Perturbation methods for ordinary and partial differential equations

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In this series of lectures various methods are introduced which allow an approximate solution of both ordinary and partial differential equations. Collectively these techniques are known as perturbation methods, and they range from the very familiar Taylor series methd to technqiues of boundary layer theory, WKB theory and the method of Multiple Scales. A sample set of problems is also given, together with worked solutions to many of them. The first 14 lectures introduce the methods, and these are applied to some problems which have formed part of my recent research.

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The following books were used in the preparation of this course:

C.M.Bender and S.A.Orszag, "Advanced Mathematical Methods of Engineers and Scientists", McGrawHill, 1978.

W.E.Boyce and R.C.DiPrima, "Elementary Differential Equations and Boundary Value Problems (6th edition)", Wiley, 1997.

A.W.Bush, "Perturbation Methods for Engineers and Scientists", CRC Press, 1992.

A.H.Nayfeh, "Problems in Perturbation", Wiley, 1985.

M. van Dyke, "Perturbation Methods in Fluid Dynamics", Parabolic Press 1978.

1. The Solution of ODEs using Taylor series.

Given an ordinary differential equation we shall assume that its solution may be written in terms of a Taylor series. Note that this assumption is not always true: the solutions of the equation

$$4xy'' + 2y' + y = 0$$

are $\cos\sqrt{x}$ and $\sin\sqrt{x}$, only one of which has a Taylor series based at $x = 0$. A Taylor series proceeds in integral powers of $(x - x_0)$. If $y = f(x)$, then its Taylor series about $x = x_0$ is

$$y = f(x_0) + (x - x_0)f'(x_0) + \tfrac{1}{2}(x - x_0)^2 f''(x_0) + \tfrac{1}{3!}(x - x_0)^3 f'''(x_0) + \cdots. \tag{1.1}$$

The correctness of this may be demonstrated easily by differentiating an appropriate number of times and substituting $x = x_0$. For example, after two differentiations we obtain

$$y''(x) = f''(x_0) + (x - x_0)f'''(x_0) + \cdots,$$

from which we see that $y''(x_0) = f''(x_0)$, as we expect. Whenever $x_0 = 0$, the series is also known by the name Maclaurin's series.

Example 1.1 $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots.$

The coefficients are obtained by substituting $x = 0$ into successive derivatives of e^x.

Example 1.2. $\ln(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} + \cdots.$

Again this may be obtained by differentiation and substitution of $x = 0$, although it is also possible to set $y = \ln x$, differentiate to get $y' = (1 + x)^{-1}$, expand y' using the Binomial theorem: $y' = 1 - x + x^2 - x^3 + \cdots$, integrate to find y, and show that the constant of integration is zero.

Example 1.3 $x^{1/2}e^x = x^{1/2} + x^{3/2} + \frac{1}{2!}x^{5/2} + \frac{1}{3!}x^{7/2} + \cdots$

This is not a Taylor series as it involves nonintegral powers of x. However, the function $x^{1/2}\sin(x^{1/2})$ does have a Taylor series about $x = 0$:

$$x^{1/2}\sin(x^{1/2}) = x - \frac{x^2}{3!} + \frac{x^3}{5!} - \frac{x^4}{7!} + \frac{x^5}{9!} + \cdots .$$

Example 1.4. Solve the equation $y' = -y$ subject to $y(0) = 1$.

If we assume that a Taylor series solution exists, then substitution of

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{1.2}$$

into the governing equation gives,

$$\sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} -a_n x^n.$$

The first sum needs to be modified as its first term is zero. Hence we have

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty} -a_n x^n.$$

Hence

$$\sum_{n=0}^{\infty}\Big[(n+1)a_{n+1} + a_n\Big]x^n = 0,$$

from which we obtain the **recurrence relation**,

$$a_{n+1} = -\frac{a_n}{n+1}.$$

There is no equation for a_0 and this remains as an arbitrary constant in general, but which may be found on applying the initial condition. From this relation we see that $a_1 = a_0$, $a_2 = -a_1/2 = a_0/2$, $a_3 = -a_2/3 = -a_0/3!$ and so on. The initial condition gives $a_0 = 1$, and therefore the solution is

$$y = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!} = e^x.$$

Second order linear equations with constant coefficients may be solved using this technique, but the solution of the *second order* difference equation (i.e. the recurrence relation) is much more difficult than solving the equation directly using the substitution $y = \exp\lambda x$. But this method is very useful for equations with variable coefficients such as Airy's equation.

Example 1.5 Solve Airy's equation, $\dfrac{d^2y}{dx^2} - xy = 0$, using a Taylor series expansion about $x = 0$.

Using (1.2) above, we get

$$[2.1.a_2 + 3.2.a_3x + 4.3.a_4x^2 + 5.4.a_5x^3 + \cdots] - [xa_0 + x^2a_1 + x^3a_2 + x^4a_3 + \cdots] = 0.$$

On equating coefficients of like powers of x we get

$$a_2 = 0 \qquad a_3 = \frac{a_0}{3.2} \qquad a_4 = \frac{a_1}{4.3} \qquad a_5 = \frac{a_2}{5.4} \cdots a_n = \frac{a_{n-3}}{n.(n-1)}.$$

Hence we have three families of cases:

$$0 = a_2 = a_5 = a_8 = a_{11} \cdots$$

$$a_3 = \frac{a_0}{3.2} \qquad a_6 = \frac{a_0}{6.5.3.2} \qquad a_9 = \frac{a_0}{9.8.6.5.3.2.} \cdots$$

and

$$a_4 = \frac{a_1}{4.3} \qquad a_7 = \frac{a_1}{7.6.4.3} \qquad a_10 = \frac{a_1}{10.9.7.6.4.3.} \cdots.$$

Therefore the solution is

$$\begin{aligned} y = a_0y_1 + a_1y_2 = a_0 &\left[1 + \frac{x^3}{3.2} + \frac{x^6}{6.5.3.2} + \frac{x^9}{9.8.6.5.3.2} + \cdots\right] \\ + a_1 &\left[x + \frac{x^4}{4.3} + \frac{x^7}{7.6.4.3} + \frac{x^{10}}{10.9.7.6.4.3} + \cdots\right]. \end{aligned} \tag{1.3}$$

These solutions cannot be written in terms of standard functions such as sines and exponentials. Therefore much work has been undertaken on cataloging their properties such as the asymptotic behaviours as $x \to \pm\infty$ (see Abramowitz and Stegun). The two independent solutions, y_1 and y_2, are shown in Figure 1.1. Also shown is the combination, $y_1 - 0.7290111y_2$, which exhibits exponential decay for large values of x.

