\{\begin{array}{l}\Xi\\ \Phi_1(y)\\ \Phi_2(y)\end{array}\right\}\exp[\mathrm{j}(\alpha x + \beta z - \alpha a t)]. \tag{4.15}$$

This has the form of a wave which travels at speed a in the x direction, i.e. along the shear layer.

Imposing the boundary conditions gives:

$$\Phi_1(y) = A_1\mathrm{e}^{ky}, \tag{4.16a}$$
$$\Phi_2(y) = A_2\mathrm{e}^{-ky}, \tag{4.16b}$$
$$k = [\alpha^2 + \beta^2]^{1/2}.$$

and, inserting this solution into the interface constraint:

$$A_1 = \mathrm{j}\alpha(U_1 - a)\Xi/k, \tag{4.17a}$$
$$A_2 = -\mathrm{j}\alpha(U_2 - a)\Xi/k. \tag{4.17b}$$

The last part of the solution is to work out the wave speed. Using the Bernoulli relation:

$$(U_1 - a)^2 = -(U_2 - a)^2, \tag{4.18}$$

and

$$a = (U_1 + U_2)/2 \pm \mathrm{j}|U_2 - U_1|/2, \tag{4.19}$$
$$= a_R \pm \mathrm{j}a_I. \tag{4.20}$$

Inserting this into the definition of the modes, gives a wave which behaves as $\exp \mathrm{j}\alpha(x - a_R t)\exp \mp\alpha a_I t$, indicating a disturbance which travels at a speed a_R along the shear layer, growing or decaying exponentially at a rate $\exp \mp\alpha a_I t$. We are most interested in those which grow exponentially, and we conclude that a shear layer with $U_1 \neq U_2$ is *always* unstable, no matter what the wavelength of a disturbance.

Figure 4.3: Kelvin-Helmholtz instability made visible by clouds (Wikimedia Commons).

What happens next?

The analysis of the previous section is based on a linearization of the problem so it can only tell us about how instability starts: the system very quickly becomes non-linear. This should be obvious, since otherwise, the disturbance would grow without bound, which is not physically valid. So what does happen?

Figure 4.3 shows a Kelvin-Helmholtz instability in the atmosphere, where clouds in the shear layer have been deformed by the flow, showing the mixing process at work. Viewing from left to right, you can see the deformed shape of the clouds (interface) and also how the deformation grows until, at the right hand side of the image, the perturbation is too great for the system to behave linearly and the flow breaks down into turbulence.

Chapter 5

The generation of sound

So far we have been talking about fluid dynamical processes without considering how they might give rise to sound or noise. The link between a flow and the noise it generates is given by the wave equation, so we now have to think about solving the wave equation for particular systems.

Pulsating sphere

The simplest three-dimensional problem we can solve is that of sound radiated by a pulsating sphere. This sphere could be, for example, a bubble, a varying heat source or an approximation to a body of varying volume. The sphere has radius a and oscillates with velocity amplitude V at frequency ω. From the linearized momentum equation (2.12b), we can find a relationship between acceleration and pressure gradient:

$$\nabla p = -\rho_0 \frac{\partial \mathbf{v}}{\partial t}. \tag{5.1}$$

Writing the radial velocity of the sphere surface as $v = V \exp[-\mathrm{j}\omega t]$, we can see that p must also have frequency ω so that we can write it as $p = P \exp[-\mathrm{j}\omega t]$ and:

$$\nabla P \mathrm{e}^{-\mathrm{j}\omega t} = \mathrm{j}\omega\rho_0 V \mathrm{e}^{-\mathrm{j}\omega t}. \tag{5.2}$$

Since p is a solution of the wave equation, we know from (2.18, page 15) that:

$$p = \frac{f(t - r/c)}{r} = \frac{A\mathrm{e}^{-\mathrm{j}\omega(t-r/c)}}{r}, \tag{5.3}$$

where A is to be found from the boundary condition at a, the sphere surface. Writing out the pressure gradient:

$$\nabla p = \frac{A}{r^2}\left[\frac{\mathrm{j}\omega r}{c} - 1\right] \mathrm{e}^{-\mathrm{j}\omega(t-r/c)}, \tag{5.4}$$

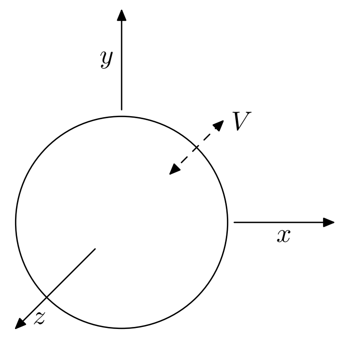


Figure 5.1: A pulsating spherical surface

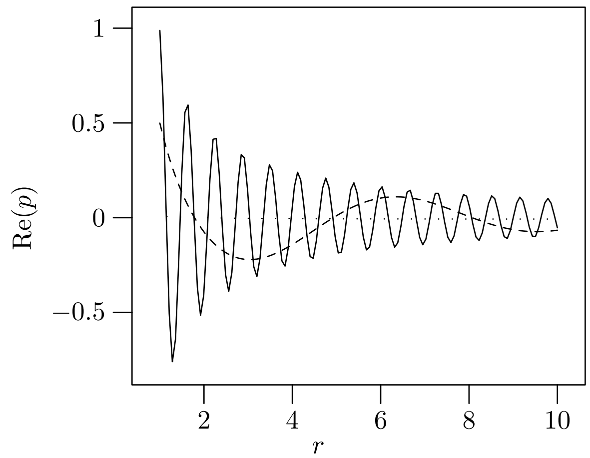


Figure 5.2: Sound field around a pulsating sphere: dotted $k = 0.1$; dashed $k = 1$; solid $k = 10$.

and applying the boundary condition:

$$\frac{A}{a^2}\left[\frac{\mathrm{j}\omega a}{c} - 1\right]\mathrm{e}^{-\mathrm{j}\omega(t-a/c)} = \mathrm{j}\omega\rho_0 V \mathrm{e}^{-\mathrm{j}\omega t}, \tag{5.5}$$

we can fix the constant A:

$$A = \frac{(ka)(ka-\mathrm{j})\rho_0 Vca}{(ka)^2+1}\mathrm{e}^{-\mathrm{j}ka}, \tag{5.6}$$

where $k = \omega/c$ is the *wavenumber*. The solution for the pressure is then:

$$p = \frac{ka}{r}\frac{ka-\mathrm{j}}{(ka)^2+1}(\rho_0 Vca)\mathrm{e}^{-\mathrm{j}k(r-a)}\mathrm{e}^{-\mathrm{j}\omega t}. \tag{5.7}$$

There are two approximations we can make which simplify this formula. When $ka \ll 1$ (i.e. when the sphere is small or it vibrates at low frequency), (5.7) can be written:

$$p \approx -\mathrm{j}\frac{\rho_0 cka^2}{r}V\mathrm{e}^{\mathrm{j}kr}\mathrm{e}^{-\mathrm{j}\omega t}; \tag{5.8}$$

when $ka \gg 1$ (i.e. when the sphere is large or vibrating at high frequency):

$$p \approx \frac{\rho_0 Vca}{r}\mathrm{e}^{-\mathrm{j}k(r-a)}\mathrm{e}^{-\mathrm{j}\omega t}. \tag{5.9}$$

The parameter ka, a non-dimensional combination of wavelength and a characteristic dimension of the body, is an important parameter of a source and is called the *compactness*. When ka is small, the source is point-like and can be treated as a simple source; when it is large, the acoustic field becomes more complicated, as in Figure 5.2.

Point sources

When we look at sound production by real systems, we cannot usually model them with simple shapes such as spheres. The solution for a sphere is useful, however, because we can use it to work out the noise radiated by a *point source*, a solution for the sound radiated by an infinitesimal element of a real system.

We start with (5.8), the result for a small oscillating sphere. We want to write this in terms of some "source strength". When the sphere oscillates, it is injecting momentum into the fluid. A sphere of radius

a has surface area $4\pi a^2$ and if it oscillates with velocity $V\exp[-\mathrm{j}\omega t]$, the momentum being injected at the surface of the sphere is:

$$M = \rho_0 4\pi a^2 V \mathrm{e}^{-\mathrm{j}\omega t} \tag{5.10}$$

and the rate of change of momentum is:

$$\frac{\partial M}{\partial t} = -\mathrm{j}\rho_0\omega 4\pi a^2 V \mathrm{e}^{-\mathrm{j}\omega t}. \tag{5.11}$$

Noting that $\omega = kc$, we can compare (5.11) to (5.8) and find that:

$$p = \frac{1}{4\pi}\frac{\partial M}{\partial t}\frac{\mathrm{e}^{\mathrm{j}kr}}{r}, \tag{5.12}$$

so that sound is generated by fluctuations in momentum. If write this in terms of a source strength $q = \rho_0 v(t)$, this equation can also be written:

$$p = \frac{\partial}{\partial t}\frac{q(t-R/c)}{4\pi R}, \tag{5.13}$$

which is the result for sound radiated by an infinitesimal point source. In a real problem, we can work out the sound from a source as a sum of contributions from point sources. This sum becomes an integral if we look at a distribution of sources over a volume V:

$$p(\mathbf{x},t) = \frac{\partial}{\partial t}\int_V \frac{q(\mathbf{y},t-R/c)}{4\pi R}\,\mathrm{d}V. \tag{5.14}$$

We can write this in a form which will be useful to us later:

$$p(\mathbf{x},t) = \frac{\partial}{\partial t}\int_V G(\mathbf{x},t;\mathbf{y},\tau)q(\tau)\,\mathrm{d}V, \tag{5.15}$$

where G is the Green's function for the problem. A Green's function is a fundamental solution, in this case the response due to a point source "firing" instantaneously. We can write the Green's function using the Dirac delta function $\delta(\,\cdot\,)$:

$$G(\mathbf{x},t;\mathbf{y},\tau) = \frac{\delta(t-\tau+R/c)}{4\pi R}, \tag{5.16}$$
$$R = |\mathbf{x}-\mathbf{y}|.$$

The delta function is a curious beast which is zero everywhere except at zero, where it jumps to an infinite value. The area under the delta function, however, is one. It has the property that:

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,\mathrm{d}x = f(x_0),$$

called the "sifting property". In the case of (5.16), this means that $t-\tau+R/c$ or, $\tau = t-R/c$. Here τ, the *retarded time*, is the time when sound leaves the source and t is the time when it arrives, so that R/c is the time delay between sound leaving a source and sound arriving at some point, which should be no surprise by now.

