Some notes on acoustics

Michael Carley
`m.j.carley@bath.ac.uk`

MARSH, Narcissus (b. Wiltshire 1638, d. Dublin 1713), Provost of Trinity College Dublin; Bishop of Bishop of Ferns and Leighlin, Archbishop of Cashel, Archbishop of Dublin (where his Dean was Jonathan Swift) and Primate of Armagh; introduced the word 'acoustics' into English and invented the word 'microphone'.

> *Being to treat of the Doctrine of* Sounds, *I hold it convenient to premise something in the general concerning this Theory; which may serve at once to engage your attention, and excuse my pains, when I shall have recommended them, as bestow'd on a subject not altogether useless and unfruitful.*

Narcissus Marsh, 1683/4, *Phil. Trans. Roy. Soc. Lond.*, **156**:472–486.

Contents

Things you should know

The aim of these notes is to introduce students who have studied fluid dynamics to some basic concepts in practical acoustics. The background knowledge which is assumed is:

- vector calculus: partial differentiation, gradient and divergence operators.
- the Navier–Stokes equations.
- use of dimensionless parameters and scaling.
- basic complex number theory.

Thanks

Thanks to Eberhard Sengpiel and Bob Cain who commented on the final draft.

Chapter 1

The basics

Acoustics is a branch of physics and, as such, anything it tells you about the world has to make sense. If it tells you something you don't believe then either it's wrong or you are. To start, it's worth looking at the things you already know about acoustics from your daily life. These are fundamental facts which also happen to be correct.

The first example we can consider is that of a lecturer droning on at a class. Everyone in the class hears the lecturer say the same thing at the same pitch: we don't have one part of the class hearing the lecturer speak with a squeaky voice while another part hears her speak in a deep bass. Furthermore, everyone hears the lecturer speak at the same speed with the words in the same order. This tells us that

$$\boxed{\text{sound travels undistorted}}$$

so, no matter where we are, as long as we can hear the speaker, we hear the same words at the same pitch and at the same rate.

Ponder now the forces of nature: the next time you are caught in a thunderstorm note the relationship between thunder and lightning. You will notice, if you have not already done so, that there is a delay between seeing the flash of the lightning and hearing the thunder:

$$\boxed{\text{sound travels with some time delay}}$$

so that we do not hear sound from a source immediately but have to wait for it to travel over the space between it and us.

Finally, bored by the lecture and scared by the storm, you go to a concert. For my purposes, I assume that you are a fan of a singer armed with a guitar. If you listen to the singer and the guitar, you will be able to distinguish the singer's voice from the sound of the guitar:

$$\boxed{\text{sound from different sources travels independently}}$$

or in other words, the sound coming from the singer does not influence the sound from the guitar—you simply hear both of them added together.

These three statements are all features of a *linear system* so we can make progress by treating acoustics as a linear problem in fluid dynamics.

1.1 The linear wave equation

From a physical or mathematical point of view, acoustics can be viewed as the study of solutions of the wave equation for a fluid. The linear wave equation, which we will derive presently, is the equation governing the propagation of small (linear) disturbances in a compressible medium. The wave equation can be applied to many different systems with different governing equations: here we apply it to fluids governed by the Navier–Stokes equations.

NAVIER, Claude (b. Dijon 10 Feb. 1785, d. Paris 21 Aug. 1836)

STOKES, George Gabriel (b. Skreen 13 Aug. 1819, d. Cambridge 1 Feb. 1903) Lucasian Professor 1849–1903, contributions in hydrodynamics and optics.

The equations of continuity and momentum for an inviscid fluid are:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0, \tag{1.1a}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \nabla p + \rho \mathbf{v} \nabla \mathbf{v} = 0. \tag{1.1b}$$

The first thing we do in deriving a wave equation is introduce the assumption that the fluctuations in the fluid dynamical quantities are small. This means that we write quantities as the sum of a mean part and a small fluctuation. These fluctuating parts are so small that their products can be neglected. Decomposing the quantities:

EULER, Leonhard (b. Basel 15 Apr. 1707, d. Saint Petersburg 18 Sep. 1783) 'the most prolific writer of mathematics of all time'; contributions in number theory, calculus, Fourier theory, mathematical physics, mechanics, fluid dynamics.

$$\begin{aligned} \rho &= \rho_0 + \rho'(t), \\ \mathbf{v} &= \mathbf{v}'(t), \\ p &= p_0 + p'(t), \end{aligned}$$

where 0 indicates a mean value and a prime symbol a fluctuation.

Applying this assumption to the equations of continuity and momentum and neglecting second order terms (products of small quantities), we find the linearized Euler equations:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla.\mathbf{v}' = 0, \tag{1.2a}$$

$$\rho_0 \frac{\partial \mathbf{v}'}{\partial t} + \nabla p' = 0. \tag{1.2b}$$

To make life easier, we can eliminate the velocity $\mathbf{v}'$ to give us a single equation:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\partial \rho'}{\partial t} + \rho_0 \nabla.\mathbf{v}'\right) - \nabla\left(\rho_0 \frac{\partial \mathbf{v}'}{\partial t} + \nabla p'\right) \\ =&\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 p' = 0. \end{aligned} \tag{1.3}$$

This is almost the wave equation except that it contains both pressure and density and we would like to deal with only one quantity at a time. To eliminate the density, we need a relationship between it and pressure. This depends on the thermodynamical properties of the fluid, as we will see below. Since we have linearized everything else, we can linearize the pressure–density relationship as well:

$$\begin{aligned} p &= p_0 + \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\rho_0} (\rho - \rho_0) + \frac{1}{2} \left.\frac{\partial^2 p}{\partial \rho^2}\right|_{\rho=\rho_0} (\rho - \rho_0)^2 + \ldots, \\ p' &= p - p_0 \approx \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\rho_0} (\rho - \rho_0) = c^2 \rho', \\ c^2 &= \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\rho_0}. \end{aligned}$$

The constant is written c^2 because it is always positive[1]. Substituting this relationship into equation 1.3, we find a wave equation for the acoustic pressure:

$$\boxed{\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0} \tag{1.4}$$

This is the most fundamental equation in acoustics. It describes the properties of a sound field in space and time and how those properties evolve. It is quite unlike the incompressible flow equations to which you may be accustomed because it describes very weak processes which happen over large distances. The most fundamental obvious property of the wave equation is that it is *linear*. This means that the sum of two solutions of the wave equation is also itself a solution[2].

---

[1] Why so?

[2] Prove it.

1.2 The speed of sound

When we come to solve the wave equation, we will find that c is the speed of sound, the speed at which a small disturbance propagates through a fluid. It depends on the thermodynamical properties of the fluid and is calculated on the assumption that sound propagation is *adiabatic*. For an adiabatic process in a gas:

$$p = k\rho^{\gamma},$$

where γ is the ratio of the specific heats. Then

$$\begin{aligned} c^2 &= \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\rho_0}, \\ &= \gamma k \rho^{\gamma-1} = \frac{\gamma p}{\rho}, \\ p &= \rho R T \end{aligned}$$

so that

$$\boxed{c^2 = \gamma R T}.$$

The speed of sound in air at STP is 343 m/s. The validity of the adiabatic assumption depends on the frequency of the sound. For low-frequency sound, there is no appreciable heat generation by conduction in the fluid and the assumption is a good one. For air, 'low frequency' means 'less than 1GHz'.

Note that if $c \to \infty$, the wave equation becomes $\nabla^2 p = 0$, the equation of incompressible flow. Saying $c \to \infty$ is the same as saying that density is independent of pressure, i.e. that the flow is incompressible. Since c is the speed at which disturbances propagate in a fluid, this is equivalent to the statement that disturbances propagate instantaneously in an incompressible flow.

NEWTON, Isaac (b. Woolsthorpe 4 Jan. 1643, d. London 31 Mar. 1727) calculated a speed of sound on the basis of isothermal propagation and 'in a monstrous exhibition of teleological data manipulation' added extra terms to get the 'right' answer.

LAGRANGE, Joseph-Louis (b. Turin 25 Jan. 1736, d. Paris 10 Apr. 1813) was honest *and* got the right answer.

1.3 Waves in one dimension

To illustrate some aspects of the solution of the wave equation, we look first at waves in one dimension. This corresponds to sound propagating in a pipe, for example. If we take x as the coordinate along the pipe, the wave properties are independent of y and z and the wave equation becomes:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = 0. \tag{1.5}$$

You can show quite easily that solutions of the form $p = f(x \pm ct)$ satisfy equation 1.5. This means that disturbances propagate as fixed shapes which shift along the x-axis at speed c. Figure 1.1 is a simple example, showing both solutions $x \pm ct$.

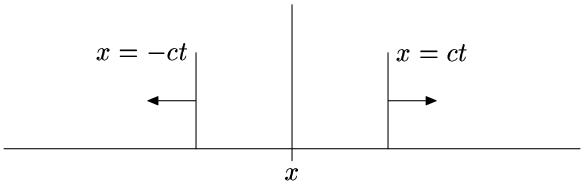


Figure 1.1: Wave propagation: right propagating wave with $x = ct$ and left propagating wave with $x = -ct$.

A pulse starts at a point $x = 0$ at time $t = 0$ so that $x \pm ct = 0$. At a later time, the wave will have moved left to a point $x = -ct$, still satisfying $x + ct = 0$ and right to a point $x = ct$, satisfying $x - ct = 0$. In both cases, the value of p will be the same as at time $t = 0$. As we might expect, the wave travels to the left or right at speed c, which is why c is called the speed of sound.

When waves propagate like this, they are called *plane waves* because their properties are constant over planes of constant x. Waves can be modelled as planar when they propagate at low frequency in pipes or ducts, such as long pipelines or engine exhausts which are often designed to enhance performance.

1.4 Waves in three dimensions

Naturally, one dimensional waves are of little interest to rounded personalities such as ourselves and we must eventually face reality in all of its three dimensions. Solving the wave equation in three dimensions is not much more difficult than doing so in one dimension. The most convenient approach is to work in spherical polar coordinates, figure 1.2.

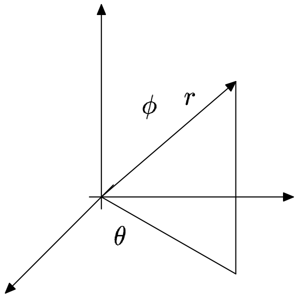


Figure 1.2: Spherical polar coordinates: $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$.