Example 1.6 Solve the equation $y'' + xy' + y = 0$.

Again substitute equation (1.2) to obtain

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=1}^{\infty} na_nx^n + \sum_{n=0}^{\infty} a_nx^n = 0.$$

If all the summations are adjusted to get the same power of x in each case, we get

$$\sum_{n=0}^{\infty}\Big[(n+2)(n+1)a_{n+2} + (1+n)a_n\Big]x^n = 0.$$

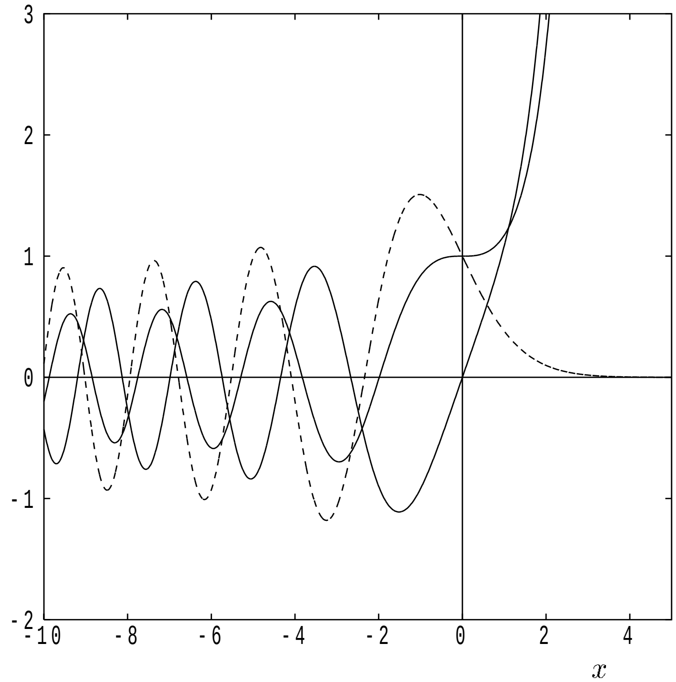

Figure 1.1. The series solutions of Airy's equation as given by (1.3). Also shown as a dashed line is the linear combination of the two series solutions which gives $y \to 0$ as $x \to \infty$.

From this we obtain the recurrence relation,

$$a_{n+2} = -\frac{a_n}{n+2}.$$

Hence the solution is

$$\begin{aligned}
y =& a_0 \left[1 - \frac{x^2}{2} + \frac{x^4}{2.4} - \frac{x^6}{2.4.6} + \frac{x^8}{2.4.6.8} + \cdots\right] \\
&+ a_1 \left[x - \frac{x^3}{3} + \frac{x^5}{3.5} - \frac{x^7}{3.5.7} + \frac{x^9}{3.5.7.9} + \cdots\right] \\
=& a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!2^n} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{n!2^n}{(2n+1)!} x^{2n+1}. \\
=& a_0 e^{-x^2/2} + a_1 e^{-x^2/2} \int_0^x e^{\xi^2/2}\, d\xi
\end{aligned}$$

The second part of this solution was obtained using the technique of variation of parameters. Although this technique is not part of the course, it consists of using one solution to obtain the other. In this case it is possible to identify the closed form version of the 'a_0' solution from its series as $\exp(-x^2/2)$. The other solution is found by substituting $y = v(x) \exp(-x^2/2)$ into the governing equation. There results a first order equation for v' which may be solved to give the solution shown above.

2. The Solution of ODEs using Frobenius series: I

In order to motivate this new form of series solution, consider Bessel's equation of zero order:

Example 2.1 $xy'' + y' + xy = 0$.

Assume a Taylor's series of the form (1.2) and hence the equation becomes,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0,$$

from which we obtain

$$\sum_{n=0}^{\infty} \Big[(n+2)^2 a_{n+2} + a_n\Big] x^{n+1} = 0,$$

and therefore the recurrence relation is

$$a_{n+2} = -\frac{a_n}{(n+2)^2}.$$

The solution is

$$\begin{aligned} y =& a_0 \left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots\right] + a_1 \left[x - \frac{x^3}{3^2} + \frac{x^5}{3^2 5^2} - \frac{x^7}{3^2 5^2 7^2} + \cdots\right] \\ =& a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! n!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} n! n!}{(2n+1)!(2n+1)!} x^{2n+1}. \end{aligned}$$

These solutions seem to be well-behaved, but the 'a_1' series is not in fact a solution of Bessel's equation. If we assume that all solutions are continuous and do not contain singularities, then setting $x = 0$ into Bessel's equation yields $y'(0) = 0$, and therefore we must have $a_1 = 0$. We have only one solution!

For this problem, a second solution involves a logarithmic term. Such difficulties often arise when the coefficient of the highest derivative is a function whose value can become zero. Logarithms do not always arise, as may be seen by Example 2.2 presented a little later. But first it is necessary to categorise equations according to the behaviour of their coefficients.

Given the second order ODE

$$y'' + P(x)y' + Q(x)y = 0 \tag{2.1}$$

we can categorise it into three classes:

1. If both $P(x)$ and $Q(x)$ are analytic at $x = x_0$ (i.e. have a Taylor series expansion) then x_0 is called an **ordinary point** of the equation, and the solution may be written in terms of a Taylor's series.
2. If one or both of $P(x)$ and $Q(x)$ are not analytic, but both $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic, then x_0 is called a **regular singular point** of the equation.
3. If one or both of $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ is/are not analytic, the x_0 is an **irregular singular point** of the equation.