Sound from a circular piston

Taking a step up in difficulty (and realism), we now look at the sound radiated by a rigid piston embedded in a wall. This is a basic model of a loudspeaker and is related to a number of other problems in the acoustics

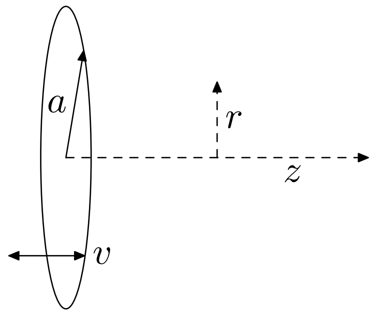


Figure 5.3: A rigid piston vibrating in a rigid wall.

of sound generation by moving surfaces. Figure 5.3 shows a rigid circular piston of radius a which vibrates periodically at frequency ω and velocity amplitude v so that its velocity is $v\exp[-\mathrm{j}\omega t]$. From (5.15):

$$p\mathrm{e}^{-\mathrm{j}\omega t} = 2\frac{\partial}{\partial t}\iint_S \frac{q(\mathbf{y},\tau)}{4\pi R}\,\mathrm{d}S,$$

where the factor 2 has been included to account for the image source in the wall and the integration is performed over the surface S of the piston. Given the velocity, the source $q = \rho_0 v\exp[-\mathrm{j}\omega t]$ so that the resulting integral for the radiated sound is:

$$p(\omega) = -\mathrm{j}\frac{\omega\rho_0}{2\pi}\iint_S \frac{\mathrm{e}^{\mathrm{j}kR}}{R}v\,\mathrm{d}S.$$

To evaluate the integral, we switch to cylindrical coordinates (r,θ,z):

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We assume that the observer is at $\theta = 0$ and the integral to be evaluated is:

$$p(\omega) = -\mathrm{j}\frac{\omega\rho_0 v}{2\pi}\int_0^{2\pi}\int_0^a \frac{\mathrm{e}^{\mathrm{j}kR}}{R}r_1\,\mathrm{d}r_1\,\mathrm{d}\theta_1,$$
$$R = (r^2 + r_1^2 - 2rr_1\cos\theta_1 + z^2)^{1/2},$$

where (r_1,θ_1) indicates a point on the piston surface.

This integral cannot be evaluated exactly for a general observer position but we can restrict it to the case where the observer is on the axis of the piston. Then $r = 0$ and $R = (r_1^2 + z^2)^{1/2}$:

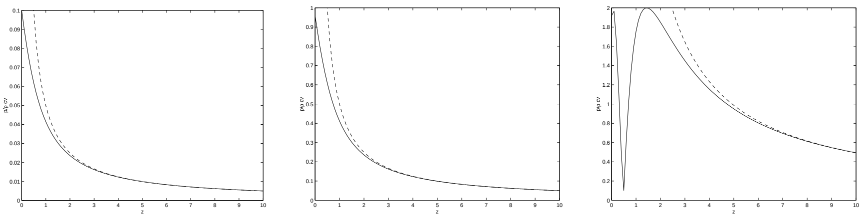
$$p = -\mathrm{j}\frac{\omega\rho_0 v}{2\pi}\int_0^{2\pi}\int_0^a \frac{\mathrm{e}^{\mathrm{j}kR}}{R}r_1\,\mathrm{d}r_1\,\mathrm{d}\theta_1,$$
$$= -\mathrm{j}\omega\rho_0 v\int_0^a \frac{\mathrm{e}^{\mathrm{j}kR}}{R}r_1\,\mathrm{d}r_1,$$

and making the transformation $r_1 \rightarrow R$,

$$p = -\mathrm{j}\omega\rho_0 v\int_{R_0}^{R_a} \mathrm{e}^{\mathrm{j}kR}\,\mathrm{d}R.$$

Here, $R_0 = z$ is the distance from the observer to the centre of the piston and $R_a = (a^2 + z^2)^{1/2}$ is the distance to the rim of the piston. The solution is then:

$$p = -\rho_0 cv(\mathrm{e}^{\mathrm{j}kR_a} - \mathrm{e}^{\mathrm{j}kz}). \tag{5.17}$$



a: $ka = 0.1$ b: $ka = 1.0$ c: $ka = 10.0$

Figure 5.4: Acoustic field (absolute value of p) along the axis of a vibrating piston. The dashed line shows the $1/z$ fit.

If we examine the acoustic field defined by (5.17) as a function of frequency, we can see that it changes quite rapidly as ka is increased. Figure 5.4 shows the absolute value of the non-dimensional pressure $|p/\rho_0 cv|$ for different values of ka. For comparison, the curve $1/R_0 = 1/|z|$ is also shown. The results for $ka = 0.1$ and $ka = 1$ are similar with a smooth $1/R_0$ decay but the $ka = 10$ curve is quite different, having a sharp drop before it begins to follow a $1/R_0$ curve. This is a result of interference between sound from different parts of the piston. When a body is large compared to the wavelength of the sound it generates, interference between different parts of the body gives rise to a complicated sound pattern, especially in the region near the body. When the body is small on a wavelength scale (or, equivalently, vibrates at low frequency), the phase difference between different parts of the source is not enough to give rise to much interference and the body radiates like a point source. The 'size' of the body at a given frequency is called its *compactness* and is characterized by the parameter ka where a is a characteristic dimension, or by the ratio of characteristic dimension to wavelength a/λ. A compact source, one with $ka \ll 1$, radiates like a point source, while non-compact bodies must be treated in more detail, as we saw in the case of a sphere.

Example: Noise from aircraft engines

The formula for sound radiated from an oscillating piston can also be used as an approximation for low frequency noise from flanged pipes. If we slightly abuse the formula, we can use it to make a guess at the noise from the end of a duct, such as an aircraft engine intake (or a cooling tower or all sorts of other things). The internal processes in an engine, such as the rotation of the fan, generate an oscillating velocity at the intake. We can pretend that this is a piston spanning the face of the intake and calculate the radiated noise using the formula derived above.

Asymmetric sources

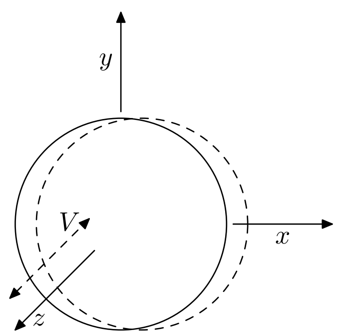


Figure 5.5: A juddering spherical surface

The next step in complexity is to consider what would happen if the sphere at the start of the chapter juddered, rather than oscillating radially. Figure 5.5 shows the geometry: the sphere oscillates along the z axis with frequency ω and velocity amplitude V so that the boundary condition for radial velocity is $v = V\cos\theta\exp[-\mathrm{j}\omega t]$. The angle θ is measured from the z axis. Since the pressure gradient and velocity are proportional we know already that:

$$\nabla p = \mathrm{j}\omega\rho_0 V\mathrm{e}^{-\mathrm{j}\omega t}\cos\theta.$$

This looks a bit like the boundary condition for the oscillating sphere so we can try using that solution to get a solution for our new problem.

First of all, we need to note that:

$$\nabla^2\frac{\partial p}{\partial z} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{\partial p}{\partial z} = 0, \tag{5.18}$$

or, in other words, given a solution p of the wave equation, $\partial p/\partial z$ is also a solution. Now, consider a solution of the wave equation $p = f(t - r/c)/r$. If we differentiate with respect to z:

$$\frac{\partial p}{\partial z} = -\frac{1}{c}\frac{f'}{r}\frac{\partial r}{\partial z} - \frac{f}{r^2}\frac{\partial r}{\partial z}, \tag{5.19}$$

$$r^2 = x^2 + y^2 + z^2,$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta,$$

$$\text{so}\quad \frac{\partial p}{\partial z} = \cos\theta\frac{\partial p}{\partial r}. \tag{5.20}$$

In other words, differentiating with respect to z gives us a factor of $\cos\theta$ which will help us meet the boundary condition.

As before, we start with a solution which satisfies the wave equation, but with the addition of differentiation with respect to z:

$$P\mathrm{e}^{\mathrm{j}\omega t} = \frac{\partial}{\partial z}\left[\frac{A\mathrm{e}^{\mathrm{j}\omega(t-r/c)}}{r}\right], \tag{5.21}$$

and apply the boundary condition:

$$\frac{\partial}{\partial r}\left(\frac{\partial}{\partial z}\left[\frac{A\mathrm{e}^{-\mathrm{j}\omega(t-r/c)}}{r}\right]\right) = \mathrm{j}\omega\rho_0 V\mathrm{e}^{-\mathrm{j}\omega t}\cos\theta, \quad r = a, \tag{5.22}$$

which gives us:

$$A\frac{\partial^2}{\partial r^2}\left[\frac{\mathrm{e}^{\mathrm{j}kr}}{r}\right] = \mathrm{j}\omega\rho_0 V, \tag{5.23}$$

and upon expanding the derivative and setting $r = a$:

$$A\mathrm{e}^{\mathrm{j}ka}\left[2 - (ka)^2 - \mathrm{j}2ka\right] = \mathrm{j}\omega\rho_0 V a^3, \tag{5.24}$$

$$A = \frac{\mathrm{j}\omega\rho_0 V a^3}{2 - (ka)^2 - \mathrm{j}2ka}\mathrm{e}^{-\mathrm{j}ka}. \tag{5.25}$$

This gives the pressure field radiated by a juddering sphere as:

$$p = \frac{\partial}{\partial z}\frac{A\mathrm{e}^{-\mathrm{j}\omega(t-r/c)}}{r} \tag{5.26}$$

Differentiation with respect to z has changed the form of the acoustic field. Whereas the pulsating sphere generated a field which is symmetric, the juddering sphere has a field which is a function of θ

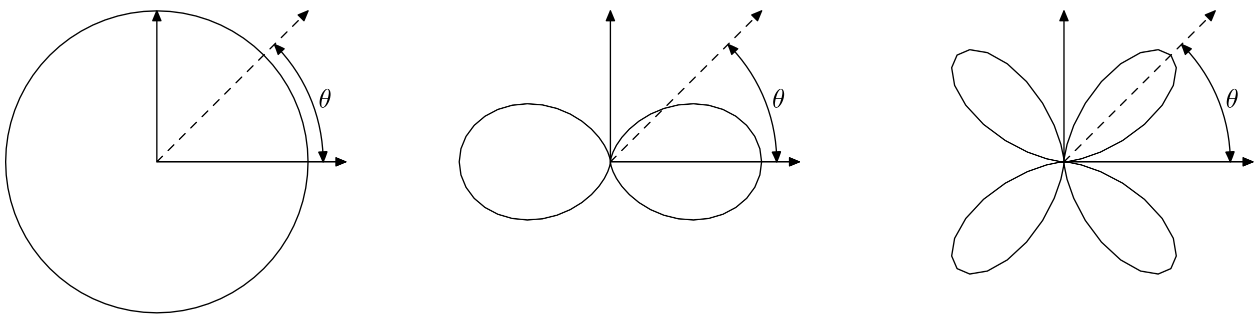


Figure 5.6: Monopole, dipole and quadrupole source directivities

with maxima at $\theta = 0$ and π and zeros at $\theta = \pm\pi/2$. We refer to this as a *dipole* source, because it radiates like two simple sources of opposite sign placed close together. Further differentiation gives higher order sources, called *quadrupoles*, *octupoles*, etc. Figure 5.6 shows the amplitude of the radiated field as a function of angle for the first three orders of source. On the whole, we are most interested in monopole and dipole sources, which correspond to applied velocities and forces on surfaces, but quadrupole sources turn out to be very important in noise from turbulence.