In this coordinate system:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\phi^2}.$$

We simplify this by considering the case of sound propagating in free space in a uniform medium. Then, by symmetry, p' is independent of ϕ and θ, so that:

$$\begin{aligned}\nabla^2 p &= \frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r}\\ &= \frac{1}{r}\frac{\partial^2}{\partial r^2}(rp) \qquad (1.6)\end{aligned}$$

and the wave equation now reads

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(rp) - \frac{\partial^2}{\partial r^2}(rp) = 0, \qquad (1.7)$$

which is identical in form to equation 1.5. Using the solution of that equation, $rp = f(r \pm ct)$, we find

$$p = \frac{f(t - r/c)}{r}.$$

Why did we not do this for one-dimensional waves?

For reasons of *causality* (things cannot happen before they have been caused), we reject the solution $rp = f(r + ct)$.

This solution contains three useful pieces of information. The first, as in the one dimensional case, is that the sound at time t depends on what happened at time $t - r/c$, the *emission time* or *retarded time*. The second, again similarly to the one dimensional case, is that the shape of the wave $f(\cdot)$ does not change. The big difference between one and three dimensional waves, however, is that the magnitude of the pressure perturbation (though not its shape) reduces as it propagates.

1.5 The linear wave equation with source terms

In keeping with the approach outlined in §1.1, we return to the fundamental fluid dynamical equations, with the addition of a source term on the right hand side:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = q(\mathbf{x}, t),$$
$$\rho \frac{\partial \mathbf{v}}{\partial t} + \nabla p + \rho \mathbf{v} \nabla \mathbf{v} = \mathbf{f}(\mathbf{x}, t).$$

The source term in the continuity equation $q(\mathbf{x}, t)$ obviously represents the addition of fluid. It has the dimensions of density per unit time and is interpreted as the rate of mass addition per unit volume. The source term in the momentum equation is, naturally, an applied force per unit volume $\mathbf{f}(\mathbf{x}, t)$. If we go through the same steps as before (equations 1.1–1.4, §1.1) and derive a wave equation:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{\partial q}{\partial t} - \nabla.\mathbf{f},$$

we see that the source terms on the right hand side are the time rate of change of the mass addition per unit volume, $\partial q/\partial t$ and the divergence of the applied force, $\nabla.\mathbf{f}$. The important thing to note is that a steady fluid dynamical process generates no noise.

1.6 Sound generation by a pulsating sphere

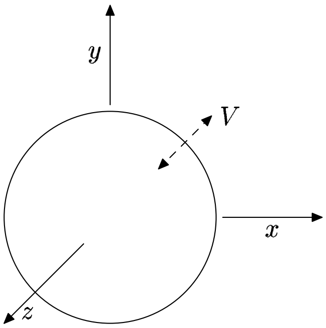


Figure 1.3: A pulsating spherical surface

The simplest physical problem we can solve is that of sound radiated by a pulsating sphere. This sphere could be, for example, a bubble, a varying heat source or an approximation to a body of varying volume. The sphere has radius a and oscillates with velocity amplitude V at frequency ω. From the linearized momentum equation (1.2b), we can find a relationship between acceleration and pressure gradient:

$$\nabla p = -\rho_0 \frac{\partial \mathbf{v}}{\partial t}. \tag{1.8}$$

Writing the radial velocity of the sphere surface as $v = V \exp[-\mathrm{j}\omega t]$, we can see that p must also have frequency ω so that we can write it as $p = P \exp[-\mathrm{j}\omega t]$ and:

$$\nabla P \mathrm{e}^{-\mathrm{j}\omega t} = \mathrm{j}\omega \rho_0 V \mathrm{e}^{-\mathrm{j}\omega t}. \tag{1.9}$$

Since p is a solution of the wave equation, we know from §1.4 that

$$p = \frac{f(t - r/c)}{r} = \frac{A\mathrm{e}^{-\mathrm{j}\omega(t-r/c)}}{r}, \tag{1.10}$$

where A is to be found from the boundary condition at a, the sphere surface. Writing out the pressure gradient:

$$\nabla p = \frac{A}{r^2}\left[\frac{\mathrm{j}\omega r}{c} - 1\right]\mathrm{e}^{-\mathrm{j}\omega(t-r/c)}, \tag{1.11}$$

and applying the boundary condition:

$$\frac{A}{a^2}\left[\frac{\mathrm{j}\omega a}{c} - 1\right]\mathrm{e}^{-\mathrm{j}\omega(t-a/c)} = \mathrm{j}\omega\rho_0 V\mathrm{e}^{-\mathrm{j}\omega t}, \tag{1.12}$$

we can fix the constant A:

$$A = \frac{(ka)(ka-\mathrm{j})\rho_0 Vca}{(ka)^2+1}\mathrm{e}^{-\mathrm{j}ka}, \tag{1.13}$$

where $k = \omega/c$ is the *wavenumber*. The solution for the pressure is then:

$$p = \frac{ka}{r}\frac{ka-\mathrm{j}}{(ka)^2+1}(\rho_0 Vca)\mathrm{e}^{-\mathrm{j}k(r-a)}\mathrm{e}^{-\mathrm{j}\omega t}. \tag{1.14}$$

There are two approximations we can make which simplify this formula. When $ka \ll 1$ (i.e. when the sphere is small or it vibrates at low frequency), (1.14) can be written:

$$p \approx -\mathrm{j}\frac{\rho_0 cka^2}{r}V\mathrm{e}^{\mathrm{j}kr}\mathrm{e}^{-\mathrm{j}\omega t}; \tag{1.15}$$

when $ka \gg 1$ (i.e. when the sphere is large or vibrating at high frequency):

$$p \approx \frac{\rho_0 Vca}{r}\mathrm{e}^{-\mathrm{j}k(r-a)}\mathrm{e}^{-\mathrm{j}\omega t}. \tag{1.16}$$

The parameter ka, a non-dimensional combination of wavelength and a characteristic dimension of the body, is an important parameter in characterizing sources and is called the *compactness*. When ka is small, the source is point-like and can be treated as a simple source; when it is large, the acoustic field becomes more complicated.

1.7 Fundamental solutions and Green's functions

We can use the result for a spherical source as a building block for the solution of more complex problems. If we assume that the sphere shrinks until it has zero radius (but still has a surface velocity), we can use the small ka approximation for the pressure, (1.15). If we define the source strength as the rate of mass injection (or volume acceleration), the total strength of the source is the surface acceleration $-\mathrm{j}kcV$ multiplied by the density and the surface area of the sphere, $4\pi a^2$:

$$q = -\mathrm{j}kcV\rho_0 4\pi a^2, \tag{1.17}$$

and p is then:

$$p(\mathbf{x},\omega) = \frac{\dot{q}(\omega)\mathrm{e}^{\mathrm{j}kR}}{4\pi R}, \tag{1.18}$$

where the sound is received at position $\mathbf{x}$ from a source at position $\mathbf{y}$ and the distance between them is $R = |\mathbf{x}-\mathbf{y}|$.

If we choose a general form for the source strength, rather than a single frequency signal, the resulting sound is:

$$p(\mathbf{x},t) = \frac{\dot{q}(\mathbf{y},t-R/c)}{4\pi R}. \tag{1.19}$$

The sound heard at a time t was generated at the source at a time $t - R/c$. This time is given the symbol τ and is called the *retarded time*. It is important to note that there is a one-to-one relationship between τ and t: the sound which leaves the source at a given time is heard once and once only.

We can view a general source distribution as a combination of point sources of varying strengths whose contributions add up to the total sound field. Because the signal which is heard is the same as the signal which is generated, we can also view the contribution from a source as made up of a sequence of short pulses in rapid succession. Then our problem reduces to calculating the sound heard if a point source generates a very short pulse. If we represent a short pulse by the symbol $\delta(\tau)$, meaning "a signal which is zero except at time $\tau = 0$", the resulting acoustic signal (the sound heard elsewhere) will be:

$$p = \frac{\delta(\tau + R/c)}{4\pi R}, \tag{1.20}$$

which is a short pulse heard at time R/c. We can write this more formally as:

$$G(\mathbf{x}, t; \mathbf{y}, \tau) = \frac{\delta(\tau - t + R/c)}{4\pi R}, \tag{1.21}$$

which is called the *Green's function* for the wave equation. It is interpreted as "the sound heard at position $\mathbf{x}$ at time t if a source at position $\mathbf{y}$ fires at time τ". The same relationship holds as before: the signal is zero everywhere except when its argument is zero or, alternatively, when $t = \tau + R/c$, which is the retarded time relationship we saw earlier.

GREEN, George (b. Sneinton, Jul. 1793, d. Sneinton 31 May 1841), developed potential theory, Green's theorem and Green's functions and made bread.

The solution of the wave equation with a source term

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = q(\mathbf{x}, t),$$

is then given by integrating the source multiplied by the Green's function:

$$\boxed{p = \int_\tau \int_V q(\mathbf{y}, t) G(\mathbf{x}, t; \mathbf{y}, \tau)\, \mathrm{d}V\, \mathrm{d}\tau.} \tag{1.22}$$

Here, V is the volume occupied by the source q. This is the solution for any problem in linear acoustics: the Green's function may change, if we include the effects of a mean flow, for example, but the procedure is the same. In these notes, we will only consider the simplest case, that of a uniform quiescent fluid for which the Green's function is equation 1.21. Equation 1.22 is a consequence of the linearity of the wave equation. Because the sum of two (or more) solutions is also a solution, we can consider any source distribution as a superposition of point sources, work out the solution for each, and add them up at the end. In the limiting case, this summation becomes the integral of equation 1.22.

1.8 Reciprocity

Some useful properties of sound fields can be found by examining the Green's function (1.21). One of the more interesting, which we will use later, comes from the symmetry of R. If we exchange the observer and source positions, $\mathbf{x}$ and $\mathbf{y}$ respectively, we can see that R is unchanged and the Green's function has the same value as before. This is an example of *reciprocity*—if we switch the observer and source positions, the sound heard by the observer is unchanged.