Now we can state Fuchs' theorem: At a regular singular point, x_0, there is at least one solution of theform

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+c}, \tag{2.2}$$

where c is not necessarily an integer.

In practice we find that if the two possible values of c differ by a noninteger value, then there are two solutions of this form. In general, when the values differ by an integer, then the second solution may involve logarithmic terms. Bessel's equations of order $n + \frac{1}{2}$ offer examples of cases where logarithms do not arise when the values of c differ by an odd integer, but do when they differ by an even integer. In general we need to use the method devised by Frobenius. This is illustrated by the next example.

Example 2.2 Solve the equation $4xy'' + 2y' + y = 0$.

By the above definitions $x = 0$ is a regular singular point since $P(x) = 1/(2x)$ and $Q(x) = 1/(4x)$, but both $xP(x)$ and $x^2Q(x)$ are analytic. (Note that all other points are ordinary points of the equation and therefore we could find two Taylor's series solutions of this equation about any other point.)

We assume a solution of the form (2.2) and substitute it into the governing equation. We get

$$\sum_{n=0}^{\infty}\left[4(n+c)(n+c-1)a_n x^{n-1+c} + 2(n+c)a_n x^{n-1+c} + a_n x^{n+c}\right] = 0.$$

Hence

$$\sum_{n=0}^{\infty} 2(n+c)(2n+2c-1)a_n x^{n-1+c} + \sum_{n=1}^{\infty} a_{n-1}x^{n+c-1} = 0.$$

Here I have made the powers of x the same in the two sums. The lowest power of x yields the **indicial equation**:

$$2c(2c-1) = 0,$$

and therefore $c = 0$ or $c = \frac{1}{2}$. These differ by a noninteger value and therefore we expect two solutions of the form (2.2).

For general values of c (we'll substitute in the specific values in a moment) the recurrence relation is

$$a_n = -\frac{a_{n-1}}{2(n+c)(2n+2c-1)} \qquad \text{for } n \geq 1$$

or

$$a_{n+1} = -\frac{a_n}{2(n+c+1)(2n+2c+1)} \qquad \text{for } n \geq 0.$$

When $c = 0$ we have $a_{n+1} = -\dfrac{a_n}{(2n+1)(2n+2)}$ which yields

$$a_1 = -\frac{a_0}{1.2} \qquad a_2 = -\frac{a_1}{4.3} = \frac{a_0}{4!} \qquad a_3 = -\frac{a_2}{6.5} = -\frac{a_0}{6!} \qquad \text{and so on,}$$

and therefore the solution is

$$y_1 = a_0\left[1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} + \cdots = a_0\cos\sqrt{x}\right].$$

When $c = \frac{1}{2}$ we have $a_{n+1} = -\dfrac{a_n}{(2n+3)(2n+2)}$ which yields

$$a_1 = -\frac{a_0}{2.3} \qquad a_2 = -\frac{a_1}{4.5} = \frac{a_0}{5!} \qquad a_3 = -\frac{a_2}{6.7} = -\frac{a_0}{7!} \qquad \text{etc.,}$$

and therefore the solution is

$$y_2 = a_0 x^{1/2}\left[1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \frac{x^4}{7!} + \cdots\right] = a_0\sin\sqrt{x}.$$

The general solution is

$$y = A\cos\sqrt{x} + B\sin\sqrt{x}.$$

In this case we were able to recognise the form of the series solutions and write the answers in closed form. This is very unusual indeed. Note firstly that since $c = 0$ is one possible value of c, it means that that solution is a Taylor series solution and can be obtained by first assuming such an expansion. The other solution cannot be obtained that way. Note also that the derivative of the $\sin\sqrt{x}$ part of the solution is infinite at $x = 0$; this means that a numerical solution would be very difficult and likely to be quite inaccurate. In such cases where numerical solutions have to be used, it often the case that a Frobenius solution can provide values of $y(x)$ and its derivative close to $x = 0$, and to use the relationship between their respective magnitudes as an alternative boundary condition.

3. The Solution of ODEs using Frobenius Series. II

When the roots of the indicial equation are equal ($c_1 = c_2$) then we cannot find two independent solutions consisting only of polynomials. Rather, the two solutions take the form,

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+c_1} \qquad y_2 = y_1 \ln x + \sum_{n=1}^{\infty} b_n x^{n+c_1}. \tag{3.1}$$

Note that the summation in the second part of the y_2 solution begins at $n = 1$; this is because we can always subtract a suitable multiple of y_1 to eliminate the x^{c_1} term.

Generally the solution y_1 is obtained straightforwardly. The second may be found either by substitution, which is usually quite lengthy, or by calculating $y(x, c)$ from the recurrence relation where c is left unspecified, and then forming

$$y_2 = \left.\frac{\partial y}{\partial c}\right|_{c=c_1}. \tag{3.2}$$

The proof of this result may be found in “Elementary Differential Equations and Boundary Value Problems” by W.E. Boyce and R.C. DiPrima (Wiley).

Example 3.1 Solve $xy'' + y' - x^3 y = 0$.

First set $y = \sum_{n=0}^{\infty} a_n x^{n+c}$ in the equation. Therefore we obtain

$$\sum_{n=0}^{\infty}\Big[a_n(n+c)(n+c-1)x^{n+c-1} + a_n(n+c)x^{n+c-1} - a_n x^{n+c+3}\Big] = 0,$$

$$\Rightarrow \qquad \sum_{n=0}^{\infty} a_n(n+c)^2 x^{n+c-1} - \sum_{n=4}^{\infty} a_{n-4} x^{n+c-1} = 0.$$

The coefficient of x^{c-1} yields $a_0 c^2 = 0$, from which we conclude that $c = 0, 0$, a repeated root.

Taking the coefficients of x^c, x^{c+1} and x^{c+2} in turn, we obtain $a_1 = a_2 = a_3 = 0$.