Questions

1. In the far field, $R \gg a$, $R \gg ka$, we can estimate the sound radiated off-axis by a piston, using the following approximations:

$$\frac{1}{R} \approx \frac{1}{R_0},$$
$$R \approx R_0 - r_1 \sin\phi \cos\theta_1$$

   where $\phi = \tan^{-1} r/z$ and $R_0 = [r^2 + z^2]^{1/2}$. Given that the *Bessel function* of zero order is:

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{j}x\cos\theta_1}\,\mathrm{d}\theta_1,$$

   and that:

$$\int xJ_0(x)\,\mathrm{d}x = xJ_1(x),$$

   where $J_1(x)$ is the Bessel function of first order, derive an approximate formula for the far field noise radiated by a piston.

2. A circular loudspeaker of radius $30\mathrm{mm}$ is driven at 200Hz. Using the result of Question 1, estimate the angle from the speaker axis at which the radiated sound is 20dB less than on axis (graphical accuracy is sufficient). What happens at 2kHz? Why?

3. Sketch the amplitude of the noise radiated from a piston as a function of θ.

4. Prove (5.18).

Chapter 6

The propagation of sound

Sound rarely propagates in free space so this chapter looks at how sound is modified by the environment, including reflection by surfaces, passage through walls, and travelling through ducts.

Reflection by a hard wall

The simplest realistic problem of interest involving the effect of a boundary on a sound field is that of the interaction of the field from a point source with a plane wall, Figure 6.1. The problem is, given a source at a point $\mathbf{x}$, near a rigid plane, to calculate the resulting overall sound field. If the wall were not present, we know that the sound field at a frequency ω would have the form:

$$p_{\mathrm{i}}\mathrm{e}^{-\mathrm{j}\omega t} = \frac{\mathrm{e}^{-\mathrm{j}\omega(t-R/c)}}{4\pi R},$$

where p_{i} is the *incident* sound field.

We will drop the factor $\exp[-\mathrm{j}\omega t]$ because it is the same for all sound fields in the problem and write:

$$p_{\mathrm{i}} = \frac{\mathrm{e}^{\mathrm{j}kR}}{4\pi R}.$$

Our problem now is to find a second acoustic field p_{s} (the 'scattered' field), such that the total field $p_{\mathrm{t}} = p_{\mathrm{i}} + p_{\mathrm{s}}$ satisfies the wave equation and the boundary conditions on the wall. By linearity, this means that p_{s} must be a valid solution of the wave equation, since the sum of two solutions is itself a solution. Now we need to decide what boundary condition to apply. As in inviscid fluid dynamics, the boundary condition is that the total velocity normal to the wall must be zero. From the momentum equation, we know that the acoustic velocity is proportional to the pressure gradient, so this boundary condition is equivalent to

$$\left.\frac{\partial p_{\mathrm{t}}}{\partial x}\right|_{x=0} \equiv 0,$$

or, in terms of the incident and scattered fields,

$$\left.\frac{\partial p_{\mathrm{s}}}{\partial x}\right|_{x=0} \equiv -\left.\frac{\partial p_{\mathrm{i}}}{\partial x}\right|_{x=0}.$$

Figure 6.1: A point source near a wall

For a source at $\mathbf{x}_0 = (x_0, y_0, z_0)$,

$$\frac{\partial p_\mathrm{i}}{\partial x} = \frac{x - x_0}{4\pi}\frac{\mathrm{e}^{\mathrm{j}kR}}{R^3}\left(\mathrm{j}kR - 1\right),$$

and at $x = 0$,

$$\left.\frac{\partial p_\mathrm{i}}{\partial x}\right|_{x=0} = -\frac{x_0}{4\pi}\frac{\mathrm{e}^{\mathrm{j}kR}}{R^3}\left(\mathrm{j}kR - 1\right),$$
$$R = [x_0^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}.$$

The solution of our problem is an acoustic field p_s with

$$\left.\frac{\partial p_\mathrm{s}}{\partial x}\right|_{x=0} = \frac{x_0}{4\pi}\frac{\mathrm{e}^{\mathrm{j}kR}}{R^3}\left(\mathrm{j}kR - 1\right).$$

A source positioned at $\mathbf{x}_- = (-x_0, y_0, z_0)$ gives just such a field so a valid solution to the problem can be found using an *image source*, the reflection of our orginal source in the rigid wall. The total field is then

$$p_\mathrm{t} = p_\mathrm{i} + p_\mathrm{s},$$
$$p_\mathrm{i} = \frac{\mathrm{e}^{\mathrm{j}kR_+}}{4\pi R_+},$$
$$p_\mathrm{s} = \frac{\mathrm{e}^{\mathrm{j}kR_-}}{4\pi R_-},$$
$$R_\pm = [(x \mp x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}.$$

One immediate result of this analysis is that the pressure generated on the wall by a source is twice that which would be generated if the wall were not present. This has two immediate applications: the first is that excessive noise in confined spaces (discotheques and clubs, for example) can be extremely damaging to hearing; the second is where the 'wall' is the ground and we want to know how noise propagates across a landscape.

Reflection by a soft wall

A concept which is very useful and we will need later on is that of *acoustic impedance*. This is like the impedance we see in mechanical systems and is defined as the ratio of acoustic pressure to acoustic velocity:

$$Z = \frac{P}{V}. \tag{6.1}$$

The acoustic impedance of a material (including gases and liquids) is a property of the material and of frequency. We usually work in terms of *specific acoustic impedance* which is simply Z/A where A is the area of material.

For a hard wall, $V = 0$ and the impedance is infinite. For a substance which is porous, the effect of flow into the pores of the material must be taken into account. We can model this by lumping the material properties together into a single impedance, which means that we do not necessarily need to know very much else about the substance. Note that Z is a function of frequency.

If we examine reflection of a plane wave from a wall with some finite impedance, we can look at the problem of acoustic treatment of rooms. In order to line a room to stop reflections (for music recording or performances, say), we want to minimize reflections so we need to know how much sound is reflected from a wall for a given impedance. Figure 6.2 shows the incoming and reflected waves. The pressure and velocity are given by:

$$P = \mathrm{e}^{\mathrm{j}k_y y}\left(\mathrm{e}^{\mathrm{j}k_x x} + R\mathrm{e}^{-\mathrm{j}k_x x}\right), \tag{6.2}$$
$$V = \frac{\mathrm{e}^{\mathrm{j}k_y y}}{\rho c}\left(\mathrm{e}^{\mathrm{j}k_x x} - R\mathrm{e}^{-\mathrm{j}k_x x}\right)\cos\theta, \tag{6.3}$$

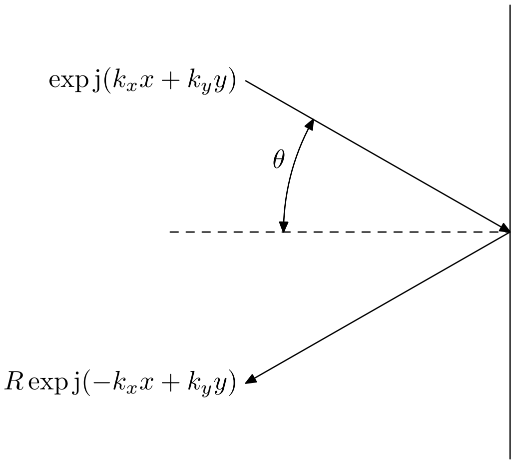


Figure 6.2: Reflection from a finite impedance wall

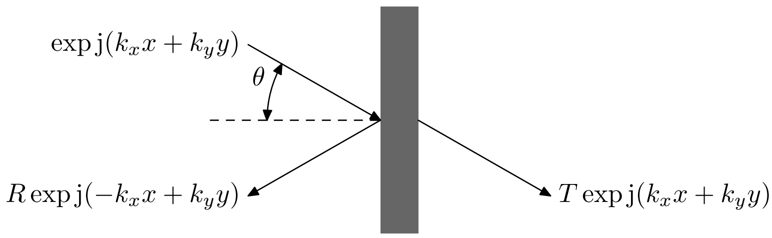


Figure 6.3: A slab of material under acoustic excitation

where the $\cos\theta$ is needed to extract the component of velocity normal to the wall—sound propagating parallel to the wall will not be affected by the impedance. The boundary condition on the wall is that $Z = P/V$ so we can write:

$$R = \frac{Z\cos\theta - \rho c}{Z\cos\theta + \rho c}. \tag{6.4}$$

Example: How to bug an embassy

One type of 'soft' wall is a slab of material which vibrates in response to acoustic pressure. Figure 6.3 shows the arrangement: a slab or sheet of material is subject to a plane wave. We want to know the complex amplitude R of the reflected wave and the amplitude T of the wave transmitted out the other side of the material. For a thin, non-deforming slab, we can assume that the velocities on each side of the slab are equal:

$$v_i = v_t, \tag{6.5}$$

and we know from the definition of impedance that:

$$P_i - P_t = Z_{\text{sl}} v_i = Z_{\text{sl}} v_t. \tag{6.6}$$

From (6.4), the reflection coefficient on the incoming wave side is

$$R = \frac{Z_i - Z_1}{Z_i + Z_1}, \tag{6.7}$$

where the local impedance $Z_1 = \rho c/\cos\theta$. This means that the velocity on side 1 is:

$$v_1 = \frac{p_1}{Z_1}(1 - R), \quad (6.8)$$
$$= \frac{2P_i}{2Z_1 + Z_{\text{sl}}}. \quad (6.9)$$

Given that the normal velocity is equal on both sides, we can work out the amplitude of the transmitted wave:

$$T = Z_1 V_2 = \frac{2\rho c/\cos\theta}{Z_{\text{sl}} + 2\rho c/\cos\theta}. \quad (6.10)$$

In 1987, *Time* reported that the Soviet Union might be using lasers to measure the vibrations of the windows of the US embassy in Moscow as a way of listening to conversations inside[1]. A modern laser vibrometer can measure velocities to a resolution of about $0.01\,\mu\text{m/s}$. If a window pane is 5mm thick, what is the quietest conversation we can listen to?