1.9 Sound from a point force

The next most basic problem we might look at is the sound radiated by an oscillating force (the lift on a propeller blade or the fluctuating lift on a cylindrical structure, say). The wave equation to be solved is:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = -\nabla.\mathbf{f}. \tag{1.23}$$

We can find a solution for this writing $p = \nabla.\mathbf{s}$ where $\mathbf{s}$ is some vector to be determined. If we do this:

$$\begin{aligned} \frac{1}{c^2}\frac{\partial}{\partial t^2}\nabla.\mathbf{s} - \nabla^2(\nabla.\mathbf{s}) &= -\nabla.\mathbf{f}, \\ \nabla.\left[\frac{1}{c^2}\frac{\partial \mathbf{s}}{\partial t^2} - \nabla^2\mathbf{s} = -\mathbf{f}\right] &. \end{aligned}$$

We already have a solution which we can use to find $\mathbf{s}$:

$$\mathbf{s} = -\frac{\mathbf{f}(\mathbf{y}, t - R/c)}{4\pi R},$$

and we can evaluate the pressure by differentiating $\mathbf{s}$:

$$p = \nabla.\mathbf{s} = \frac{\nabla R}{4\pi R} \cdot \left(\frac{\mathbf{f}(\tau)}{R} + \frac{1}{c}\frac{\partial \mathbf{f}(\tau)}{\partial \tau} \right).$$

We can rewrite this by noting that $\nabla R = \hat{\mathbf{r}}$ is a unit vector pointing from the source to the observer:

$$p = \frac{\hat{\mathbf{r}}}{4\pi R} \cdot \left(\frac{\mathbf{f}(\tau)}{R} + \frac{1}{c}\frac{\partial \mathbf{f}(\tau)}{\partial \tau} \right),$$

and noting that $\hat{\mathbf{r}}.\mathbf{f} = |\mathbf{f}|\cos\theta$ where θ is the angle between the source–observer vector and the direction of the force,

$$p = \frac{\cos\theta}{4\pi R}\left(\frac{f}{R} + \frac{\dot{f}}{c} \right). \qquad (1.24)$$

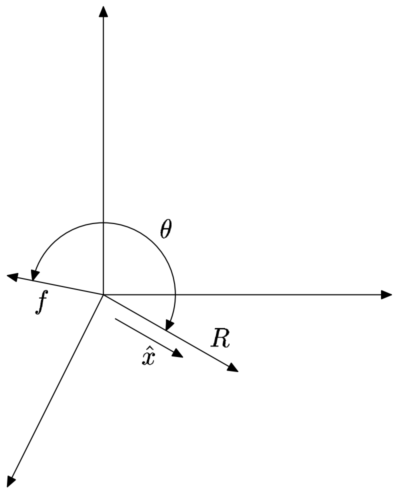


Figure 1.4: Dipole coordinate system

The solution for the sound radiated by a point volume and force source is then:

$$p = \frac{\dot{q}(\tau)}{4\pi R} - \frac{\cos\theta}{4\pi R}\left(\frac{f(\tau)}{R} + \frac{\dot{f}(\tau)}{c} \right), \qquad (1.25)$$

so that the radiated noise depends on the time rate of change of the fluid injection and the applied force, and on the orientation of the observer.

1.10 Multipole sources

Equation 1.25 has a different form for volume and force sources. The sound radiated by the volume source does not depend on θ and decays as $1/R$. The sound radiated by the point force does depend on the observer orientation as well as on distance. The rate of decay of the sound depends on the observer distance. Close to the source, $1/R^2 \gg 1/R$ and the sound decays as $1/R^2$. Far from the source, in the *far field*, $1/R^2$ is small and the pressure is like:

$$p(\mathbf{x},t) \approx -\frac{\dot{f}(\tau)\cos\theta}{4\pi Rc}.$$

The dependence on the observer orientation is perhaps not surprizing. A volume source is symmetric, while a force is a vector quantity, so that we might expect the acoustic field of a volume source to be symmetric while the sound radiated by a force is not. The strongest radiation from a force is along its axis ($\theta = 0$) while no sound at all is radiated at right angles ($\theta = \pi/2$).

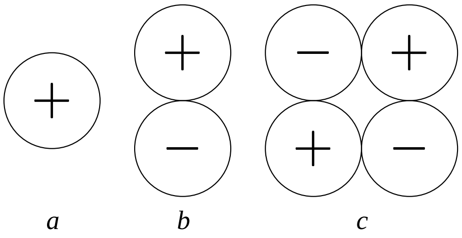


Figure 1.5: Multipole sources formed by arranging simple volume sources; *a*: monopole; *b*: dipole; *c*: quadrupole.

There is another way of thinking about this variation in directivity which can also be generalized to more complex source processes and field shapes. If we take two point volume sources of opposite strength and place them very close to each other, their acoustic fields partially cancel out and the remaining term depends on $\cos\theta$ where θ is measured from the line joining the two sources. Because it is formed using two symmetric sources, the resulting source is called a *dipole*. Similarly, a volume source is called a *monopole*. Figure 1.5 shows how general multipole sources can be devised. In principle, we can generate sources of as high an order as we like, but in practice, monopoles and dipoles are the most important. The exception to this is in studying noise generation by turbulence, where the *quadrupole* source, figure 1.5*c*, is dominant.

The form of the acoustic field for a dipole system can be derived from first principles. If we start with two sources of equal and opposite strength, separated by a small distance a, their positions are $(\pm a/2, 0, 0)$. Then the total sound at some point is:

$$\begin{aligned} p &= \frac{q(t-R_+/c)}{4\pi R_+} - \frac{q(t-R_-/c)}{4\pi R_-}, \\ R_\pm &= [(x \mp a/2)^2 + y^2 + z^2]^{1/2}. \end{aligned} \tag{1.26}$$

We want to calculate the total radiated sound for (very) small values of a assuming that $f = aq$ the *dipole moment* remains constant. The easiest way to do this is to expand p in a Taylor series:

$$p \approx p|_{a=0} + \left.\frac{\mathrm{d}p}{\mathrm{d}a}\right|_{a=0} a + \ldots. \tag{1.27}$$

Differentiating (1.26):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}a}\frac{q(t-R_\pm/c)}{4\pi R_\pm} &= -\frac{\partial R_\pm}{\partial a}\left(\frac{\dot{q}(t-R_\pm/c)}{4\pi R_\pm c} + \frac{q(t-R_\pm/c)}{4\pi R_\pm^2}\right), \\ \left.\frac{\partial R_\pm}{\partial a}\right|_{a=0} &= \mp\frac{1}{2}\frac{x}{R}, \\ R &= (x^2+y^2+z^2)^{1/2}. \end{aligned}$$

Using these results in (1.26):

$$p \approx a\frac{x}{R}\left(\frac{\dot{q}(t-R/c)}{4\pi Rc}+\frac{q(t-R/c)}{4\pi R^2}\right). \tag{1.28}$$

We can rewrite this by noting that $f = aq$ and $x/R = \cos\theta$:

$$p = \left(\frac{\dot{f}(t-R/c)}{c}+\frac{f(t-R/c)}{R}\right)\frac{\cos\theta}{4\pi R}, \tag{1.29}$$

which is the same as the noise radiated by a point force in §1.9.

As the order of a source increases, its radiation efficiency decreases. All things being equal, a monopole is stronger than a dipole which is stronger than a quadrupole. The exceptions occur when some source terms are absent (for example in a turbulent flow, where the absence of monopole and dipole sources means that quadrupoles are dominant) and, naturally, when the *strengths* of the sources are such that the relative inefficiency of one source type is overcome by the fact that it is much stronger than the others.

1.11 Application of reciprocity: Microphone arrays

We can use the principle of reciprocity to alter the response of a *microphone system* to be able to say something about the position of noise sources. When a microphone detects sound from a dipole source, the sound it detects depends on the orientation of the dipole. Can we set up a microphone system so that the sound depends on the orientation of the microphone? This would allow us to detect sound coming from particular *directions* or to reject interference (in an aircraft headset, for example).

If we apply reciprocity, we can devise just such a system. An approximation to a dipole system is made up of two sources 180° out of phase and a microphone. The sound detected at the microphone depends on the angle between the dipole axis and the vector to the microphone. If we switch microphones and sources, we have one source and two microphones close together. We take the signals from the microphones, shift one by 180° and add them up. Then, by reciprocity, the microphone system response has a dipole characteristic. This means that it detects nothing from sources at 90° to its axis and has a maximum response from sources along its axis. Such a system is called a *dipole microphone* and is a simple example of a *microphone array*. More complex arrays with more microphones are used to make measurements of source distributions in space and to characterize the noise generation processes in complicated systems.

Chapter 2

Describing sound

Before we continue, we need some terminology for describing waves in general and acoustic waves in particular. Most of the time, we are interested in waves of constant frequency, partly because many systems generate discrete tones and partly because it is easier to make calculations for one frequency at a time. If a time-dependent solution is needed, we can always assemble the different frequency components into an overall solution.

2.1 Waves of constant frequency

If we write $p = P\exp[-\mathrm{j}\omega t]$ where ω is the radian frequency, the wave equation becomes the *Helmholtz* equation:

$$\boxed{\nabla^2 P + k^2 P = 0.} \tag{2.1}$$

Note that t has disappeared, reducing the order of the equation by one. The *wavenumber* $k = \omega/c$ and we have already seen it in §1.6.

When we are dealing with waves of constant frequency, the sound field is a sinusoidal pattern which propagates in space. Ignoring the decay term $1/R$, we can see that the field is periodic in space as well as in time; in the same way that it has a time period $1/f$ where f is the frequency, it has a spatial period λ, the *wavelength*. Because the wave propagates at constant speed, the frequency and wavelength are linked by the relation $c = \lambda f$.

HELMHOLTZ, Hermann Ludwig Ferdinand von (b. Potsdam 31 Aug. 1821, d. Berlin 8 Sep. 1894) contributions in acoustics, fluid dynamics, electromagnetism, thermodynamics and optics.

2.2 Acoustic pressure and velocity

When we derived the wave equation, we chose to eliminate velocity and density and concentrated on pressure as our dependent variable. There are two main reasons for doing this: the first is that pressure is a scalar and so is conceptually easier to work with than velocity. In practice, given that we could use a velocity potential, this is not a huge advantage. The second, and more important, reason is that pressure is what we hear and what we measure. Our ears and the microphones we use to measure sound are sensitive to pressure fluctuations, so that is what we choose as our main quantity.

There are times, however, when we will need to use some other quantity. The fundamental theory of aerodynamically generated noise is actually based on density fluctuations (which are usually converted to pressure variations using a linear relationship). A more important relationship is that between pressure and velocity because the acoustic velocity is often used as a boundary condition in calculations involving solid bodies. Remember that acoustics is a branch of fluid dynamics and it is a fluid-dynamical boundary condition that must be satisfied, i.e. usually a velocity.