At the next term we get relations involving two terms, and the recurrence relation is

$$a_n = \frac{a_{n-4}}{(n+c)^2}, \tag{3.3}$$

or, when $c = 0$,

$$a_n = \frac{a_{n-4}}{n^2}. \tag{3.4}$$

From this latter relation we obtain,

$$a_4 = \frac{a_0}{4^2} \qquad a_8 = \frac{a_4}{8^2} = \frac{a_0}{4^2 8^2} \qquad \cdots \qquad a_{4n} = \frac{a_0}{4^2 8^2 \cdots (4n)^2} = \frac{a_0}{2^{4n} n! n!}.$$

All other coefficients are zero. Hence one solution is

$$y_1 = a_0 \sum_{n=0}^{\infty} \frac{x^{4n}}{2^{4n} n! n!}.$$

Here, the value of a_0 is arbitrary and we may set it equal to unity.

To find the second solution we follow the procedure given in (3.2) and use an arbitrary value of c to begin with. From (3.3) we get that

$$a_{4n} = \frac{a_0}{(4+c)^2 (8+c)^2 \cdots (4n+c)^2}$$

$$\Rightarrow \qquad y(x,c) = a_0 \Big[x^c + \sum_{n=1}^{\infty} \frac{x^{4n+c}}{(4+c)^2 (8+c)^2 \cdots (4n+c)^2} \Big].$$

Now we need to differentiate with respect to c and set $c = 0$. But first recall that

$$\frac{\partial x^c}{\partial c} = \frac{\partial e^{c \ln x}}{\partial c} = e^{c \ln x} \ln x = x^c \ln x.$$

Hence,

$$\frac{\partial y}{\partial c} = y(x,c) \ln x + \sum_{n=1}^{\infty} \Big[\frac{x^{4n}}{(4+c)^2 (8+c)^2 \cdots (4n+c)^2} \Big] \Big[-\frac{2}{4+c} - \frac{2}{8+c} - \cdots - \frac{2}{4n+c} \Big].$$

On setting $c = 0$ we get

$$\begin{aligned} y_2 &= y_1 \ln x + \sum_{n=1}^{\infty} \frac{x^{4n}}{2^{4n} n! n!} \Big[-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots - \frac{1}{2n} \Big] \\ &= y_1 \ln x - \tfrac{1}{2} \sum_{n=1}^{\infty} \frac{x^{4n}}{2^{4n} n! n!} H_n, \end{aligned}$$

where we have introduced

$$H_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \tfrac{1}{n} \tag{3.5}$$

for notational convenience; this will be used in later examples.

Example 3.2 Solve $xy'' + y' + y = 0$.

Letting $y = \sum_{n=0}^{\infty} a_n x^{n+c}$ yields

$$\sum_{n=0}^{\infty} \left[(n+c)(n+c-1)a_n x^{n+c-1} + (n+c)a_n x^{n+c-1} + a_n x^{n+c} \right] = 0$$

$$\Rightarrow \qquad \sum_{n=0}^{\infty} (n+c)^2 a_n x^{n+c-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+c-1} = 0.$$

The indicial equation is again $c^2 = 0$. Although this is the same as for Example 3.1, there is no procedural difference between these two cases and any other involving equal roots.

The recurrence relation is now

$$a_n = -\frac{a_{n-1}}{(n+c)^2}$$

from which we obtain

$$a_n = \frac{(-1)^n a_0}{(1+c)^2(2+c)^2 \dots (n+c)^2},$$

or

$$a_n = \frac{(-1)^n}{n!n!}$$

when $c = 0$. For general values of c we have

$$y(x,c) = x^c + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+c}}{(1+c)^2(2+c)^2 \dots (n+c)^2}$$

where I have set $a_0 = 1$. Therefore the two solutions are

$$y_1 = y(x;0) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!n!}, \tag{3.6}$$

$$y_2 = \frac{\partial y}{\partial c}(x,c)\Big|_{c=0} = y_1 \ln x - 2\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!n!} H_n, \tag{3.7}$$

and the general solution is $y = Ay_1 + By_2$.

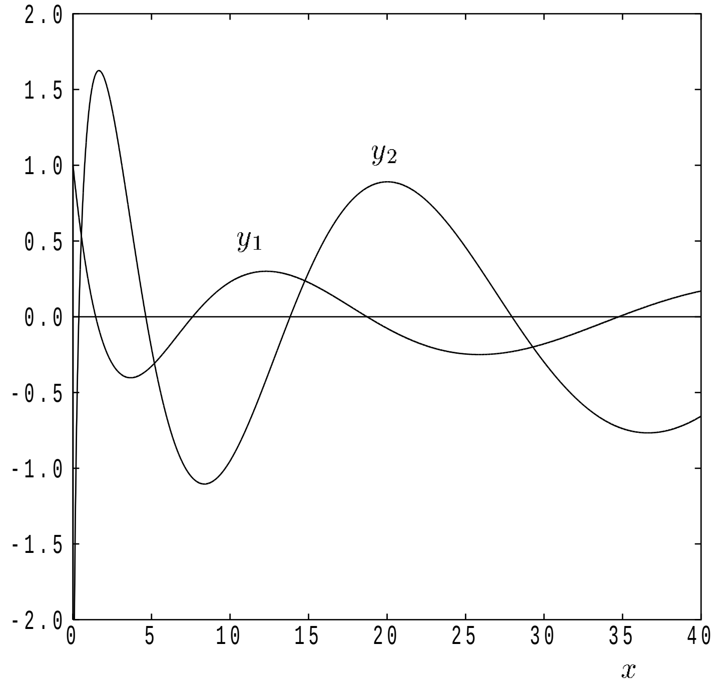

Figure 3.1. The series solutions (3.6) and (3.7).