A simple assumption is that the glass acts as a *limp plate* and the only resistance to motion is the slab inertia. Then, for a plate of mass per unit area m moving at a frequency ω

$$-\text{j}\omega V m = P_i - P_t \quad (6.11)$$

and $Z_{\text{sl}} = -\text{j}\omega m$. The transmitted wave then has amplitude:

$$|T| = \left[1 + \left(\frac{\omega m}{2\rho c}\right)^2 \cos^2\theta\right]^{-1/2}.$$

From (6.8), and assuming $\theta = 0$,

$$v = \frac{2P_i}{2\rho c - \text{j}\omega m}.$$

If we are interested in sound at around 3kHz (roughly in the middle of the range of human speech), given that the density of glass is about 2500kg/m^3, $m = 12.5\text{kg/m}^2$ and:

$$v = \frac{2}{1.2 \times 340 - \text{j}2\pi \times 3000 \times 12.5} P_i = \frac{1}{204 - \text{j}1.178 \times 10^5} P_i$$

and

$$|v| = |P_i|/1.178 \times 10^5.$$

If we assume we can measure the velocity over a range of $1\,\mu\text{m/s}$,

$$|P_i| = 1.178 \times 10^5 \times 10^{-6}\text{Pa} = 75\text{dB}.$$

For comparison, the *sound* transmitted on the other side of the window would be TP_i which has magnitude:

$$|TP_i| = \left[1 + \left(\frac{\omega m}{2\rho c}\right)^2 \cos^2\theta\right]^{-1/2} P_i,$$
$$= 1.178 \times 10^5 \times 10^{-6}/289\text{Pa} = 26.2\text{dB}.$$

It might be possible to measure this signal very close to the window, but at a distance of 100m it would be impossible. A sophisticated laser system, however, could measure the window's vibrations from a distance of hundreds of meters. It is interesting to know that the American embassy in Moscow was surrounded by four taller buildings, and it was reported that:

[1]The article is available online at: `http://www.bugsweeps.com/info/hitech_snooping.html`

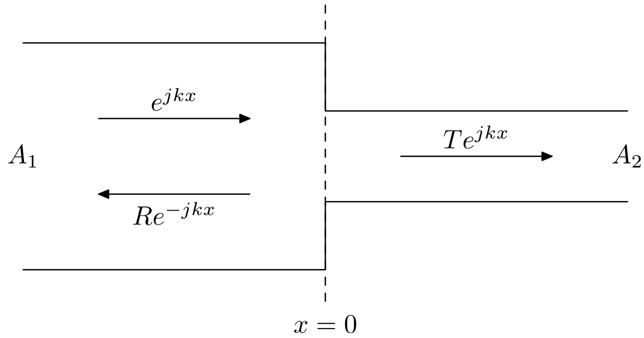


Figure 6.4: Change in duct section

> The Soviets' new Washington embassy, built on a high hill, is perfectly placed to beam laser light from a generator as small as a flashlight toward windows, catching the conversations going on behind them. John Pike of the Federation of American Scientists says that "the White House has put little noisemakers on its windows" to foil the eavesdropping, which can also be hindered by heavy drapes.[2]

Ducts and silencers

Figure 6.4 shows a simple example of propagation along a duct whose section changes suddenly. If a wave of the form $\exp(\mathrm{j}kx)$ propagates to the right and hits the change in section, there is a reflected wave $R\exp(-\mathrm{j}kx)$ which propagates to the left and a transmitted wave $T\exp(\mathrm{j}kx)$ which carries on to the right past the change in section.

For low-frequency applications, we can assume that the only thing that matters is the change in area going from one section to the next. If the initial part of the duct has area A_1 and the second part area A_2, the boundary conditions at the change in section $x = 0$ are continuity of pressure and conservation of mass. The first of these conditions is simple; the second requires that the volume flow rate be conserved across the interface, so that $A_1U_1 = A_2U_2$ where U is acoustic velocity, which we can relate to the acoustic pressure using the momentum equation. Setting $x = 0$, the boundary conditions are then:

$$1 + R = T, \tag{6.12a}$$

$$A_1(1 - R) = A_2T. \tag{6.12b}$$

Solving for R and T, we find that:

$$R = \frac{A_1 - A_2}{A_1 + A_2}, \tag{6.13a}$$

$$T = \frac{2A_1}{A_1 + A_2}. \tag{6.13b}$$

Note that when $A_2 \to \infty$, $R \to -1$ and $T \to 0$ so that, on this theory, an open-ended duct reflects the whole signal back from the end and no sound escapes. As might be expected, when $A_2 = A_1$, $R = 0$ and $T = 1$ so the sound travels unaffected.

An application of changes in duct section is the exhaust muffler, such as those seen on the motorcycles of thoroughly respectable acoustics lecturers or on the exhaust pipes of noisy brats. The simplest form of muffler, Figure 6.5, is simply a section of pipe with a greater cross-sectional area than the rest of the pipe.

A muffler has two functions: to reduce the noise radiated into the surroundings (which is why vehicles are obliged to have them) and to increase the engine power (which is why people fit new ones). The first

[2]*Newsweek*, 20 April 1987, http://www.bugsweeps.com/info/battle_of_bugs-newsweek-04-20-87.html

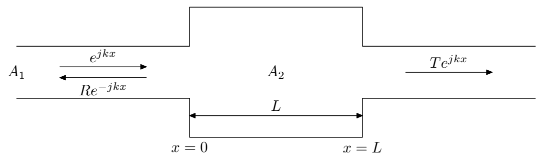


Figure 6.5: A simple exhaust muffler

function is fulfilled by modifying the pressure field which reaches the open end of the exhaust, the second by imposing a reflected wave which alters slightly the exhaust characteristics of the engine cylinder.

The muffler shown in Figure 6.5 is the simplest device we can imagine but it will give us an idea of the behaviour of a realistic system. We need boundary conditions at $x = 0$ and at $x = L$. The pressure and continuity conditions at $x = 0$ are:

$$1 + R = T_2 + R_2, \tag{6.14a}$$
$$A_1(1 - R) = A_2(T_2 - R_2), \tag{6.14b}$$

and at $x = L$:

$$T_2 e^{jkL} + R_2 e^{-jkL} = Te^{jkL}, A_2(T_2 e^{jkL} - R_2 e^{-jkL}) = A_1 Te^{jkL}. \tag{6.15a}$$

Rearranging these equations, we can eliminate T_2 and R_2 (we are not very interested in what happens inside the muffler) to find T, the transmitted wave. Combining (6.14) yields:

$$(A_2 + A_1) - (A_1 - A_2)R = 2A_2T_2,$$
$$(A_2 - A_1) + (A_2 + A_1)R = 2A_2R_2,$$

and, writing $m = A_2/A_1$:

$$(m + 1) + (m - 1)R = 2mT_2,$$
$$(m - 1) + (m + 1)R = 2mR_2.$$

Similarly (6.15) can be combined:

$$2mT_2 e^{jkL} = (m + 1)Te^{jkL},$$
$$2mR_2 e^{-jkL} = (m - 1)Te^{jkL}.$$

We can eliminate R_2 and T_2 to find the transmitted wave:

$$T = \frac{\cos kL - j \sin kL}{\cos kL - j(m + m^{-1})/2 \sin kL} \tag{6.16}$$

The most interesting thing to know from an environmental point of view is the magnitude of the transmitted wave:

$$|T| = \left(1 + \frac{(m - m^{-1})^2}{4} \sin^2 kL\right)^{-1} \tag{6.17}$$

Looking at this equation, we can see that the transmitted wave amplitude is minimized for certain values of kL, if we take m fixed. The net effect is that the muffler acts as a low pass filter.

We can also calculate the reflected wave amplitude:

$$R = \frac{m + 1}{m - 1}(T - 1), \tag{6.18}$$

showing that quite a strong wave is reflected back into the engine. With the correct timing, which depends on the length of the exhaust pipe leading up to the muffler, this can increase the engine power slightly.

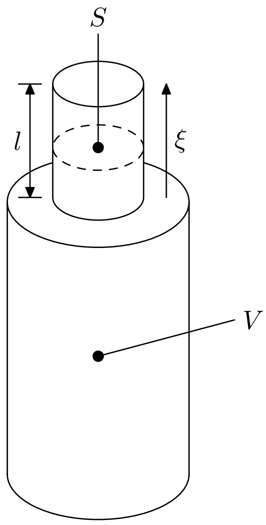


Figure 6.6: Helmholtz' bottle

The Helmholtz resonator

One of the most important resonant systems is the *Helmholtz resonator*, the classic example of which is the wine or beer bottle. It is modelled, Figure 6.6, as a volume V connected to the outside world by a neck of length l and cross-sectional area S. We can estimate the resonant frequency of the system by considering the motion of a 'plug' of fluid in the neck of the bottle under the action of an external force and an internal restoring force due to the compressibility of the fluid in the bulb.

Assuming that the process is adiabatic, the density and pressure in the bulb are related by:

$$p = k\rho^\gamma; \quad \frac{\mathrm{d}p}{\mathrm{d}\rho} = c^2.$$

If the plug of fluid in the neck of the bottle is displaced by an amount ξ (assumed positive out of the neck), the volume of fluid inside the bulb changes by an amount $S\xi$. Using subscript 0 to indicate mean values, the resulting change in density is:

$$\begin{aligned}\frac{\rho}{\rho_0} &= \frac{V}{V - S\xi},\\ &= \frac{1}{1-(S/V)\xi},\\ &\approx 1 - \frac{S}{V}\xi,\end{aligned}$$

by the binomial theorem and the corresponding change in pressure is:

$$p - p_0 = -\rho_0\frac{c^2S}{V}\xi.$$

The equation of motion for the plug can then be written, noting that its mass $m = \rho_0 Sl$:

$$\rho_0 Sl\ddot{\xi} + \rho_0\frac{c^2S^2}{V}\xi = -p_a S,$$

where p_a is the externally applied pressure. This is the equation of motion for an oscillator with a resonant frequency:

$$\omega = \sqrt{\frac{c^2S}{Vl}}.$$

Helmholtz resonators can be used whenever you want to reduce noise at some known frequency. One of the main applications is in acoustic liners used in aircraft engines, which are made up of a large number of small Helmholtz resonators with dimensions chosen to absorb noise at a specified frequency.

Sound from a wine bottle

A wine bottle has internal volume $V \approx 7.5 \times 10^{-4}\mathrm{m}^3$ and a neck of length $l \approx 0.05\mathrm{m}$ and cross-sectional area $S \approx 7.854 \times 10^{-5}\mathrm{m}^2$. The resonant frequency is then about $492\mathrm{rad/s}$, or 78Hz.

Sound in circular ducts

One of the main acoustical systems which appears in applications is the circular duct: ventilation systems, pipelines, and aircraft engines can all be represented, at least approximately, by a duct of circular cross section. This makes the system important in its own right, and as a model problem for the three-dimensional field inside a channel.