The linearized momentum equation (1.2b) gives us the relationship we need:

$$\frac{\partial \mathbf{v}'}{\partial t} = -\frac{\nabla p'}{\rho_0},$$

in other words, the acoustic velocity is proportional to the pressure gradient. If we write the solution of the wave equation in terms of a velocity potential $\phi = f(t - R/c)$, the pressure and radial velocity are related via:

$$\begin{aligned} p &= -\rho_0 \frac{\partial \phi}{\partial t}, \quad \mathbf{v} = \nabla \phi, \\ v &= \frac{p}{\rho_0 c} + \frac{f(t - R/c)}{\rho_0 R^2}. \end{aligned} \tag{2.2}$$

For a wave of constant frequency, the acoustic velocity amplitude V is related to the acoustic pressure by

$$V = -\mathrm{j} \frac{\nabla P}{\rho_0 \omega}.$$

For a plane wave $\nabla \to \partial/\partial x$ and $V = P/\rho_0 c$. For large R, the pressure–velocity relationship for a spherical wave reduces to this form, as seen in equation 2.2.

2.3 Acoustic intensity and power

A basic characteristic of a source is the rate at which it transfers energy. If we multiply equation 1.2a by $c^2 \rho'$,

$$c^2 \rho' \frac{\partial \rho'}{\partial t} + \rho_0 c^2 \rho' \frac{\partial v}{\partial x} = 0 \tag{2.3}$$

and note that $\rho' \partial \rho' / \partial t = \frac{1}{2} (\partial / \partial t) {\rho'}^2$ and that $c^2 \rho' = p'$,

$$\frac{c^2}{\rho_0} \frac{1}{2} \frac{\partial}{\partial t} {\rho'}^2 + p' \frac{\partial v}{\partial x} = 0.$$

Multiplying the momentum equation 1.2b by v gives

$$\rho_0 v \frac{\partial v}{\partial t} + v \frac{\partial p'}{\partial x} = 0,$$

which can be rearranged:

$$\frac{1}{2} \rho_0 \frac{\partial}{\partial t} v^2 + v \frac{\partial p'}{\partial x} = 0. \tag{2.4}$$

Adding equations 2.3 and 2.4 gives a result for the energy transport in the sound field:

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho_0 v^2 + \frac{1}{2} \frac{c^2}{\rho_0} {\rho'}^2 \right) + \frac{\partial}{\partial x} (p' v) = 0. \tag{2.5}$$

In equation 2.5, $\rho_0 v^2 / 2$ is the *kinetic energy* per unit volume, $c^2 / \rho_0 {\rho'}^2 / 2$ is the *potential energy* per unit volume and $p' v$ is the *acoustic intensity* I which is the rate of energy transport across unit area. Equation 2.5 is a statement of energy conservation for the system and says that the rate of change of energy in a region is equal to the net rate at which energy is carried into the region.

If insert the relationship between pressure and velocity, equation 2.2, the acoustic intensity is

$$I = \frac{p^2}{\rho c} + \frac{\partial}{\partial t} \left(\frac{f^2 (t - R/c)}{2 \rho R^3} \right).$$

If we average I over time for a periodic wave, the second term has a mean value of zero and the resulting mean intensity is:

$$\bar{I} = \frac{\overline{p^2}}{\rho c}. \tag{2.6}$$

2.4 Measures of sound

In practice, we only ever refer to the ‘amplitude’ of a sound: ‘loudness’ has a particular technical meaning related to our *perception* of noise.

Before going any further, you will need to know how to describe a sound or sound field. We characterize noise by its pitch (frequency) and its ‘volume’ (amplitude). To describe the amplitude of a sound we usually use the root mean square (rms) pressure:

$$p_{\text{rms}} = \left(\overline{p^2}\right)^{1/2}$$

where the bar denotes ‘time average’. This is a useful measure but suffers from the problem that acoustic pressures of interest vary over a huge range. The threshold of human hearing is at $p_{\text{rms}} = 20\mu$Pa while the threshold of pain and the onset of hearing damage are at about $p_{\text{rms}} = 200$mPa, a difference of seven orders of magnitude. To keep the numbers manageable, we use a logarithmic scale. On this scale, the ‘difference’ in *sound pressure level* between two pressures p_1 and p_2 is:

$$\Delta_{\text{SPL}} = 10 \log \frac{\overline{p_1^2}}{\overline{p_2^2}}.$$

When we want to talk about only one signal, we use a standard reference pressure. Then the sound pressure level is

$$\text{SPL} = 10 \log \frac{\overline{p^2}}{p_{\text{ref}}^2}. \tag{2.7}$$

The reference level is the nominal threshold of human hearing 20μPa. The ‘units’ of SPL are decibels, dB.

Level/dB	Example
140	3m from a jet engine
130	Threshold of pain
120	Rock concert
110	Accelerating motorcycle at 5m
60	Two people talking
80	Vacumn cleaner
10	3m from human breathing

Table 2.1: Some sample approximate noise levels

Table 2.1 shows levels for some typical noises. A good rule of thumb is that if you have to raise your voice to speak, the noise level is greater than 80dB, and if you have to shout, the noise level is greater than 85dB and you risk hearing damage.

Chapter 3

Sound fields around solid bodies

Many interesting problem in engineering acoustics will involve solid bodies, either as passive boundaries which modify the sound field or as noise generators in their own right. This chapter looks at some simple examples of sound being modified by reflection and sound being generated by vibrating bodies.

3.1 Solid surfaces and boundary conditions

The first thing we need to consider when a boundary is imposed on a problem is the appropriate boundary condition. Usually, this will be specified in terms of the pressure or velocity at the boundary. The two boundary conditions we will consider are a velocity or pressure gradient condition and a pressure condition. In the first case, we specify the velocity normal to a surface (zero in the case of a stationary body) and write the boundary condition in terms of the pressure gradient, as in (1.11). In the second case, we specify a pressure at the surface: this will often be zero, as on a *pressure release surface*, such as the open end of a duct or the surface of water.

3.2 Reflection from an infinite plane

The simplest realistic problem of interest involving the effect of a solid body on a sound field is that of the interaction of the field from a point source with a plane wall, figure 3.1.

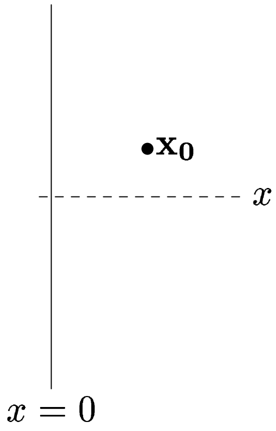


Figure 3.1: A point source near a wall

The problem is, given a source at a point $\mathbf{x}$, near a rigid plane, to calculate the resulting overall sound field. If the wall were not present, we know that the sound field at a frequency ω would have the form:

$$p_{\mathrm{i}} = \frac{\mathrm{e}^{-\mathrm{j}\omega(t-R/c)}}{4\pi R},$$

where p_{i} is the *incident* sound field. We will drop the factor $\exp[-\mathrm{j}\omega t]$ because it is the same for all sound fields in the problem and write:

$$p_{\mathrm{i}} = \frac{\mathrm{e}^{\mathrm{j}kR}}{4\pi R}.$$

Our problem now is to find a second acoustic field p_{s} (the 'scattered' field), such that the total field $p_{\mathrm{t}} = p_{\mathrm{i}} + p_{\mathrm{s}}$ satisfies the wave equation and the boundary conditions on the wall. By linearity, §1.1, this means that p_{s} must be a valid solution of the wave equation, since the sum of two solutions is itself a solution.

Now we need to decide what boundary condition to apply. As in inviscid fluid dynamics, the boundary condition is that the total velocity normal to the wall must be zero. We know that the acoustic velocity is proportional to the pressure gradient, §2.2, so this boundary condition is equivalent to

$$\left.\frac{\partial p_{\mathrm{t}}}{\partial x}\right|_{x=0} \equiv 0,$$

or, in terms of the incident and scattered fields,

$$\left.\frac{\partial p_{\mathrm{s}}}{\partial x}\right|_{x=0} \equiv -\left.\frac{\partial p_{\mathrm{i}}}{\partial x}\right|_{x=0}.$$

For a source at $\mathbf{x}_0 = (x_0, y_0, z_0)$,

$$\frac{\partial p_{\mathrm{i}}}{\partial x} = \frac{x - x_0}{4\pi}\frac{\mathrm{e}^{\mathrm{j}kR}}{R^3}\left(\mathrm{j}kR - 1\right),$$

and at $x = 0$,

$$\left.\frac{\partial p_{\mathrm{i}}}{\partial x}\right|_{x=0} = -\frac{x_0}{4\pi}\frac{\mathrm{e}^{\mathrm{j}kR}}{R^3}\left(\mathrm{j}kR - 1\right),$$
$$R = [x_0^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}.$$

The solution of our problem is an acoustic field p_{s} with

$$\left.\frac{\partial p_{\mathrm{s}}}{\partial x}\right|_{x=0} = \frac{x_0}{4\pi}\frac{\mathrm{e}^{\mathrm{j}kR}}{R^3}\left(\mathrm{j}kR - 1\right).$$

A source positioned at $\mathbf{x}_- = (-x_0, y_0, z_0)$ gives just such a field so a valid solution to the problem can be found using an *image source*, the reflection of our orginal source in the rigid wall. The total field is then

$$\begin{aligned}
p_{\mathrm{t}} &= p_{\mathrm{i}} + p_{\mathrm{s}},\\
p_{\mathrm{i}} &= \frac{\mathrm{e}^{\mathrm{j}kR_+}}{4\pi R_+},\\
p_{\mathrm{s}} &= \frac{\mathrm{e}^{\mathrm{j}kR_-}}{4\pi R_-},\\
R_\pm &= [(x \mp x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}.
\end{aligned}$$

One immediate result of this analysis is that the pressure generated on the wall by a source is twice that which would be generated if the wall were not present.

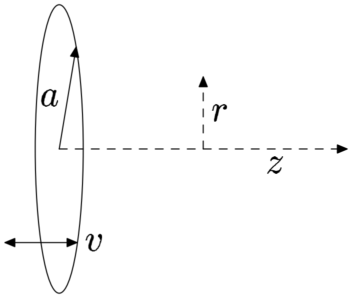


Figure 3.2: A rigid piston vibrating in a rigid wall.

3.3 Sound generation by a piston

Taking a step up in difficulty (and realism), we now look at the sound radiated by a rigid piston embedded in a wall. This is a basic model of a loudspeaker and is related to a number of other problems in the acoustics of sound generation by moving surfaces.