4. <u>The Solution of ODEs using Frobenius Series. III</u>

When the roots of the indicial equation differ by an integer ($c_1 > c_2$) then we cannot find two independent solutions consisting only of polynomials, as in the case where $c_1 = c_2$. But two solutions of the form,

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+c_1} \qquad y_2 = A y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+c_2}. \tag{4.1}$$

exist where A may be zero. An example of when $A = 0$ is given in Example 4.1, below, but in general $A \neq 0$.

Again the solution y_1 is obtained straightforwardly. Likewise, the second may be found either by substitution, or by calculating $y(x,c)$ and forming

$$y_2 = \left.\frac{\partial y}{\partial c}\right|_{c=c_2}. \tag{4.2}$$

In many circumstances it turns out to be necessary to differentiate $(c - c_2)y(x,c)$, as the expression (4.2) can yield infinities due to the presence of the factor $(c - c_2)$ in the denominators.

Example 4.1 Solve $xy'' + y = 0$.

Following the last example we have

$$\sum_{n=0}^{\infty}(n+c)(n+c-1)a_n x^{n+c-1} + \sum_{n=1}^{\infty} a_{n-1}x^{n+c-1} = 0.$$

The coefficient of x^{c-1} yields $a_0 c(c-1) = 0$, from which we conclude that $c = 0, 1$ are the roots.

The recurrence relation for general values of c is

$$a_n = -\frac{a_{n-1}}{(n+c)(n+c-1)}. \tag{4.3}$$

From this we obtain

$$y(x,c) = x^c - \frac{x^{c+1}}{c(c+1)} + \frac{x^{c+2}}{c(c+1)^2(c+2)} - \frac{x^3}{c(c+1)^2(c+2)^2(c+3)} + \cdots. \tag{4.4}$$

Hence the solution for $c = 1$, the larger of the two values of c, is

$$\begin{aligned} y_1 = y(x,1) &= x - \frac{x^2}{1.2} + \frac{x^3}{1.2^2.3} - \frac{x^4}{1.2^2.3^2.4} + \cdots \\ &= \sum_{n=1}^{\infty}\frac{n(-1)^{n+1}x^n}{n!n!} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n!(n-1)!}. \end{aligned}$$

Clearly, if we naively set $c = 0$ into equation (4.4) we obtain infinite coefficients. But this is also true if we form y_2 using (4.2) and then set $c = 0$, for then each term in the Frobenius series has quadratic poles at $c = 0$. Therefore we use

$$\begin{aligned} \frac{\partial}{\partial c}\Big[cy(x,c)\Big] &= \frac{\partial}{\partial c}\Big[cx^c - \frac{x^{c+1}}{c+1} + \frac{x^{c+2}}{(c+1)^2(c+2)} - \frac{x^{c+3}}{(c+1)^2(c+2)^2(c+3)} + \cdots\Big] \\ &= cy_1(x)\ln x + x^c + \frac{x^{c+1}}{(c+1)}\frac{1}{(c+1)} - \frac{x^{c+2}}{(c+1)^2(c+2)}\Big(\frac{2}{(c+1)} + \frac{1}{(c+2)}\Big) \\ &\quad + \frac{x^3}{(c+1)^2(c+2)^2(c+3)}\Big(\frac{2}{(c+1)} + \frac{2}{(c+2)} + \frac{1}{(c+3)}\Big) + \cdots \end{aligned}$$

Now we can set $c = 0$ to obtain,

$$\begin{aligned} y_2 &= 1 + x - \frac{x^2}{1^2.2}\Big(\tfrac{2}{1} + \tfrac{1}{1}\Big) + \frac{x^3}{1^2.2^2.3}\Big(\tfrac{2}{1} + \tfrac{2}{2} + \tfrac{1}{3}\Big) - \frac{x^4}{1^2.2^2.3^2.4}\Big(\tfrac{2}{1} + \tfrac{2}{2} + \tfrac{2}{3} + \tfrac{1}{4}\Big) + \cdots \\ &= 1 + x + \sum_{n=2}^{\infty}\frac{(-1)^n x^n}{n!(n-1)!}\Big(H_n + H_{n-1}\Big). \end{aligned}$$

There is no logarithmic term in this case. The next Example is a case where a logarithmic term appears.

Example 4.2 Solve Bessel's equation of order 1: $x^2y'' + xy' + (x^2-1)y = 0$.

We begin as always in this type of problem with $y = \sum_{n=0}^{\infty} a_n x^{n+c}$. Substitution into the governing equation yields

$$\sum_{n=0}^{\infty}\Big[(n+c)^2 - 1\Big]a_n x^{n+c} + \sum_{n=2}^{\infty} a_{n-2}x^{n+c} = 0.$$

The coefficient of x^c gives the indicial equation $c^2 - 1 = 0$ from which we find that $c = -1, +1$.

The recurrence relation for general values of c is

$$a_n = -\frac{a_{n-2}}{(n+c+1)(n+c-1)},$$

from which we obtain

$$y(x,c) = x^c\Big[1 - \frac{x^2}{(c+1)(c+3)} + \frac{x^4}{(c+1)(c+3)^2(c+5)} - \frac{x^6}{(c+1)(c+3)^2(c+5)^2(c+7)} + \cdots\Big]$$

The first solution is given by substituting in the upper value of c, $c = 1$:

$$y_1 = x\Big[1 - \frac{x^2}{2.4} + \frac{x^4}{2.4^2.6} - \frac{x^6}{2.4^2.6^2.8} + \cdots\Big], = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n}n!(n+1)!}.$$

We find the second solution by beginning with

$$\begin{aligned}
&\frac{\partial}{\partial c}\Big[(c+1)y(x,c)\Big]\\
&\quad= \frac{\partial}{\partial c}\Big[(c+1)x^c - \frac{x^{c+2}}{(c+3)} + \frac{x^{c+4}}{(c+3)^2(c+5)} - \frac{x^{c+6}}{(c+3)^2(c+5)^2(c+7)} + \cdots\Big]\\
&\quad= \Big[(c+1)x^c - \frac{x^{c+2}}{(c+3)} + \frac{x^{c+4}}{(c+3)^2(c+5)} - \frac{x^{c+6}}{(c+3)^2(c+5)^2(c+7)} + \cdots\Big]\ln x\\
&\quad+ \Big[x^c + \frac{x^{c+2}}{(c+3)}\Big(\frac{1}{c+3}\Big) - \frac{x^{c+4}}{(c+3)^2(c+5)}\Big(\frac{2}{c+3} + \frac{1}{c+5}\Big)\\
&\quad+ \qquad \frac{x^{c+6}}{(c+3)^2(c+5)^2(c+7)}\Big(\frac{2}{c+2} + \frac{2}{c+5} + \frac{1}{c+7}\Big) + \cdots\Big].
\end{aligned}$$