For a circular duct, it makes sense to write the Helmholtz equation (2.14, page 13) in cylindrical coordinates (r, θ, z) where z is displacement along the duct axis and r and θ are polar coordinates. Using the definition of ∇^2 in cylindrical coordinates (page 60):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2} + \frac{\partial^2 P}{\partial z^2} + k^2 P = 0. \tag{6.19}$$

We can derive solutions for this equation by separation of variables, writing $P = R(r)\Theta(\theta)Z(z)$ and:

$$\frac{\Theta Z}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{ZR}{r^2}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} + R\Theta\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} + k^2 R\Theta Z = 0, \tag{6.20}$$

$$\frac{1}{R}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{\Theta}\frac{1}{r^2}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} + \frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} + k^2 = 0. \tag{6.21}$$

Each term in this equation is a function of one variable only, or a constant. If we differentiate with respect to z:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} + k^2\right) = 0, \tag{6.22}$$

so $\mathrm{d}^2 Z/\mathrm{d}z^2/Z + k^2$ is a constant which we will call α_n^2. Then:

$$\frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} + k^2 = \alpha_n^2, \tag{6.23}$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} + (k^2 - \alpha_n^2)Z = 0, \tag{6.24}$$

$$Z = \mathrm{e}^{\pm \mathrm{j}\beta_n z}, \tag{6.25}$$

where $\beta_n = (k^2 - \alpha_n^2)^{1/2}$. We know that the solution in θ must be periodic, so we can write $\Theta = \cos m\theta$ or $\Theta = \sin m\theta$ and $\mathrm{d}^2\Theta/\mathrm{d}\theta^2 = -m^2$. This gives:

$$\frac{1}{R}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{m^2}{r^2} + \alpha_n^2 = 0, \tag{6.26}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \left(\alpha_n^2 - \frac{m^2}{r^2}\right)\right]R = 0. \tag{6.27}$$

Happily, (6.27) is a standard differential equation with a solution that has a name: the Bessel function $J_m(\alpha_n r)$.

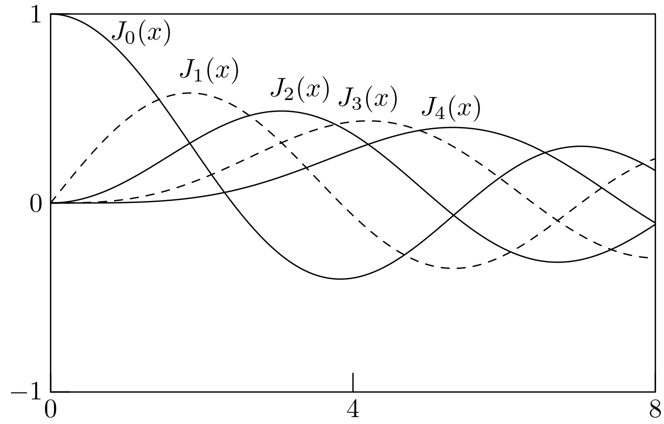


Figure 6.7: The first few Bessel functions $J_n(x)$

The first few Bessel functions are plotted in Figure 6.7, where you can see that they look a bit like decaying sinusoids. The only requirement we must impose is that the pressure gradient be zero at $r = a$, the duct wall. This means that only certain values of α_n are permissible in the solution, defined by $\alpha_n J'_m(\alpha_n a) = 0$. If we define the zeros as α_{mn} with $\alpha_{mn} J'_m(\alpha_{mn}) = 0$, the solution for the field in the duct is:

$$P = J_m(\alpha_{mn} r/a) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} \mathrm{e}^{\pm \mathrm{j}\beta_n z}. \tag{6.28}$$

From (6.28), we can see that some modes propagate with amplitude $\exp \mathrm{j}|\beta|z$ and are oscillatory in z. Others have amplitude $\exp[-|\beta|z]$ and decay exponentially with z. These modes are said to be 'cut-off' and to all intents and purposes do not propagate. The frequency below which they do not propagate is called the *cut-off frequency*. In most applications, it is only necessary to consider the 'cut-on' modes since the cut-off modes do not propagate within, or out of, the duct.

Radiation from ducts

The question we are usually most interested in is how much noise escapes from the pipe. The full solution of this problem, calculating the field radiated from a mode which reaches the end of the duct, is a hard piece of work in applied mathematics and beyond the scope of these lecture notes. There is, however, a reasonably good approximation in which we model the end of the duct as a piston with an oscillating velocity distribution, the *Rayleigh approximation*, which you will work out in Question 3.

Source filtering

A question of some philosophical and practical importance is that of the relationship between a source and its acoustic field. The propagation of sound has a filtering effect on the source, in the sense that only 'part' of the 'source' contributes to the radiated sound. For a concrete example, consider a source $\exp[\pm \mathrm{j}k_x x]$ with wavenumber k distributed over the plane $z = 0$. The resulting acoustic field (Junger and Feit, 1993, p131–2), has the form:

$$p \propto \frac{\mathrm{e}^{\mathrm{j}(k^2 - k_x^2)^{1/2} z}}{(k^2 - k_x^2)^{1/2}}, \tag{6.29}$$

which is obviously exponentially small when $k < |k_x|$. In other words, only source terms with supersonic phase speed, $|k_x| > k$, propagate. Components of the source with subsonic phase speed are 'filtered' out and make no contribution to the field. This is a very general result and applies to sources of many kinds, including, as we will see later, turbulent jets, which are very inefficient radiators. It also throws into question the idea of a 'source'. If we consider a source made up of a combination of terms of the form $\exp[\pm \mathrm{j}k_i x]$, we can add as many terms as we like with $|k_i| > k$ without changing the acoustic field. In what sense, then, can we define the source of the field? Alternatively, given an acoustic field, can we define its source using only acoustic measurements?

Questions

1. A point source of wavenumber k is placed near a pressure release surface, on which the boundary condition is that the pressure be zero. Calculate the effect of the boundary on the radiated sound.

2. The density of Perspex is about $1200\mathrm{kg/m^3}$. Estimate the attenuation of a normal wave of frequency 100Hz, transmitted through an aircraft window of thickness $5\mathrm{mm}$. Perform the same calculation for an aluminium (density $2700\mathrm{kg/m^3}$) wall of thickness $2\mathrm{mm}$. Which path reduces the cabin noise most and what would be the first easy way to reduce the noise inside the aircraft? What happens to noise at 1kHz?

3. Given that the Bessel function of order n can be represented:

$$J_n(x) = \frac{\mathrm{j}^n}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{j}(n\theta_1 - x\cos\theta_1)}\, \mathrm{d}\theta_1,$$

   estimate the acoustic field radiated by a piston with velocity distribution $J_n(\alpha_{mn} r/a)\exp[n\theta_1]$, as in Question 1 of Chapter 5. Sketch the distribution of noise as a function of angle from the piston axis. This is the Rayleigh approximation for the propagation of sound from a duct termination and is a reasonably good approximation for points past the end of the duct.

Chapter 7

The Proper Orthogonal Decomposition

One of the most important modern techniques in the study of turbulence is the Proper Orthogonal Decomposition (POD), which is a method for giving us information about a system, which is in some sense optimal. The POD is a *decomposition*: it breaks the flow down into a set of modes; its is *orthogonal*: the modes are unique; and it is *proper*: the modes which it finds are ordered by the energy which they contain. The POD gives a modal decomposition of the flow, capturing a given proportion of the flow energy with the smallest possible number of modes. A standard introduction to the POD is given by Berkooz et al. (1993), but we will start with some basic ideas and examples.

Inner products and orthogonality

The first concept which we have to deal with is orthogonality. You have probably come across this idea in the past without being told what it was, so take a simple example. If we have two vectors in the plane, $\mathbf{x}$ and $\mathbf{y}$, we can take their dot product, $\mathbf{x}.\mathbf{y}$. If this is identically zero, the vectors are orthogonal, with the usual geometrical meaning of being perpendicular to each other.

We can say the same thing in three dimensions, and in as many dimensions as we like, if we define the dot product:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{N} x_i y_i, \tag{7.1}$$

where N is the dimension of the space, and we now call $\langle \mathbf{x}, \mathbf{y} \rangle$ the *inner product*. As before, the vectors are orthogonal if $\langle \mathbf{x}, \mathbf{y} \rangle \equiv 0$. If we want, we can add some weighting to different components of the vectors:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{N} w_i x_i y_i, \tag{7.2}$$

and we can define the magnitude of a vector in the obvious way:

$$|\mathbf{x}| = \left(\langle \mathbf{x}, \mathbf{x} \rangle\right)^{1/2}, \tag{7.3}$$

so that the vector $\mathbf{x}/|\mathbf{x}|$ will have unit length.

If we have a set of vectors $\mathbf{x}_i$ which are orthogonal to each other, and which are all of unit length, $|\mathbf{x}_i| \equiv 1$, we have an *orthonormal basis*, which can be used to represent any vector in our domain:

$$\mathbf{y} = \sum_{i}^{N} a_i \mathbf{x}_i. \tag{7.4}$$

In two dimensions, $(1, 0)$ and $(0, 1)$, or $(2^{-1/2}, 2^{-1/2})$ and $(-2^{-1/2}, 2^{-1/2})$, would be an orthonormal basis and in three, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ would likewise form a basis. Because the basis is orthonormal, we can find the coefficients of any vector using an inner product. Rearranging (7.4),

$$a_i = \langle \mathbf{x}_i, \mathbf{y} \rangle. \tag{7.5}$$

The net result of this theory is that if we have a suitable basis, we can specify the state of a system using the coefficients a_i, as long as we know the basis vectors $\mathbf{x}_i$.

One final point to note is that we have so far limited ourselves to systems defined by vectors in N-space. We can also have systems where the vectors are replaced by functions, and the inner product is defined as an integral:

$$\langle f, g \rangle = \int_\Omega f(\mathbf{x}) g(\mathbf{x}) w(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \tag{7.6}$$

where Ω is the domain we are interested in (the region of a flow, say), $f(\mathbf{x})$ and $g(\mathbf{x})$ are functions defined over the region (velocity, for example), and $w(\mathbf{x})$ is a weighting function, as before.