Figure 3.2 shows a rigid circular piston of radius a which vibrates periodically at frequency ω and velocity amplitude v so that its velocity is $v\exp[-\mathrm{j}\omega t]$. From equation 1.22:

$$p = 2\frac{\partial}{\partial t}\iint_S \frac{q(\mathbf{y},\tau)}{4\pi R}\,\mathrm{d}S,$$

where the factor 2 has been included to account for the image source in the wall and the integration is performed over the surface S of the piston. Given the velocity, the source $q = \rho_0 v\exp[-\mathrm{j}\omega t]$ so that the resulting integral for the radiated sound is:

$$p(\omega) = -\mathrm{j}\frac{\omega\rho_0}{2\pi}\iint_S \frac{\mathrm{e}^{\mathrm{j}kR}}{R}v\,\mathrm{d}S.$$

To evaluate the integral, we switch to cylindrical coordinates (r,θ,z):

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We assume that the observer is at $\theta = 0$ and the integral to be evaluated is:

$$\begin{aligned} p(\omega) &= -\mathrm{j}\frac{\omega\rho_0 v}{2\pi}\int_0^{2\pi}\int_0^a \frac{\mathrm{e}^{\mathrm{j}kR}}{R}r_1\,\mathrm{d}r_1\,\mathrm{d}\theta_1, \\ R &= (r^2 + r_1^2 - 2rr_1\cos\theta_1 + z^2)^{1/2}, \end{aligned}$$

where (r_1,θ_1) indicates a point on the piston surface.

This integral cannot be evaluated exactly for a general observer position but we can restrict it to the case where the observer is on the axis of the piston. Then $r = 0$ and $R = (r_1^2 + z^2)^{1/2}$:

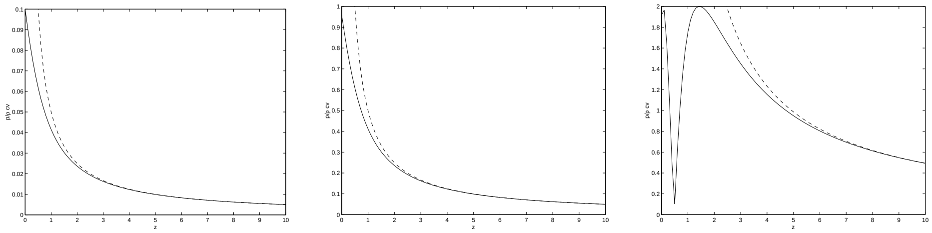
$$\begin{aligned} p &= -\mathrm{j}\frac{\omega\rho_0 v}{2\pi}\int_0^{2\pi}\int_0^a \frac{\mathrm{e}^{\mathrm{j}kR}}{R}r_1\,\mathrm{d}r_1\,\mathrm{d}\theta_1, \\ &= -\mathrm{j}\omega\rho_0 v\int_0^a \frac{\mathrm{e}^{\mathrm{j}kR}}{R}r_1\,\mathrm{d}r_1, \end{aligned}$$

and making the transformation $r_1 \to R$,

$$p = -\mathrm{j}\omega\rho_0 v\int_{R_0}^{R_a}\mathrm{e}^{\mathrm{j}kR}\,\mathrm{d}R.$$

Here, $R_0 = z$ is the distance from the observer to the centre of the piston and $R_a = (a^2 + z^2)^{1/2}$ is the distance to the rim of the piston. The solution is then:

$$p = -\rho_0 cv(\mathrm{e}^{\mathrm{j}kR_a} - \mathrm{e}^{\mathrm{j}kz}). \tag{3.1}$$



a: $ka = 0.1$ b: $ka = 1.0$ c: $ka = 10.0$

Figure 3.3: Acoustic field (absolute value of p) along the axis of a vibrating piston. The dashed line shows the $1/z$ fit.

If we examine the acoustic field defined by equation 3.1 as a function of frequency, we can see that it changes quite rapidly as ka is increased. Figure 3.3 shows the absolute value of the non-dimensional pressure $|p/\rho_0 cv|$ for different values of ka. For comparison, the curve $1/R_0 = 1/z$ is also shown. The results for $ka = 0.1$ and $ka = 1$ are similar with a smooth $1/R_0$ decay but the $ka = 10$ curve is quite different, having a sharp drop before it begins to follow a $1/R_0$ curve. This is a result of interference between sound from different parts of the piston. When a body is large compared to the wavelength of the sound it generates, interference between different parts of the body gives rise to a complicated sound pattern, especially in the region near the body. When the body is small on a wavelength scale (or, equivalently, vibrates at low frequency), the phase difference between different parts of the source is not enough to give rise to much interference and the body radiates like a point source. The ‘size’ of the body at a given frequency is called its *compactness* and is characterized by the parameter ka where a is a characteristic dimension, or by the ratio of characteristic dimension to wavelength a/λ. A compact source, one with $ka \ll 1$, radiates like a point source, while non-compact bodies must be treated in more detail, as we saw in the case of a sphere in §1.6.

Chapter 4

Sound in pipes and ducts

The propagation of sound along ducts or pipes is an area of acoustics which is important in many fields, including engine exhaust systems, fuel systems, oil and gas pipelines and musical instruments. In the case of musical instruments and engine exhausts, we are concerned with the noise which escapes from the open end of the duct. In pipelines, the propagation of pulses within the duct is of special interest, especially since such pipelines can be hundreds or thousands of kilometres in length. Exhausts are often designed acoustically to give a small boost in engine power at the design speed (which is why motorcyclists often change the exhaust can on their bike).

4.1 One-dimensional propagation in ducts

At low frequency (i.e. for sound whose wavelength is much greater than the duct diameter) waves are planar and the propagation can be modelled as one dimensional. A wave of constant frequency and complex amplitude P then has the form $P\exp[\mathrm{j}kx]$ if it propagates to the right and $P\exp[-\mathrm{j}kx]$ if it propagates to the left. As before a factor $\exp[-\mathrm{j}\omega t]$ is assumed. From the solution of the one-dimensional wave equation, §1.3, we know that sound does not decay as it does in three dimensions, so any disturbance will propagate unchanged unless the duct changes. This can mean that the duct either changes form or terminates.

4.2 Reflection from duct terminations

The first simple problem to consider is how the termination of a duct affects the sound field inside it. As an example, we take the case of a hard wall termination. The boundary condition is the same as in §3.2: the acoustic velocity on the end wall must be zero, or $\nabla p_{\mathrm{t}} = 0$.

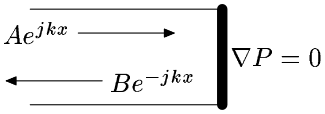


Figure 4.1: Duct with hard wall termination

Figure 4.1 shows the system. A right-travelling wave $p_{\mathrm{i}} = A\exp[\mathrm{j}kx]$ is incident on the wall. We want to find a left-travelling wave $p_{\mathrm{r}} = B\exp[-\mathrm{j}kx]$ such that the total field inside the duct $p_{\mathrm{i}} + p_{\mathrm{r}}$ satisfies the boundary condition on the duct termination. In one dimension, $\nabla = \partial/\partial x$ so the boundary condition is:

$$\frac{\partial p_{\mathrm{t}}}{\partial x} = \mathrm{j}kA\mathrm{e}^{\mathrm{j}kx} - \mathrm{j}kB\mathrm{e}^{-\mathrm{j}kx} = 0.$$

Evaluating at $x = 0$, the position of the termination,

$$A = B,$$

so that the reflected wave has the same amplitude and phase as the incident wave. We might have expected this from §3.2: this is the result we would have found by using an image source.

4.3 Resonant systems

Resonance in ducts

When we take into account both ends of a duct, the behaviour of a constant frequency sound field changes. As an example, consider a duct of length L with a hard wall termination at $x = 0$ and at $x = L$. Now the boundary condition has to be applied at both ends:

$$\begin{aligned} \mathrm{j}kA\mathrm{e}^{\mathrm{j}kL} - \mathrm{j}kB\mathrm{e}^{-\mathrm{j}kL} &= 0 \quad (x = L), \\ \mathrm{j}kA - \mathrm{j}kB &= 0 \quad (x = 0). \end{aligned}$$

As before the $x = 0$ boundary condition tells us that $A = B$ but this is not enough information to satisfy the boundary condition at $x = L$. Substituting for B, this now reads:

$$\mathrm{e}^{\mathrm{j}kL} - \mathrm{e}^{-\mathrm{j}kL} = 0,$$

which we can only satisfy for certain values of kL. The length of the duct is fixed, which means that we can find a valid solution only for certain wavenumbers. In this case, the requirement is that $kL = m\pi$, where m is an integer. Rearranging this condition:

$$\lambda = \frac{2L}{m},$$

meaning that there are $m/2$ wavelengths contained in the duct length. The frequencies corresponding to these values of wavenumber are *resonant frequencies* of the duct and are the only frequencies at which the duct can support a constant frequency sound field.

Helmholtz hits the bottle

One of the most important resonant systems is the *Helmholtz resonator*, the classic example of which is the wine or beer bottle. It is modelled, figure 4.2, as a volume V connected to the outside world by a neck of length l and cross-sectional area S. We can estimate the resonant frequency of the system by considering the motion of a 'plug' of fluid in the neck of the bottle under the action of an external force and an internal restoring force due to the compressibility of the fluid in the bulb.

Assuming that the process is adiabatic, the density and pressure in the bulb are related by:

$$p = k\rho^{\gamma}; \quad \frac{\mathrm{d}p}{\mathrm{d}\rho} = c^2,$$

as in §1.2. If the plug of fluid in the neck of the bottle is displaced by an amount ξ (assumed positive out of the neck), the volume of fluid inside the bulb changes by an amount $S\xi$. Using subscript 0 to indicate mean values, the resulting change in density is:

$$\begin{aligned} \frac{\rho}{\rho_0} &= \frac{V}{V - S\xi}, \\ &= \frac{1}{1 - (S/V)\xi}, \\ &\approx 1 - \frac{S}{V}\xi, \end{aligned}$$

by the binomial theorem and the corresponding change in pressure is:

$$p - p_0 = -\rho_0 \frac{c^2 S}{V}\xi.$$

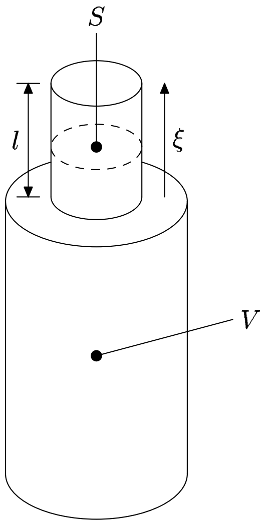


Figure 4.2: Helmholtz' bottle

The equation of motion for the plug can then be written, noting that its mass $m = \rho_0 S l$:

$$\rho_0 S l \ddot{\xi} + \rho_0 \frac{c^2 S}{V} \xi = -p_a S,$$

where p_a is the externally applied pressure. This is the equation of motion for an oscillator with a resonant frequency:

$$\omega = \sqrt{\frac{c^2 S}{V l}}.$$

Helmholtz resonators can be used whenever you want to reduce noise at some known frequency. One of the main applications is in acoustic liners used in aircraft engines, which are made up of a large number of small Helmholtz resonators with dimensions chosen to absorb noise at a specified frequency.