At $c = -1$ we obtain

$$\begin{aligned}
y_2 &= \Big[-\frac{x}{2} + \frac{x^3}{2^2.4} - \frac{x^5}{2^2.4^2.6} + \cdots\Big]\ln x\\
&+ \Big[x^{-1} + \frac{x}{2}\Big(\frac{1}{2}\Big) - \frac{x^3}{2^2.4}\Big(\frac{2}{2} + \frac{1}{4}\Big) + \frac{x^5}{2^2.4^2.6}\Big(\frac{2}{2} + \frac{2}{4} + \frac{1}{6}\Big)\\
&- \frac{x^7}{2^2.4^2.6^2.8}\Big(\frac{2}{2} + \frac{2}{4} + \frac{2}{6} + \frac{1}{8}\Big) + \cdots\Big].
\end{aligned}$$

This lengthy expression may be reduced to the more compact form:

$$y_2 = -\tfrac{1}{2}y_1 \ln x + x^{-1} + \frac{x}{4} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}x^{2n}}{2^{2n-1}n!(n-1)!}\Big(H_n + H_{n-1}\Big).$$

In this case we have a logarithmic term; this is true of all Bessel's equations of integer order:

$$x^2y'' + xy' + (x^2 - a^2)y = 0$$

where a is the order of the equations. But when $a = n + \frac{1}{2}$, where n is an integer, the indicial equation has two roots which differ by an odd integer, and, given the nature of the recurrence relation which involves a_n and a_{n-2}, we have two independent series solutions without logarithmic terms. An example of this is given in the exercises.

Concluding remarks

These methods may be extended to inhomogeneous systems, and to higher order systems. In the latter case we need to consider how the presence of three or more identical roots modifies the solution procedure. To see the expected form the solutions might take, we can return to a typical Cauchy-Euler equation:

$$x^3y''' + xy' - y = 0. \tag{4.5}$$

The general solution of this equation is $y = [A + B\ln x + C(\ln x)^2]x$. Therefore we expect the presence of a log-squared term when three roots of the indicial equation are identical. Therefore, for illustrative purposes, let us modify equation (4.5) to

$$x^3y''' + xy' - (x+1)y = 0.$$

The indicial equation is now $(c-1)^3$, which yields $c = 1, 1, 1$, a three-times repeated root. The recurrence relation is very straightforward:

$$a_n = \frac{a_{n-1}}{n^3}.$$

Therefore

$$y(x,c) = x^c + \sum_{n=1}^{\infty} \frac{x^{n+c}}{c^3(c+1)^3(c+2)^3\ldots(c+n-1)^3}.$$

The three solutions are formed according to

$$y_1 = y(x,1) \qquad y_2 = \frac{\partial}{\partial c}y(x,c)\Big|_{c=1} \qquad y_3 = \frac{\partial^2}{\partial c^2}y(x,c)\Big|_{c=1}.$$

Hence we find that

$$y_1 = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!n!n!},$$

$$y_2 = y_1 \ln x - 3\sum_{n=1}^{\infty} \frac{x^{n+1}}{n!n!n!}H_n \quad \equiv \quad y_1 \ln x - 3y_2^*,$$

$$y_3 = y_1(\ln x)^2 - 6y_2^* \ln x + 9\sum_{n=1}^{\infty} \frac{x^{n+1}}{n!n!n!}H_n^2 + 4\sum_{n=1}^{\infty} \frac{x^{n+1}}{n!n!n!}G_n,$$

where
$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \qquad G_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$
More general cases are more complicated, and it will often take some ingenuity to obtain the solutions in cases where, for example, there are two sets of repeated roots which differ by an integer! Again there are some examples of these in the exercises.

Finally it is necessary to reiterate the point that such solutions may be useful when computing the solutions of Ordinary Differential Equations where a function is expected to display a fractional power dependence on x (e.g. $x^{1/2}$) or even a logarithmic behaviour near the origin. Even with nonlinear equations it will often be sufficient to use the first few terms of the series to obtain a numerical solution from $x = \epsilon \ll 1$ onwards.

5. Irregular Singular Points. I

We have found that linear equations where the coefficients are analytic (i.e. have Taylor series expansions about x_0) could be expressed in terms of Taylor series about x_0, which is an ordinary point. When x_0 is a regular singular point solutions may not take the form of Taylor series. Often fractional powers of $(x - x_0)$ and/or logarithms are involved. In these latter cases we assume a Frobenius expansion. Such an expansion is closely related to the solution of a corresponding Cauchy-Euler equation such as $x^2y'' + axy' + by = 0$. However, when we deal with irregular singular points there will be at least one solution which is not of Frobenius form.

Example 5.1 Solve the equation $y' = x^{1/2}y$.

This is an interesting case, for although the coefficient of the highest derivative does not vanish as $x \to 0$, nevertheless the coefficient of y does not have a Taylor series expansion about $x = 0$. Hence $x = 0$ is an irregular singular point. The solution of the equation may be found analytically:
$$y = Ae^{(\frac{2}{3}x^{3/2})} \quad = \quad A\sum_{n=0}^{\infty} \frac{(\frac{2}{3}x^{3/2})^n}{n!}.$$
In this case the solution is well-behaved at the origin although its series expansion does not proceed in integer powers of x. Although $y'(x)$ is also zero at the origin, $y'' \to \infty$ as $x \to 0^+$.