POD of real flows

It has been estimated, using dimensional analysis, that a three-dimensional turbulent flow has of the order of $\mathrm{Re}^{9/4}$ degrees of freedom (Tennekes and Lumley, 1972; Smith et al., 2005), which is a large, but not infinite number. The power of the POD is that, firstly, it identifies the modes which can be used to form a basis for the system, and, secondly, it puts those modes in order of importance, based on how much they contribute to the kinetic energy of the flow[1]. The details can be found elsewhere (Berkooz et al., 1993; Smith et al., 2005), but the core of the theory is that in order to find the basis functions, denoted ϕ_i, we have to solve the integral equation:

$$\int_\Omega R(\mathbf{x}, \mathbf{x}') \phi_i(\mathbf{x}') \, \mathrm{d}\mathbf{x}' = \lambda \phi_i(\mathbf{x}), \tag{7.7}$$

where R is the two-point velocity correlation, and λ is an eigenvalue for the problem. There are infinitely many solutions to this integral equation (think of the eigenvalue problems for a homogeneous system in linear algebra) and there are well-established procedures for solving it. These procedures give a set of eigenfunctions ϕ_i and corresponding eigenvalues, ordered by the magnitude of λ, so that we are guaranteed to have the smallest number of modes possible which still capture the flow energy. A component of velocity, for example can be represented by the sum

$$u(\mathbf{x}) \approx \sum_{i=1}^{N} a_i \phi_i(\mathbf{x}), \tag{7.8}$$

and for any value of N, there is no other decomposition of the velocity which captures more of the flow energy. In effect, we have used the flow to tell us how best to represent the system.

Finally, for now, we have made no assumption about the variables we decompose in the POD: we could just as easily have applied the method to the acoustic pressure field, and a number of people have. We will return to this question when we consider jet noise, but for now have a look at Figures 6 and 7 in the paper by Lele et al. (2010), which show POD modes for a subsonic and supersonic jet.

[1] Other measures are available, but kinetic energy is the most widely used.

Chapter 8

Turbulent jets

A jet is one of the most important basic flows which we have to deal with. Jets occur in many forms and sizes from volcanoes to chimneys to ink jet printers. The basic flow structure is reasonably well understood and has been summarized in a number of books such as Abramovich (1963), and in an extensive literature in the research journals.

Jet flows

The axisymmetric turbulent jet has a wide range of applications. The most obvious is aircraft engines, but there are many others: hairdryers, hand-dryers, chimneys and volcanoes all have some characteristics of a circular jet. The structure of the jet has been intensively studied and there are numerous standard references which describe it. The basic form is shown in Figure 8.1, derived from the published data of Wygnanski and Fiedler (1969), Bogusławski and Popiel (1979) and Hussein et al. (1994).

The flow exits the nozzle with axial velocity U and keeps this velocity in a region called the *potential core*, just beyond the exit plane. The potential core shrinks, as the region of turbulent flow at the outer edge of the jet expands, until at some point the core has shrunk onto the jet axis. Further upstream, there is a *self-similar* region in which the jet velocity distribution is described by functions of a single variable η which depends on the radial displacement r and on the axial displacement z. Between the initial region and the self-similar region is a transition region, where the flow is not so well-defined, as it changes from one self-similar form to another.

The downstream self-similar region has been most extensively studied over the last half century. The data shown here are based on the results of Hussein et al. (1994), who conducted a detailed, very careful, set of measurements in a turbulent jet and fitted functions for the velocity and other profiles. Figure 8.2 shows the axial velocity U_c on the jet axis, scaled on the jet initial velocity U. Past a point $z \approx 15D$, U_c/U is very well approximated by the formula:

$$\frac{U_c(z)}{U} = \frac{B_u}{z/D - z_0/D}, \tag{8.1}$$

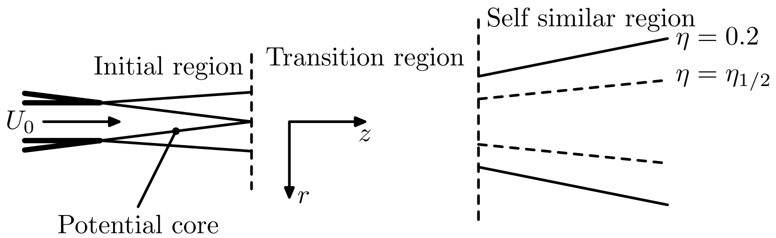


Figure 8.1: The basic structure of a circular jet. The origin of the cylindrical coordinates (r, z) is taken on the jet exit plane.

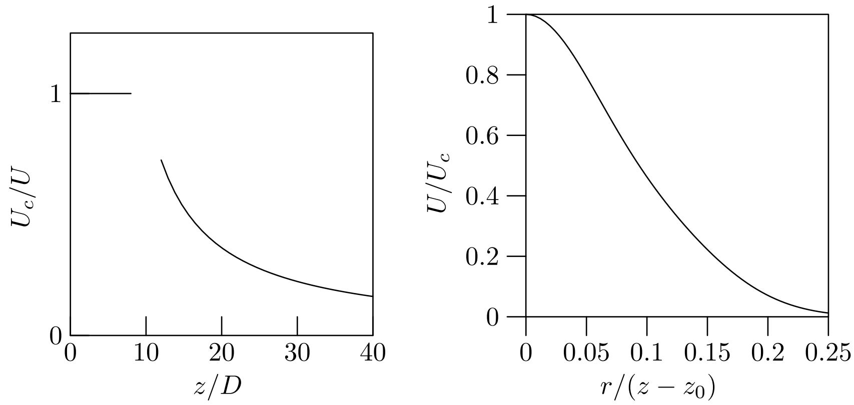


Figure 8.2: Mean axial velocity in a turbulent jet (Hussein et al., 1994)

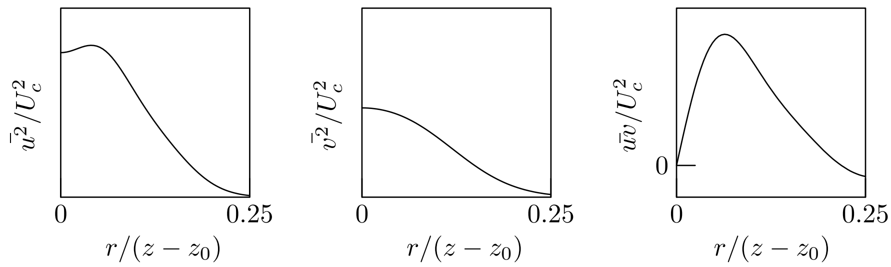


Figure 8.3: Fluctuating velocities in a turbulent jet (Hussein et al., 1994)

where good values for z_0/D and B_u are 4.0 and 5.8 respectively. Similar, though more complicated, approximations are given for the mean and fluctuating velocity profiles as a function of radius, and axial displacement, plotted in Figure 8.3.

Lighthill's eighth power law for jet noise

Solving Lighthill's equation for different sources is more than we can manage in these notes (or anywhere else), but we can derive a scaling law for jet noise which was one of the first great successes of the theory. The solution of (2.27) is:

$$p = -\frac{\partial^2}{\partial x_i \partial x_j}\int_V \frac{T_{ij}(\mathbf{y}, t - R/c_0)}{4\pi R}\,\mathrm{d}V.$$

In the far field, we can approximate this integral by differentiating it: when we do this, we will retain only terms which depend on $1/R$ (everything else decays much more rapidly). Setting coordinates so that the origin is inside the source region, $\mathbf{x}-\mathbf{y}\approx\mathbf{x}$, and:

$$p \approx \frac{1}{4\pi}\frac{x_i x_j}{x^3}\int_V \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}T_{ij}(\mathbf{y}, t - R/c_0)\,\mathrm{d}V.$$

There is no general solution for this equation, but we can derive a scaling law for the radiated acoustic power. Figure 8.4 shows the labelling for a jet characteristic quantities. We take a characteristic length L,

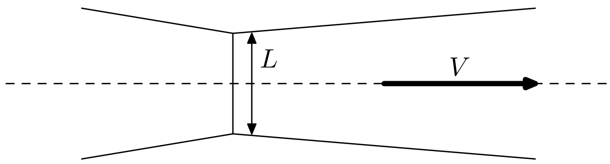


Figure 8.4: Parameters for jet noise.

characteristic velocity V and a mean density ρ_0. Then:

$$T_{ij} \sim \rho_0 V^2, \quad \frac{\partial}{\partial t} \sim \frac{V}{L},$$
$$p \sim \frac{1}{4\pi}\frac{1}{x}\frac{1}{c_0^2}\left(\frac{V}{L}\right)^2 \rho_0 V^2 L^3,$$

and the pressure scales as:

$$p \sim \rho_0 \frac{V^4}{c_0^2}\frac{L}{x}.$$

From (2.24), the intensity scales as

$$\bar{I} \sim \rho_0 \frac{V^8}{c_0^5}\left(\frac{L}{x}\right)^2.$$

The total acoustic power W is the intensity integrated over a spherical surface of radius x and

$$W \sim \rho_0 \frac{V^8}{c_0^5} L^2. \tag{8.2}$$

The acoustic power thus scales on the eighth power of jet velocity. This is Lighthill's eighth power law and was derived before experimental data were available to confirm it: it is one of the few major scientific predictions to have been made before the data were available. It is strictly only true for low speed flows, because we have implicitly assumed the source to be compact. At higher speeds, the characteristic frequency of the source increases and interference and convection effects become important.

Example: Modern aircraft

Using Lighthill's scaling law, we can estimate the difference in noise from a twin-engine and four-engine aircraft. We know that the thrust from an engine is proportional to $\rho V^2 D^2$. The total thrust F is the same in both cases, and:

$$F = 4\rho V_4^2 D_4^2 = 2\rho V_2^2 D_2^2,$$

and the total noise W is:

$$W_4 = 4V_4^8 D_4^2,$$
$$W_2 = 2V_2^8 D_2^2.$$

We can calculate the ratio of the total noise, by calculating the ratio of the jet velocities:

$$\frac{F/4}{F/2} = \left(\frac{V_4}{V_2}\right)^2 \left(\frac{D_4}{D_2}\right)^2,$$
$$V_2 = \sqrt{2}\frac{D_4}{D_2} V_4,$$

Figure 8.5: Trends in aircraft design: the Boeing 777 has two engines providing almost as much thrust as the four engines of the Boeing 747, a quieter, more fuel-efficient solution.

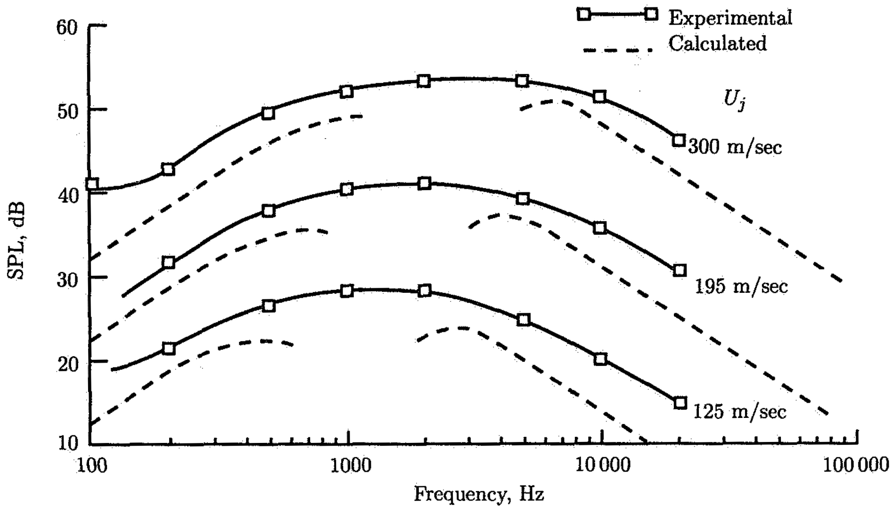


Figure 8.6: Jet noise spectra at 120 jet diameters, in the jet exhaust plane (Lilley, 1995)

and, if we assume that $D_2 = 2D_4$,

$$\frac{W_2}{W_4} = \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^8 (2)^2,$$
$$= 1/8,$$

which is a noise reduction of $9\mathrm{dB}$.