4.4 Combustion oscillations

A very good introduction to this area—and the basis of this section—is DOWLING, A. 2000, "Vortices, sound and flames—a damaging combination", *Aeronautical Journal*, 104(1033):105–116.

Another important application of one-dimensional acoustics is in combustion instability in engines. In order to model such a problem, we need to look at the *thermodynamics* of the system in order to model the effects of heat release. When we derived the wave equation in §1.1, we assumed that the system was adiabatic—no heat was added or removed. Obviously, if we want to look at a problem involving heat addition, this assumption is wrong so we have to include some extra information.

From thermodynamics, we know that:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{1}{c^2}\frac{\mathrm{D}p}{\mathrm{D}t} + \left.\frac{\partial \rho}{\partial s}\right|_p \frac{\mathrm{D}s}{\mathrm{D}t}, \tag{4.1}$$

which is what we derived in §1.1 but we now include a term which depends on s the *entropy* of the fluid. When, as we assumed previously, the flow is isentropic, the second term disappears. When we include heat release in the problem, however, we cannot ignore the entropy variations.

When we ignore viscosity and heat conduction, the heat input q per unit volume is given by

$$q(\mathbf{x}, t) = \rho T \frac{\mathrm{D}s}{\mathrm{D}t}.$$

For a perfect gas,

$$\left.\frac{\partial \rho}{\partial s}\right|_p = -\frac{\rho}{c_p} = -\frac{\rho T(\gamma - 1)}{c^2},$$

where c_p is the specific heat at constant pressure and γ the ratio of the specific heats. We can substitute this relation into equation 4.1:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{1}{c^2}\left[\frac{\mathrm{D}p}{\mathrm{D}t} - (\gamma - 1)q\right]. \tag{4.2}$$

If we assume that perturbations are small and that there is no mean heat addition (otherwise the speed of sound and other thermodynamic properties would change), we can linearize this equation:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{1}{c_0^2}\left[\frac{\partial p'}{\partial t} - (\gamma - 1)q\right], \tag{4.3}$$

where c_0 is the mean speed of sound. If we now return to equation 1.3,

$$\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 p' = 0,$$

we can insert this new relationship between p' and ρ' to find:

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = \frac{\gamma - 1}{c_0^2}\frac{\partial q}{\partial t}, \tag{4.4}$$

and we end up with a linear wave equation with a source term on the right hand side which is related to the heat input per unit volume. If we reduce this to the one-dimensional case,

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = \frac{\gamma - 1}{c_0^2}\frac{\partial q}{\partial t}, \tag{4.5}$$

we can look at some simple problems related to combustion in aero-engines.

If we think of combustion happening in a tube of length L open at both ends, the pressure inside the tube has to be of the form

$$p(x,t) = P(t)\sin\frac{n\pi x}{L}$$

and the wave equation becomes

$$\left[\frac{\ddot{P}}{c_0^2} + \frac{n^2\pi^2}{L^2}P\right]\sin\frac{n\pi x}{L} = \frac{\gamma - 1}{c_0^2}\frac{\partial q}{\partial t}.$$

If we now assume that the unsteady heat release is related to the unsteady pressure, we can see how it affects the acoustics.

The first simple assumption is that the heat release is proportional to pressure,

$$q = \frac{-\alpha c_0^2 p'}{\gamma - 1},$$

which leads to the equation for pressure,

$$\frac{\ddot{P}}{c_0^2} + \alpha\dot{P} + \frac{n^2\pi^2}{L^2}P = 0,$$

which is the equation for a damped oscillator (think of the spring-mass-dashpot system you saw in mechanics). If α is positive, the response P decays with time. If, however, α is negative, the response grows over time: the combustion is unstable. The case where α is positive corresponds to heat addition 180° out of phase with the pressure; negative α means that the heat addition is in phase with the pressure. This is *Rayleigh's criterion*: heat must be added in phase with pressure if energy is to be transferred into the acoustic waves. Remember that the heat release is proportional to the pressure, so if the pressure is unstable, so is the heat release and your engine blows up.

This is a very simple example which ignores the mechanism of heat addition—the combustion of fuel—but it illustrates how the combustion depends on the relationship between the acoustics and the heat generated in the system.

Chapter 5

Sound from moving sources

> Yeeeeeeeeehaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaw.
>
> Major T.J. "King" Kong (Slim Pickens) in *Doctor Strangelove or: How I learned to stop worrying and love the bomb*.

As you may be aware from the movies and the scream of Major Kong as he plummets to his doom astride a bomb, the sound heard from a source changes if the source is moving. As Major Kong falls Russia-wards, he accelerates (Isaac Newton says he has to). This acceleration changes the frequency of his shout as he falls.

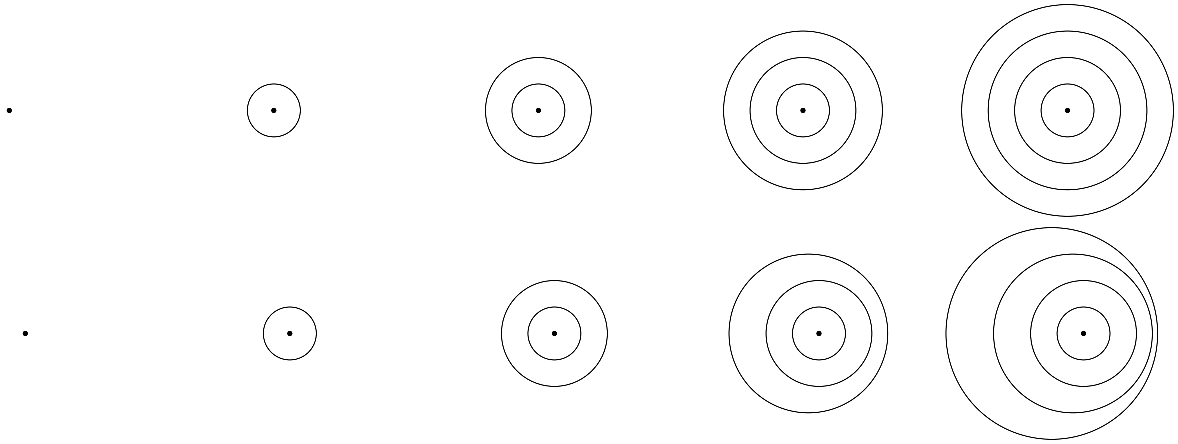


Figure 5.1: A simple model for the Doppler effect, *a*: stationary source; *b*: moving source.

Figure 5.1 shows what is happening. Figure 5.1*a* shows the wavefronts radiating from a stationary source. They propagate at the speed of sound and along any line from the source, they are equidistant. In figure 5.1*b*, the source moves to the right at some velocity V. The wavefronts still travel at the speed of sound, but each is generated a point successively further to the right. This causes the wavefronts to bunch up ahead of the source and stretch out behind it. This obviously leads to a change in the frequency of the sound at some observer position but also to a change in the amplitude, as more or fewer wavefronts arrive per unit time.

To quantify the effect of motion on the sound radiated by a source, we use the solution of the wave equation, equation 1.22, with a moving point source:

$$q(\mathbf{y},t) = q(t)\delta(\mathbf{y}-\mathbf{y}_0(t)).$$

This represents a point source which is at $\mathbf{y} = \mathbf{y}_0$ at time t. Inserting this into equation 1.22:

$$p = \int_\tau \int_V q(t) \frac{\delta(\tau - t + R/c)}{4\pi R} \,\mathrm{d}V \,\mathrm{d}\tau. \tag{5.1}$$

This can be solved using the normal relationship for the delta function, but with the change of variables $\tau \to g$ where $g(\tau) = \tau - t + R/c$:

$$\int \delta(g(\tau)) f(\tau) \,\mathrm{d}\tau = \left. \frac{f(\tau)}{|dg/d\tau|} \right|_{g(\tau)=0}.$$

Integrating over τ in equation 5.1

$$\int_\tau \frac{\delta(\tau - t + R/c)}{4\pi R} \,\mathrm{d}\tau = \frac{1}{4\pi R |1 + \partial R/\partial\tau/c|}$$

where

$$\frac{\partial R}{\partial \tau} = -\frac{\partial \mathbf{y}_0}{\partial \tau} . \frac{\mathbf{x} - \mathbf{y}_0}{R},$$
$$\frac{1}{c}\frac{\partial \mathbf{y}_0}{\partial \tau} = \mathbf{M},$$

the source (vector) Mach number and

$$M_r = -\mathbf{M} . \frac{\mathbf{x} - \mathbf{y}_0}{R},$$

the relative Mach number of the source in the direction of the observer, so that

$$p = \int_V \frac{q(\tau)}{4\pi R |1 - M_r|} \,\mathrm{d}V.$$

Because q is a point source, we can integrate over V to find:

$$p = \frac{q(\tau)}{4\pi R |1 - M_r|}.$$

Finally, for a moving source with monopole strength q and dipole strength $\mathbf{f}$:

DOPPLER, Christian Andreas (b. Salzburg, 29 Nov. 1803, d. Venice, 17 March 1853) ‘experiments were conducted with musicians on railway trains playing instruments and other trained musicians writing down the apparent note as the train approached them and receded from them.’

$$\boxed{p = \frac{\partial}{\partial t} \frac{q(\tau)}{4\pi R |1 - M_r|} + \nabla . \frac{\mathbf{f}(\tau)}{4\pi R |1 - M_r|}.} \tag{5.2}$$

The important thing to note here is that the sound is amplified by a factor $1/|1 - M_r|$, the *Doppler factor*. For a supersonic source, it can happen that $1 - M_r = 0$ and the pressure p is infinite. It is also important to realize that a source which is steady in its own reference frame (the loading on a propeller blade, for example) can still radiate noise if it is moving, due to variations in the Doppler factor.