If we had naively assumed a Taylor series expansion, the presence of the $x^{1/2}$ term in the equation would have forced us very quickly to reassess that assumption. Substitution of a power series solution in powers of $x^{1/2}$ would of course work, although many of the coefficients would be zero. In such equations we may assume an expansion of the form
$$y = \sum_{n=0}^{\infty} a_n x^{n\alpha}$$
where α is determined during the subsequent analysis.

Example 5.2 Solve the equation $x^3y'' - y = 0$.

This is a more obvious case of an irregular singular point at $x = 0$. Let us try a Frobenius series to see what happens. On setting $y = \sum_{n=0}^{\infty} a_n x^{n+c}$ into the equation, we obtain

$$\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c+1} - \sum_{n=0}^{\infty} a_n x^{n+c} = 0.$$

Contrary to previous practice we expand these sums explicitly:

$$[-a_0]x^c + [a_0c(c-1) - a_1]x^{c+1} + [a_1(c+1)c - a_2]x^{c+2} + \cdots = 0.$$

Hence we obtain $a_0 = 0$, a contradiction, since we must have $a_0 \neq 0$ in order to have such a series. With regular singular points we obtain an indicial equation at this stage, but with this type of irregular singular point we see immediately that a Frobenius solution is not possible.

This contradiction has occurred because the term involving the highest derivative no longer dominates the equation, as it does for ordinary points or regular singular points. This situation happens frequently in equations arising in many areas of mechanics and physics.

We will return to this example below, but it is now essential to introduce commonly used notation for asymptotic and perturbation theories.

Notation

The symbols, $\ll$, $\sim$ and $O(\ )$ are used frequently to describe the relative sizes of functions in varying degrees of precision. In addition we also introduce the symbols, $o(\ )$ and $\mathrm{Ord}(\ )$.

- $f(x) \ll g(x) \quad (x \to x_0)$. This means that $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = 0$. An alternative way of writing this is $f(x) = o(g(x))$ as $x \to x_0$.

- $f(x) \sim g(x) \quad (x \to x_0)$. This is equivalent to $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = 1$. That is, the difference between the functions is asymptotically smaller than the size of the functions themselves.

- $f(x) = O(g(x)) \quad (x \to x_0)$. This means that $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} < \infty$. It is less precise than the '$\sim$' notation, but it means that the ratio of the two functions approaches a constant value. Strictly speaking this constant value could be zero, in which case the 'O' notation includes the 'o' notation as a subset.

- In some texts there is the more precise version, $f(x) = \mathrm{Ord}(g(x))$ as $x \to x_0$. In this case we have $0 < \lim_{x\to x_0} \dfrac{f(x)}{g(x)} < \infty$, and this notation specifically eliminates the 'o' possibility .

For example:

- $x \ll x^{-1}$ as $x \to 0^+$ or, alternatively, $x = o(x^{-1})$ as $x \to 0^+$.
- $x = O(x^{-1})$ as $x \to 0$. The ratio approaches zero which is less than infinity!
- $x \ll e^x$ as $x \to \infty$.
- $\sin x \sim x$ as $x \to 0$.
- $1 - \cos^2 x \sim \frac{1}{2}x^2$ as $x \to 0$.
- $1 - \cos^2 x = O(x^2)$ as $x \to 0$.
- $\sin 3x = O(x)$ as $x \to 0$. The ratio of the functions approaches a finite constant.
- $e^x + x \sim e^x$ as $x \to \infty$. Although the difference grows, the relative difference does not.

The '$\simeq$' notation is nothing to do with asymptotic theory. It means that two numbers or a variable and a number have approximately the same numerical value.

A general method for irregular singular points

A fairly general method for solving systems such as Example 5.2 was developed independently by Carlini, Green and Liouville in the early 19th century. It centres around the use of the exponential

$$y(x) = e^{S(x)} \tag{5.1}$$

as a substitution. If we use this in the general linear second order ordinary differential equation,

$$y'' + p(x)y' + q(x)y = 0 \tag{5.2}$$

we obtain

$$e^{S(x)}\Big[S'' + (S')^2 + p(x)S' + q(x)\Big] = 0,$$

and hence that

$$S'' + (S')^2 + p(x)S' + q(x) = 0. \tag{5.3}$$

When $x = x_0$ is an irregular singular point it is often the case that $S'' \ll (S')^2$ as $x \to x_0$. This fact simplifies the equation for S considerably and allows an asymptotic expansion to proceed more easily. Let us return to Example 5.2 where we will undertake a comprehensive asymptotic analysis.

Example 5.2 (again) Using the substitution (5.1) on our ODE gives

$$e^{S(x)}\left[(S''+S'S')x^3-1\right]=0,$$

from which we get

$$S''+S'S'=x^{-3}. \tag{5.4}$$

We will assume that $S'' \ll S'S'$, which we will check afterwards, and hence $S'S' \sim x^{-3}$. Therefore

$$S' \sim \pm x^{-3/2} \tag{5.5}$$

from which we get

$$S \sim \pm 2x^{-1/2}.$$

Now we need to do an *a posteriori* check that this solution satisfies our original assumption. We find that $S'' = \text{Ord}(x^{-5/2})$ and $S'S' = \text{Ord}(x^{-3})$ as $x \to 0$, which is consistent with the assumption.

Hence the first approximation to $S(x)$ is known. Given the presence of the square root we will assume that $x > 0$ just to avoid complex numbers!