Jet noise fields

As noted above, the general problem of jet noise is a bit too complicated for this course, but we can have a look at some general features. Some typical spectra, taken from a NASA publication, are shown in Figure 8.6, and these show most of the typical features of subsonic jet noise. The frequency range scales on jet diameter and speed, through a Strouhal number, but the essential features are quite similar: there is a peak in the spectrum above which the spectrum decays quite rapidly, with the acoustic power generally being concentrated in a frequency range $2\pi fa/c \lessapprox 2$, where a is the exhaust radius.

More detailed studies of the noise field of jets have been conducted (Lilley, 1995, for more details), to attempt to reveal details of the source location and characteristics, and to acquire some understanding of the structure of the acoustic field. As we know from page 43, the relationship between a 'source' and its acoustic field is a tenuous one, and certainly not one which can be uniquely defined from field measurements alone. On the other hand, certain definite statements can be made, on the basis of correlation measurements of the acoustic pressure.

It is known, for example, that the acoustic field is dominated by only two or three azimuthal modes, i.e., the form of the acoustic field is very simple as a function of angle (answer Question 4 to see why), which gives us one first clue that the acoustic field is much simpler than the jet flow field. A turbulent flow has appreciable energy in a large number of azimuthal modes, so obviously they are not all radiating.

Studies using the POD, applied to the flow and to the acoustic field, have found that the flow requires 350 modes to capture 50% of the flow energy, while the acoustic field only required 24 modes to capture 90% of the acoustic energy. The effect of source filtering, or cancellation, is very powerful. There is some argument about the precise reasons for this disparity in complexity, but it is agreed that it might open possibilities for active and passive noise control.

Questions

1. Find a domestic jet such as a hairdryer or toilet hand-dryer. Switch it on and probe the flow by moving your hand in and out along radii at different distances from the nozzle. How does the force on your hand vary as a function of axial and radial displacement? The cooling rate? What should equations for the velocity profile look like?

2. Sketch the structure (velocity profiles, etc.) of a turbulent jet, using the data given by Hussein et al. (1994).

3. Using the same jet as in Question 1, try to sketch how the noise varies as a function of orientation with respect to the jet axis. You might find this easiest to do with a hairdryer which you can rotate to vary the orientation.

4. In Question 3 of Chapter 6, you derived an expression for the sound radiated to the far field by a circular source with azimuthal variation. Look up some properties of the Bessel function (graphically or otherwise) and say which azimuthal modes you think will radiate for a source of wavenumber k. Can you estimate the cross-spectrum and auto-spectrum for two points in the acoustic field?

Bibliography

Abramovich, G. N. (1963). *The theory of turbulent jets*. Cambridge, Massachusetts: MIT Press.

Berkooz, G., P. Holmes, and J. L. Lumley (1993). The proper orthogonal decomposition in the analysis of turbulent flows. *Annual Review of Fluid Mechanics 25*, 539–575.

Bogusławski, L. and C. O. Popiel (1979). Flow structure of the free round turbulent jet in the initial region. *Journal of Fluid Mechanics 90*(3), 531–539.

George, W. K. (2007). Lectures in turbulence for the 21st century. `http://www.turbulence-online.com`.

Hussein, H. J., S. P. Capp, and W. K. George (1994). Velocity measurements in a high-Reynolds-number momentum-conserving, axisymmetric, turbulent jet. *Journal of Fluid Mechanics 258*, 31–75.

Junger, M. C. and D. Feit (1993). *Sound, structures, and their interaction*. Acoustical Society of America.

Lele, S. K., S. Mendez, J. Ryu, J. Nichols, M. Shoeybi, and P. Moin (2010). Sources of high-speed jet noise: analysis of LES data and modeling. *Procedia Engineering 6*, 84–93.

Lighthill, M. J. (1952). On sound generated aerodynamically: I General theory. *Proceedings of the Royal Society of London. A. 211*, 564–587.

Lighthill, M. J. (1954). On sound generated aerodynamically II. Turbulence as a source of sound. *Proceedings of the Royal Society of London. A. 222*, 1–32.

Lilley, G. M. (1995). Jet noise classical theory and experiments. In *Aeroacoustics of flight vehicles*, pp. 211–290. Acoustical Society of America.

Panton, R. L. (2005). *Incompressible flow* (Third ed.). John Wiley and Sons, Inc.

Smith, T. R., J. Moehlis, and P. Holmes (2005). Low-dimensional modelling of turbulence using the proper orthogonal decomposition: A tutorial. *Nonlinear Dynamics 41*, 275–307.

Tennekes, H. and J. Lumley (1972). *A first course in turbulence*. MIT Press.

Wygnanski, I. and H. Fiedler (1969). Some measurements in the self-preserving jet. *Journal of Fluid Mechanics 38*(3), 577–612.

Appendix A

Further reading

These notes are based on a number of standard texts and lecture notes by other people, which are worth reading for a deeper knowledge, or for a different point of view. In particular:

- CRIGHTON, D. G., DOWLING A. P., FFOWCS WILLIAMS, J. E., HECKL, M. & LEPPINGTON, F. G. 1992, *Modern methods in analytical acoustics*, Springer-Verlag. Very mathematical but covers a lot of material.
- DOWLING, A. P. & FFOWCS WILLIAMS, J. E. 1983, *Sound and sources of sound*, Butterworth. This is quite a slim book compared to Pierce but it covers more of the things in these notes.
- FFOWCS WILLIAMS, J. E. 1984, The acoustic analogy—thirty years on, *IMA Journal of Applied Mathematics*, **32**:113–124. A readable account (not many equations) of the development and application of Lighthill's acoustic analogy.
- GRADSHTEYN, I. & RYZHIK, I. M. 1980, *Table of integrals, series, and products*, Academic, London. A big book of all the mathematical formulae anyone could ever need.
- GEORGE, W. K., *Lectures in Turbulence for the 21st Century*, lecture notes available from `http://www.turbulence-online.com`.
- HUBBARD, H. H. ed 1995, *Aeroacoustics of flight vehicles*, Acoustical Society of America. This is a two volume review of almost everything connected to noise from aircraft.
- LIGHTHILL, M. J. 1952, On sound generated aerodynamically: I General theory, *Proceedings of the Royal Society* A, **211**:564–587. This is the foundation of modern aeroacoustics and is very readable for a paper of such fundamental importance.
- PANTON, R. L. 2005, *Incompressible flow*, John Wiley, Hoboken. This is a very good, comprehensive textbook on the fluid dynamics you need for this course.
- PIERCE, A. 1994, *Acoustics: An introduction to its physical principles and applications*, American Institute of Physics, New York. This is the standard modern reference for acoustics. If you want to buy one comprehensive book on acoustics, this is the one. It doesn't really cover aerodynamically generated noise so you might want to look at Dowling & Ffowcs Williams as well.
- TENNEKES, H. & LUMLEY, J. 1972, *A first course in turbulence*, MIT Press. This has long been one of the most highly-regarded textbooks on turbulence and is a very good place to start.
- Radio 4's The Life Scientific featured an interview with Dame Ann Dowling, one of the world's leading researchers into aircraft noise, talking about her life and work in the field, and about the Silent Aircraft Initiative. You can download the recording from: `http://www.bbc.co.uk/podcasts/series/tls`

Fiction

There are not nearly enough novels about turbulence, but one very good one, with a main character loosely based on Lewis Fry Richardson is:

- FODEN, G. 2009, *Turbulence*, Faber & Faber. This is available in paperback (get it at Mr B's Emporium of Reading Delights opposite the Salamander; tell them I sent you) and is an excellent fictional account of deciding whether or not the weather would allow the 1944 Normandy landings to go ahead.

Appendix B

Basic equations

Potential flow:

$$\mathbf{u} = \nabla\phi,$$
$$u_i = \frac{\partial\phi}{\partial x_i}.$$

Bernoulli equation:

$$\frac{\partial\phi}{\partial t} + \frac{\nabla\phi.\nabla\phi}{2} + \frac{p}{\rho} = \text{const}$$

Appendix C

Some useful mathematics

Complex variables

We often use complex variable notation to make life easier. If we write a complex number $z = x + \mathrm{j}y$ where $\mathrm{j} = \sqrt{-1}$, then:

$$z = |z|\mathrm{e}^{\mathrm{j}\phi},$$
$$|z| = (x^2 + y^2)^{1/2},$$
$$\phi = \tan^{-1} y/x.$$

In dealing with constant frequency waves, we can use the relation:

$$\mathrm{e}^{-\mathrm{j}\omega t} = \cos\omega t - \mathrm{j}\sin\omega t$$

and if we wish to consider a general wave p of a fixed frequency, say, this can be written:

$$p(t) = P\mathrm{e}^{-\mathrm{j}\omega t},$$

where now P contains information about the amplitude and the phase.