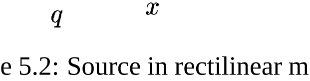


Figure 5.2: Source in rectilinear motion

We now look again at the problem of a monopole source moving in a straight line, figure 5.2. The position of the source is $x = vt$. The general problem is left as an exercise, and here we will only look at the sound radiated to an observer on the axis of motion. To work out the radiated noise for an observer ahead of the source, we need the following quantities:

$$
\begin{aligned}
R &= c(t-\tau) = x - v\tau, \\
\tau &= \frac{t - x/c}{1-M}, \\
R &= \frac{x - vt}{1-M}, \\
M_r &= M.
\end{aligned}
$$

The source-observer Mach number M_r is equal to the source Mach number M for observer positions ahead of the source ($x > vt$) and $-M$ for observer positions behind the source ($x < vt$). Inserting the various quantities into equation 5.2:

$$
p = \frac{\partial}{\partial t}\frac{1}{4\pi}\frac{q(\tau)}{x - vt}.
$$

To look at the effect of motion on the frequency of the noise, consider a source with $q = \exp[-\mathrm{j}\omega t]$. The sound heard by an observer will be proportional to $\exp[-\mathrm{j}\omega\tau]$. Since $\tau = (t - x/c)/(1-M)$, the sound at the observer will be proportional to

$$
\exp[-\mathrm{j}\omega(t - x/c)/(1-M)]
$$

and the perceived frequency will be $\omega/(1-M)$. For points behind the source, $R = x + v\tau$ and the perceived frequency is $\omega/(1+M)$.

Chapter 6

Aerodynamically-generated noise: propellers and rotors

The calculation of the noise generated by a general body in arbitrary motion is a hard problem. The sound radiated by a source undergoing motion as simple as pure rotation is qualitatively different from that of a source moving in a straight line. This is partly because the calculation of the retarded time and the Doppler factor is not as simple as in the linear motion case and partly because of the difficulty of calculating the source terms, the force and volume sources of equation 5.2.

6.1 Sound from rotating sources

To keep things as simple as possible without making them unrealistic, we will look at the problem of the sound radiated by a rotating point source. This is a very simple system but contains most of the behaviour of real rotors and will spare us the agonies of dealing with superfluous difficulties. The arrangement is shown in figure 6.1: a point source at radius a rotates at frequency Ω. We assume that there is no forward motion, so this system corresponds to a stationary propeller, or a helicopter rotor in hover.

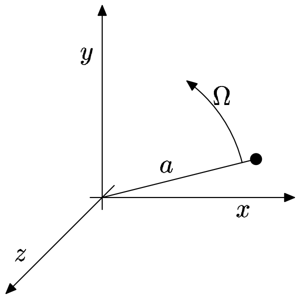


Figure 6.1: A rotating source

We will use cylindrical coordinates (r, θ, z) and assume that the observer is positioned at a point $(r, 0, z)$. Changing the angular position of the observer will only affect the phase of the sound and not its overall shape. To make things easier for ourselves, we will work in terms of the retarded time rather than the observer time. The position of the source at time τ is:

$$(a \cos \Omega\tau, a \sin \Omega\tau, 0).$$

Differentiating, its velocity is:

$$(-a\Omega \sin \Omega\tau, a\Omega \cos \Omega\tau, 0).$$

For more information, you might like to look at CHAPMAN, C. J. 1990, “The spiral Green function in acoustics and electromagnetism”, *Proc. R. Soc. Lond.* A, **431**:157–167.

The source observer distance is (remember the observer does not move):

$$R^2 = R_0^2 + a^2 - 2ar\cos\Omega\tau,$$

where R_0 is the distance of the observer from the centre of rotation,

$$R_0 = [r^2 + z^2]^{1/2}.$$

We have the source-observer distance, but to calculate the Doppler factor we need to know the source–observer Mach number M_r:

$$\begin{aligned} M_r &= -\frac{1}{c}\frac{\partial R}{\partial \tau}, \\ \frac{\partial R}{\partial \tau} &= a\frac{r}{R}\Omega\sin\Omega\tau, \\ M_r &= -\frac{r}{R}M_t\sin\Omega\tau. \end{aligned}$$

Here $M_t = a\Omega/c$ is the rotational Mach number of the source. The Doppler factor is:

$$\frac{1}{|1-M_r|} = \frac{R}{|R + rM_t\sin\theta|},$$

where $\theta = \Omega\tau$ is the position of the source at time τ. The first obvious thing is to check if and when the Doppler factor becomes (nominally) infinite:

$$R = -rM_t\sin\theta.$$

This can be solved by squaring both sides and remembering that $\sin^2\theta = 1 - \cos^2\theta$:

$$M_t^2r^2\cos^2\theta - 2ar\cos\theta + R_0^2 + a^2 - M_t^2r^2 = 0.$$

If we now scale all lengths on the source radius a, the equation becomes:

$$M_t^2r^2\cos^2\theta - 2r\cos\theta + R_0^2 + 1 - M_t^2r^2 = 0, \tag{6.1}$$

which has two solutions:

$$\cos\theta = \frac{1}{M_t^2 r} \pm \frac{1}{M_t^2 r}[(1-M_t^2)(1-M_t^2r^2) - M_t^2z^2]^{1/2}. \tag{6.2}$$

If the source is to approach the observer at sonic velocity, the solution for $\cos\theta$ must be real. This means that the term inside the square root must not be negative:

$$(1-M_t^2)(1-M_t^2r^2) - M_t^2z^2 \geq 0.$$

Solving with this term set to zero:

$$z^2 = (M_t^2 - 1)\left(r^2 - \frac{1}{M_t^2}\right), \tag{6.3}$$

which defines a curve in the r–z plane dividing points where the source approaches at sonic velocity from points where it does not. For z^2 to be positive (i.e. a valid point in the plane) $M_t > 1$ and $r > 1/M_t$. This means that a source must be travelling supersonically if it is to approach an observer position at sonic velocity (hardly a surprise) and the observer position must lie outside the *sonic radius* $1/M_t$, which is the radius where the source has, or would have, sonic rotation velocity. Figure 6.2 shows the dividing curves for different values of M_t. The region inside the curve, labelled 'subsonic', never experiences the source approaching at sonic velocity, while the points in the outer region, labelled 'sonic', do.

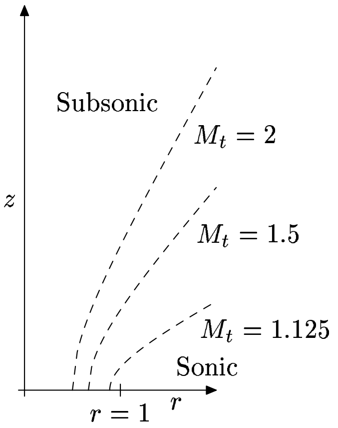


Figure 6.2: Points subject to Doppler radiation from a rotating source. The dashed lines indicate the curve $z^2 = (M_t^2 - 1)(r^2 - 1/M_t^2)$ for different tip Mach numbers.

We have managed to get this far without ever calculating the noise heard at some observation point. If we now calculate the quantities we need to work out the noise:

$$\begin{aligned}
R &= [1 + r^2 + z^2 - 2r\cos\theta]^{1/2}, \\
1 - M_r &= 1 + M_t \frac{r}{R}\sin\theta, \\
\Omega t &= \theta + M_t R, \\
\frac{1}{4\pi R|1 - M_r|} &= \frac{1}{4\pi|R + rM_t\sin\theta|},
\end{aligned}$$

where lengths are still scaled on a and θ is still the source position at the retarded time τ.

To calculate the radiated noise, we simply take different values of θ, ranging from 0 to 2π and calculate the corresponding values of R and the arrival times Ωt. If the values of θ are evenly spaced, we do not expect the values of Ωt to be evenly spaced, but they will cover a range of 2π.

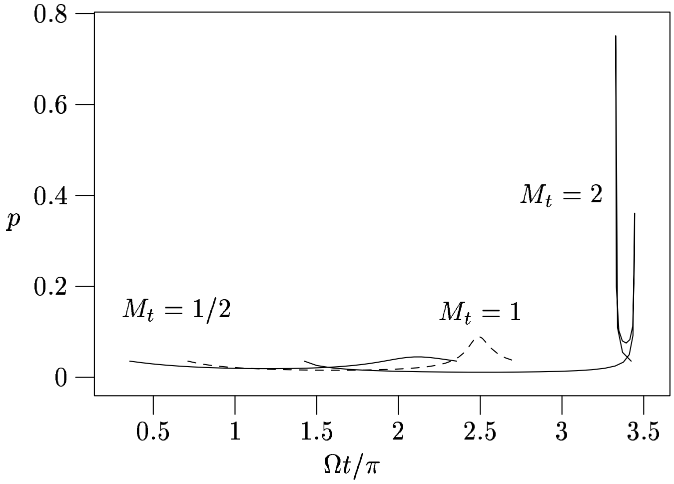


Figure 6.3: Time records for rotating source.

Figure 6.3 shows $1/4\pi R|1 - M_r|$ plotted against $\Omega t/\pi$ for three different values of M_t. Note that in each case, Ωt covers a range of 2π. As you might expect, the noise for $M_t = 0.5$ is weaker (though not much weaker) than that for $M_t = 1$ which is very much weaker than that for $M_t = 2$. This is not unexpected but there is something strange about the noise record for $M_t = 2$: there are three values of pressure for some time points.

"In or near the plane of rotation the noise has peculiar and indescribably unpleasant physiological effects.", E. J. H. Lynam, *Preliminary report of experiments on a high tip-speed airscrew at zero advance*, Aeronautical Research Committee, Reports and Memoranda, 596, 1919.

The reason for this is shown in figure 6.4 which shows the position θ as a function of Ωt. For $M_t = 2$, there is a range of Ωt for which there are three values of τ, meaning that the sound received at each time has a contribution from three different source positions. This is a feature unique to supersonically rotating sources and illustrates the manner in which noise from such sources is *qualitatively* different and is not just a louder version of subsonic source noise. For higher rotation speeds, there can be five, seven or more retarded times for a given arrival time.

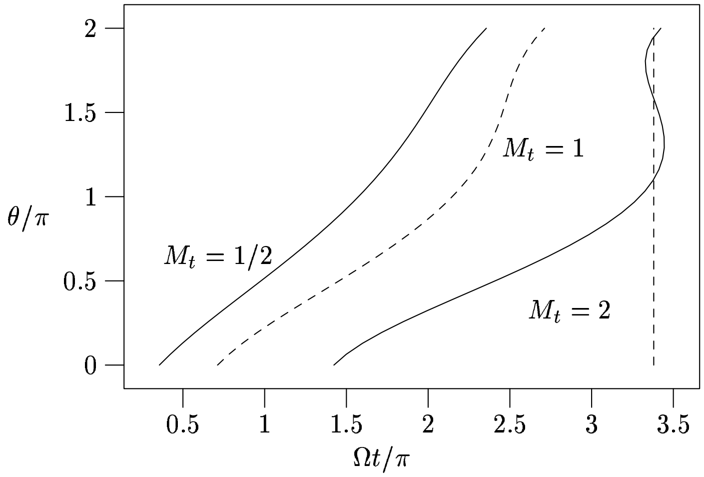


Figure 6.4: Retarded times for rotating source: the vertical dashed line indicates a value of t for which there are three values of τ.

Chapter 7

Aerodynamically-generated noise: jets

The approach to sound generation by sources in a flow is that of Lighthill who developed the basis of modern aeroacoustics in the 1950s, as civil jet engines were being developed. The derivation given here follows Lighthill's original approach but is closer to that of Powell who developed a theory of sound generation by vorticity. The idea is to go through the motions of §1.1 but without linearizing the equations. The exact equations of inviscid fluid motion are:

"Theory of vortex sound", *Journal of the Acoustical Society of America*, **36**(1):177–195.

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\mathbf{v}) = 0, \tag{7.1a}$$

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}\nabla\mathbf{v} + \nabla p = 0. \tag{7.1b}$$

As in §1.1, we differentiate equation 7.1a with respect to time, equation 7.1b with respect to space and subtract one from the other:

$$\nabla^2 p - \frac{\partial^2 \rho}{\partial t^2} = \nabla.\left(\nabla p + \frac{\partial}{\partial t}(\rho\mathbf{v})\right). \tag{7.2}$$

Lighthill, Michael James (b. Paris 23 Jan 1924, d. Sark 17 July 1998) Lucasian Professor 1969–1980, contributions in aeroacoustics, biofluiddynamics and Fourier theory. Swam around islands for fun.

To simplify this equation, we can rearrange equations 7.1. Multiplying equation 7.1b by $\mathbf{v}$ and adding it to equation 7.1a:

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla.(\rho\mathbf{v}\mathbf{v}) + \nabla p = 0.$$

Inserting this into equation 7.2:

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = -\nabla.\left(\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla.(\rho\mathbf{v}\mathbf{v}) - \frac{\partial}{\partial t}(\rho\mathbf{v})\right),$$

which includes the usual approximation for the relationship between ρ and p. The product $\rho\mathbf{v}\mathbf{v}$ is to be read as a tensor (like a matrix, or vector of vectors) which can be written:

$$\mathbf{T} = \begin{bmatrix} \rho v_x v_x & \rho v_y v_x & \rho v_z v_x \\ \rho v_x v_y & \rho v_y v_y & \rho v_z v_y \\ \rho v_x v_z & \rho v_y v_z & \rho v_z v_z \end{bmatrix},$$

or, more compactly, $T_{ij} = \rho v_i v_j$. The net result is then:

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = -\nabla\nabla(\rho\mathbf{v}\mathbf{v}), \tag{7.3}$$

which is an approximation to Lighthill's theory of aerodynamically generated sound.

If you must know, it should read $\partial^2\rho/\partial t^2 - c_0^2\nabla^2\rho = \nabla\nabla T_{ij}$, $T_{ij} = \rho v_i v_j + p_{ij} - c_0^2\rho\delta_{ij}$.

7.1 Lighthill's eighth power law for jet noise

Solving Lighthill's equation for different sources is more than we can manage in these notes, but we can derive a scaling law for jet noise which was one of the first great successes of the theory. The 'solution' of equation 7.3 is

$$p = -\nabla\nabla \int_V \frac{\mathbf{T}(\mathbf{y}, t - R/c_0)}{4\pi R}\,\mathrm{d}V,$$

where $\mathbf{T} = \rho\mathbf{v}\mathbf{v}$. In the far field, we can approximate this integral by differentiating it, as we did with the point force in §1.9. When we do this, we will retain only terms which depend on $1/R$ (everything else decays much more rapidly). Setting coordinates so that the origin is inside the source region, $\mathbf{x} - \mathbf{y} \approx \mathbf{x}$ and

$$p \approx \frac{1}{4\pi}\frac{\mathbf{x}\mathbf{x}}{x^3}\int_V \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\mathbf{T}(\mathbf{y}, t - R/c_0)\,\mathrm{d}V.$$

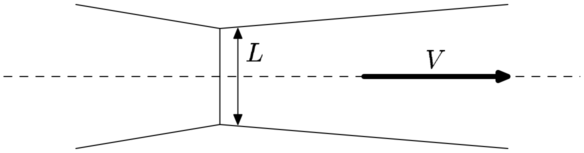


Figure 7.1: Parameters for jet noise.

There is no general solution for this equation, but we can derive a scaling law for the radiated acoustic power. Figure 7.1 shows a simple jet flow with the relevant parameters indicated. We take a characteristic length L, characteristic velocity V and a mean density ρ_0. Then:

$$\mathbf{T} \sim \rho_0 V^2, \quad \frac{\partial}{\partial t} \sim \frac{V}{L},$$

$$p \sim \frac{1}{4\pi}\frac{1}{x}\frac{1}{c_0^2}\left(\frac{V}{L}\right)^2 \rho_0 V^2 L^3,$$

and the pressure scales as:

$$p \sim \rho_0 \frac{V^4}{c_0^2}\frac{L}{x}.$$

From equation 2.6, the intensity scales as

$$\bar{I} \sim \rho_0 \frac{V^8}{c_0^5}\left(\frac{L}{x}\right)^2.$$

The total acoustic power W is the intensity integrated over a spherical surface of radius x and

$$W \sim \rho_0 \frac{V^8}{c_0^5} L^2. \tag{7.4}$$

The total acoustic power thus scales on the eighth power of jet velocity. This is Lighthill's eighth power law and was derived before experimental data were available to confirm it: it is one of the few scientific predictions to have been a genuine prediction. It is strictly only true for low speed flows, because we have implicitly assumed the source to be compact. At higher speeds, the characteristic frequency of the source increases and interference effects become important.

References

These notes only cover some of the basic elements of acoustics. Recommended texts if you want a different view or to deepen your knowledge:

- DOWLING, A. P. & FFOWCS WILLIAMS, J. E. 1983, *Sound and sources of sound*, Butterworth. This is quite a slim book compared to Pierce but it covers more of the things in these notes.

- CRIGHTON, D. G., DOWLING A. P., FFOWCS WILLIAMS, J. E., HECKL, M. & LEPPINGTON, F. G. 1992, *Modern methods in analytical acoustics,* Springer-Verlag. Very mathematical but covers a lot of material.

- HUBBARD, H. H. ed 1995, *Aeroacoustics of flight vehicles*, Acoustical Society of America. This is a two volume review of almost everything connected to noise from aircraft.

- LIGHTHILL, M. J. 1952, On sound generated aerodynamically: I General theory, *Proceedings of the Royal Society* A, **211**:564–587. This is the foundation of modern aeroacoustics and is surprizingly readable for a paper of such fundamental importance.

- PIERCE, A. 1994, *Acoustics: An introduction to its physical principles and applications*, American Institute of Physics, New York. This is the standard modern reference for acoustics. If you want to buy one comprehensive book on acoustics, this is the one. It doesn't really cover aerodynamically generated noise so you might want to look at Dowling & Ffowcs Williams as well.

If you want to know more about the history of acoustics and how it developed, you could start with:

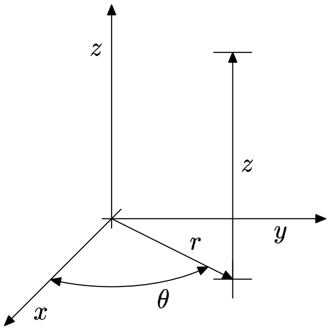
- the biographies of mathematicians at http://www-groups.dcs.st-andrews.ac.uk/ history/BiogIndex.html which provided much of the information in the side panels of these notes.

- HUNT, F. V. 1992, *Origins in acoustics,* Acoustical Society of America. This short book contains a good history of acoustics and where it came from.

Some useful mathematics

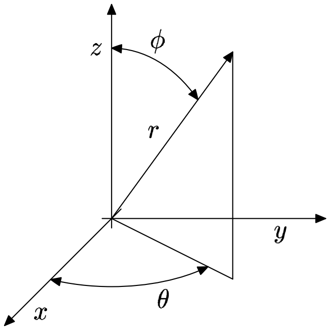
Coordinate systems

Cylindrical coordinates:



$$\begin{aligned} x &= r\cos\theta, \quad y = r\sin\theta; \\ r &= (x^2+y^2)^{1/2}, \quad \theta = \tan^{-1} y/x. \end{aligned}$$

Spherical coordinates:



$$\begin{aligned} & x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta, \\ & z = r\cos\phi; \\ & r = (x^2+y^2+z^2)^{1/2}, \quad \theta = \tan^{-1} y/x, \\ & \phi = \tan^{-1} z/(x^2+y^2)^{1/2}. \end{aligned}$$

Differential operators

In Cartesian coordinates:

$$\begin{aligned} \nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right), \\ \nabla.\mathbf{f} &= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}, \\ \nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \end{aligned}$$

In cylindrical coordinates:

$$\begin{aligned} \nabla f &= \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right), \\ \nabla.\mathbf{f} &= \frac{1}{r}\frac{\partial}{\partial r}(r f_r) + \frac{1}{r}\frac{\partial f_\theta}{\partial \theta} + \frac{\partial f_z}{\partial z}, \\ \nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \end{aligned}$$

In spherical coordinates:

$$\begin{aligned} \nabla f &= \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \phi}, \frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\right), \\ \nabla.\mathbf{f} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f_r\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}(f_\phi \sin\phi) \\ &\quad + \frac{1}{r\sin\phi}\frac{\partial f_\theta}{\partial \theta}, \\ \nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) \\ &\quad + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}. \end{aligned}$$

He has never again encountered the most esteemed Arkady Apollonovich Sempleyarov in connection with acoustical problems. The latter was quickly transferred to Bryansk and appointed director of a mushroom-growing center. Nowadays, Moscow residents eat pickled saffron milk caps and marinated white mushrooms with endless relish and praise, and never stop rejoicing in the lucky transfer. Since it is all a matter of the past now, we feel free to say that Arkady Apollonovich never did make any headway with acoustics, and, for all his efforts to improve the sound, it remained as bad as it was.

The Master and Margarita, Mikhail Bulgakov