The Dirac delta

The basic rule for integrating the delta function is:

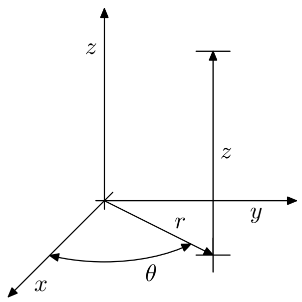
$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\,\mathrm{d}x = f(x_0),$$

and in the more complicated case where the argument of the delta function is itself a function:

$$\int_{-\infty}^{\infty} f(x)\delta(g(x))\,\mathrm{d}x = \frac{f(x_{g=0})}{|\mathrm{d}g/\mathrm{d}x_{g=0}|}.$$

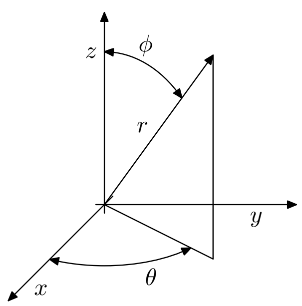
Coordinate systems

Cylindrical coordinates:



$$x = r\cos\theta, \quad y = r\sin\theta;$$
$$r = (x^2 + y^2)^{1/2}, \quad \theta = \tan^{-1} y/x.$$

Spherical coordinates:



$$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta,$$
$$z = r\cos\phi;$$
$$r = (x^2 + y^2 + z^2)^{1/2}, \quad \theta = \tan^{-1} y/x,$$
$$\phi = \tan^{-1} z/(x^2 + y^2)^{1/2}.$$

Differential operators

In Cartesian coordinates:

$$
\begin{aligned}
\nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right), \\
\nabla . \mathbf{f} &= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}, \\
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.
\end{aligned}
$$

In cylindrical coordinates:

$$
\begin{aligned}
\nabla f &= \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right), \\
\nabla . \mathbf{f} &= \frac{1}{r}\frac{\partial}{\partial r}(r f_r) + \frac{1}{r}\frac{\partial f_\theta}{\partial \theta} + \frac{\partial f_z}{\partial z}, \\
\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.
\end{aligned}
$$

In spherical coordinates:

$$
\begin{aligned}
\nabla f &= \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \phi}, \frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\right), \\
\nabla . \mathbf{f} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f_r\right) \\
&\quad + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}(f_\phi \sin\phi) \\
&\quad + \frac{1}{r\sin\phi}\frac{\partial f_\theta}{\partial \theta}, \\
\nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) \\
&\quad + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) \\
&\quad + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}.
\end{aligned}
$$

Tensors

Summation convention for repeated indices:

$$
\begin{aligned}
a_{ii} &= a_{11} + a_{22} + a_{33}, \\
a_i b_i &= a_1 b_1 + a_2 b_2 + a_3 b_3.
\end{aligned}
$$

Kronecker delta:

$$
\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}
$$

Double inner product:

$$
\mathbf{S} : \mathbf{T} = S_{ij} T_{ji},
$$

with the summation convention applied.

UNIVERSITY OF BATH

Faculty of Engineering & Design

DEPARTMENT OF MECHANICAL ENGINEERING

ME40328: Turbulence and noise

Semester 1

Day/date 2011/Time

Candidates should attempt **all questions** from Section A, **one question** from Section B, and **one question** from Section C.

Air density at sea level, $\rho = 1.225\mathrm{kg/m^3}$;
Acceleration due to gravity at sea level, $g = 9.81\mathrm{m/s^2}$;
Speed of sound at sea level, $c = 340\mathrm{m/s}$.

Section A

There will be five questions in this section, all worth ten marks each

1. Outline the main features of a turbulent jet flow. How does the flow become turbulent after leaving the nozzle?
2. With suitable sketches, describe the process of shear layer instability and transition to turbulence in a circular jet. What main features would you expect to see immediately after transition?
3. Give a definition of a 'coherent structure' in turbulence. What form would these structures take in a turbulent jet?
4. With the aid of sketches, describe qualitatively the essential features of sound propagation in a circular duct and the characteristics of the sound radiated from the duct termination. In an aero engine, how does the modal content of the sound field relate to the engine configuration?
5. State Lighthill's eighth power law for noise from a turbulent jet. Describe the implications of this law for the development of aircraft engines and aircraft configurations.
6. Describe the operation of a Helmholtz resonator. How might such a device be used in practice, in high and low frequency applications?
7. The image below is a Wikimedia Commons photograph of clouds in Australia. Why have they taken this particular form and what physical process is at work? What other examples of this process are common in nature and how does it end?

8. Turbulence is a phenomenon which occurs over a range of length and velocity scales. What are suitable scales for the description of turbulence and how are they related in any given problem? How is turbulent kinetic energy transferred between these scales and dissipated?

9. What are the statistical properties of homogeneous, isotropic turbulence? Under what circumstances does turbulence deviate from these assumptions?

10. With reference to noise generated by a vibrating surface, describe the role of the Helmholtz number ka in determining the nature of the acoustic field. Refer to the field complexity and radiation efficiency.

Section B

There will be four questions in this section, all worth twenty five marks each

11. A plane acoustic wave of frequency ω is incident at angle θ on one side of a slab of material of acoustic impedance Z_{sl}.

    (a) Show that the transmitted wave amplitude is given by:

    $$T = \frac{2\rho c/\cos\theta}{Z_{\text{sl}} + 2\rho c/\cos\theta},$$

    where ρ and c are the density and speed of sound for the fluid.

    [10 marks]

    (b) If the slab of material can be assumed limp, and has mass m per unit area, show that:

    $$|T| = \left[1 + \left(\frac{\omega m}{2\rho c}\right)^2 \cos^2\theta\right]^{-1/2}.$$

    [8 marks]

    (c) It is believed that laser velocity measurements of window vibrations have been used to detect speech inside buildings. If the density of glass is 2500$\mathrm{kg/m^3}$, and a standard laser unit can measure velocities of the order of 1$\mathrm{\mu m/s}$, estimate the lowest SPL for speech at 3kHz which could be detected using such a method on a window of thickness 5mm.

    [7 marks]

12. (a) A circular piston of radius a set in a rigid wall vibrates at frequency ω with velocity amplitude v. Show that in the far field, the acoustic pressure is:

    $$p(\omega) \approx -\mathrm{j}\frac{\rho c v k a^2}{2}\frac{2J_1(ka\sin\phi)}{ka\sin\phi}\frac{\mathrm{e}^{\mathrm{j}kR_0}}{R_0},$$

    where k is wavenumber, R_0 is distance from the centre of the source, ϕ is the angular separation from the source axis and $J_1(\,\cdot\,)$ is the first order Bessel function. You may use the relations:

    $$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{j}x\cos\theta_1}\,\mathrm{d}\theta_1,$$

and:

$$\int x J_0(x)\,\mathrm{d}x = x J_1(x),$$

and assume that in the far field:

$$\frac{1}{R} \approx \frac{1}{R_0} \text{ and } R \approx R_0 - r_1 \sin\phi \cos\theta_1.$$

[15 marks]

(b) The noise radiated from the intake of an aircraft engine can be approximated as that due to a piston set in the intake. Sketch the directivity pattern, indicating the directions of zero radiation, for the noise radiated from the intake of a Trent 900 with the fan rotating at 1500rpm. The intake radius is 1.5m and there are 24 fan blades. A graph of $J_1(x)$ is shown in Figure Q12.

[10 marks]

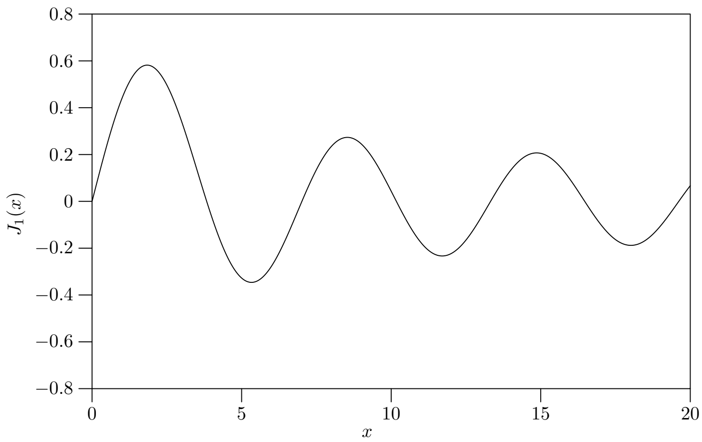


Figure Q12: First order Bessel function

13. In the turbulent energy cascade, it can be assumed that the energy dissipation rate ϵ is independent of wavenumber κ.

(a) Show that the energy distribution E is given by:

$$E(\kappa) = C\kappa^{-5/3}\epsilon^{2/3},$$

where C is some constant.

[8 marks]

(b) With the aid of suitable sketches and equations, describe the Kelvin-Helmholtz instability and the process of transition to turbulence.

[8 marks]

(c) In a jet which has undergone transition, describe the role of vortex ring dynamics and its effect on the turbulence statistics.

[9 marks]

14. A point source rotates at radius a and angular velocity Ω.

(a) Given that the Bessel function of order n can be represented:

$$J_n(x) = \frac{\mathrm{j}^n}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{j}(n\theta_1 - x\cos\theta_1)}\,\mathrm{d}\theta_1,$$

derive an expression for the acoustic pressure of the nth harmonic in the far field. You may assume that in the far field:

$$\frac{1}{R} \approx \frac{1}{R_0} \text{ and } R \approx R_0 - r_1 \sin\phi\cos\theta_1.$$

[15 marks]

(b) For a point source rotating at a Mach number of 0.5, sketch the spectrum of the radiated noise at a polar angle $\phi = \pi/4$, to within a scaling factor. The first few Bessel functions are sketched in Figure Q14.

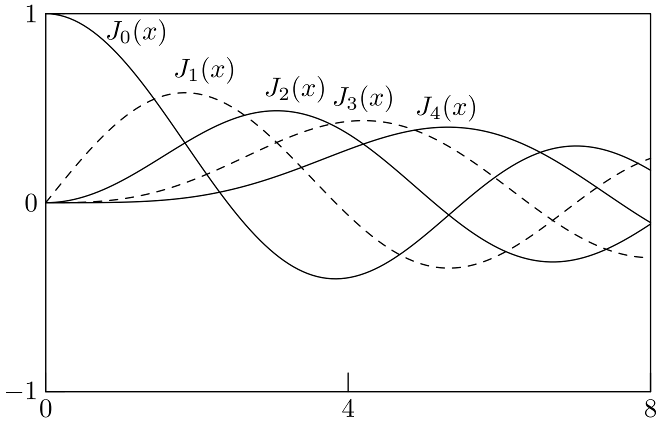


Figure Q14

[10 marks]

15. (a) Sketch the structure of a turbulent jet, clearly identifying all regions, with appropriate notation.

[8 marks]

(b) For the self-similar region of the jet, sketch and describe the mean and turbulent velocity profiles, clearly indicating the coordinates used.

[10 marks]

(c) You have been given the job of measuring the two-point axial velocity correlations along the axis of a turbulent jet. Sketch the correlations which you expect to see, being careful to show the correct behaviour for magnitude and time delay.

[7 marks]

16. Far field noise from a turbulent flow in a volume V can be approximated by:

$$p(\mathbf{x}, t) = A\frac{\partial^2}{\partial t^2}\int_V \frac{T_{ij}(\mathbf{y}, t - R/c)}{R}\,\mathrm{d}V,$$

where the factor A contains constants of proportionality.

(a) Discuss the significance of the term T_{ij} and how it is related to the turbulence statistics?

[8 marks]

(b) Show that the pressure autocorrelation is given by an integral of the form:

$$\overline{p(\mathbf{x}, t)p(\mathbf{x}, t + \tau)} = B\int_V\int_{V'} \frac{\overline{T_{ij}(\mathbf{y}, t - R/c)T_{ij}(\mathbf{y}', t - R'/c + \tau)}}{RR'}\,\mathrm{d}V'\,\mathrm{d}V.$$

[10 marks]

(c) What elements of the flow make up the term $\overline{T_{ij}(\mathbf{y}, t)T_{ij}(\mathbf{y}', t + \tau)}$ and what is their physical significance?

[7 marks]