Let us continue and try to find a second term in a series for $S(x)$. It is best not to make any assumptions at all about the size of this new term except that it must be asymptotically smaller than the first. Therefore we will take

$$S = S_0 + S_1 + S_2 + \cdots$$

where $S_{n+1} \ll S_n$ as $x \to 0$. We will take $S_0 = 2x^{-1/2}$ as the first term, and hence we set $S = 2x^{-1/2} + S_1$ into (5.4). We obtain

$$\tfrac{3}{2}x^{-5/2} + S_1'' + \left(x^{-3} - 2S_1'x^{-3/2} + S_1'S_1'\right) = x^{-3}.$$

The x^{-3} terms cancel. Guided by our previous experience we balance the $x^{-5/2}$ term with the $S_1'x^{-3/2}$ term to get

$$S_1' \sim \tfrac{3}{4}x^{-1}. \tag{5.6}$$

Hence $S_1 = \frac{3}{4}\ln x$ which is asymptotically smaller than S_0 as $x \to 0$, and which also confirms the order of magnitude assumptions used in deriving equation (5.6). So far we have

$$S \sim 2x^{-1/2} + \tfrac{3}{4}\ln x \qquad \Rightarrow \qquad y \sim x^{3/4}e^{2x^{-1/2}}.$$

We can continue further with this analysis by setting

$$S = 2x^{-1/2} + \tfrac{3}{4}\ln x + S_2$$

into the governing equation. Many terms cancel and we eventually obtain

$$S_2' \sim -\tfrac{3}{32}x^{-1/2} \tag{5.7}$$

where neglected terms will turn out to be asymptotically smaller than those retained. We find that

$$S_2 = -\tfrac{3}{16}x^{1/2}.$$

To summarise, we have found that

$$S(x) \sim 2x^{-1/2} + \tfrac{3}{4}\ln x - \tfrac{3}{16}x^{1/2} + c,$$

where the constant of integration, c, could have been introduced into any of the solutions to the three equations (5.5), (5.6) and (5.7). Hence the solution is

$$y(x) \sim Ax^{3/4}e^{[2x^{-1/2}-\frac{3}{16}x^{1/2}]}.$$

The first **two** terms of $S(x)$ are known as the "leading behaviour" of $y(x)$ since the third term in $S(x)$ vanishes in the limit $x \to 0$. It is more usual at this stage to proceed, should one wish, by using the substitution

$$y(x) = Ax^{3/4}e^{[2x^{-1/2}-\frac{3}{16}x^{1/2}]}\Big(1+\epsilon(x)\Big)$$

and by trying to find an power series representation for $\epsilon(x)$ which is $o(1)$ as $x \to 0$. This power series may not involve integer powers of x.

6. Irregular Singular points. II

Dominant Balance

In our detailed analysis of the leading behaviour of the equation given in Example 5.2 we employed a device which is called the **method of dominant balance**. It is a frequently-used technique in asymptotic theory where some terms are neglected because they are asymptotically small, while others are retained to balance orders of magnitude. In that Example we neglected S_1'' in favour of $S_1'x^{-3/2}$, and we balanced the orders of magnitude of $S_1'x^{-3/2}$ and $x^{-5/2}$. This balance allowed us not only to find the asymptotic size of S_1 as $x \to 0$, but also to determine S_1. Finally it is essential to check that the terms retained are indeed asymptotically larger than those neglected.

Singular points at infinity

The method we have used may also be applied to the solution of equations in the $x \to \infty$ limit. But first we need to categorise the point at ∞. We do this by transforming the point at ∞ to a point at 0, and by categorising the resulting equation. We take

$$y'' + p(x)y' + q(x)y = 0 \tag{6.1}$$

and introduce the transformation, $x = t^{-1}$. We find that

$$\frac{dy}{dx} = -t^2\frac{dy}{dt} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt},$$

and therefore (6.1) transforms to

$$t^4\frac{d^2y}{dt^2} + \left(2t^3 - t^2p(t^{-1})\right)\frac{dy}{dt} + q(t^{-1})y = 0. \tag{6.2}$$

We may now classify the point at infinity of (6.1) by examining the point at 0 of (6.2).

Example 6.1 Classify the point at infinity of $\dfrac{d^2y}{dx^2} - y = 0$.

This equation transforms into

$$t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt} - y = 0.$$

Therefore $t = 0$ and hence $x = \infty$ is an irregular singular point. The solution in this case is

$$y \quad = \quad Ae^x + Be^{-x} \quad = \quad Ae^{(t^{-1})} + Be^{-(t^{-1})}.$$

The "A" part of this solution approaches infinity faster than any power of t as $t \to 0$ and is called an **essential singularity**, as opposed to a pole. Essential singularities often arise at irregular singular points.

Many of the standard equations of mechanics (Bessel's equation, Airy's equation, Hermite's equation etc.) have irregular singular points at infinity.

Example 6.2 Find the leading behaviour of Airy's equation, $y'' - xy = 0$, as $x \to \infty$.

It is straightforward to find Taylor series solutions of this equation about $x = 0$ for that is an ordinary point, but the point at infinity is an irregular singular point. We follow the procedure introduced in (5.1) for the case of irregular singular points at finite values of x which is to set

$$y = e^{S(x)}$$

in Airy's equation. This gives

$$e^{S(x)}\left[S'' + (S')^2 - x\right] = 0. \tag{6.3}$$

If we balance the terms S'' and x to get $S'' \sim x$, we obtain $S \sim x^3/6$. But this implies that $(S')^2 = x^4/4$ which is asymptotically larger than the terms we balanced.

So we balance $(S')^2$ and x to obtain $S' \sim \pm x^{1/2}$. Hence $S \sim \pm\frac{2}{3}x^{3/2}$. Again we check the size of the neglected term: $S'' \sim \frac{1}{2}x^{-1/2}$, which is asymptotically smaller than x as $x \to \infty$. So we may continue with the analysis.

We will retain both signs ($\pm$) in the following analysis as these correspond to two possible solutions of the given second order equation. Now let us substitute

$$S = \pm\tfrac{2}{3}x^{3/2} + S_1(x)$$

into (6.3) where all we should assume for now is that $S_1 \ll x^{3/2}$ as $x \to \infty$. Hence we obtain

$$\left[\pm\tfrac{1}{2}x^{-1/2} + S_1''\right] + \left[\pm x^{1/2} + S_1'\right]^2 - x = 0.$$

The leading terms (x and $-x$) cancel leaving the exact equation

$$\pm\tfrac{1}{2}x^{-1/2} + S_1'' \pm 2x^{1/2}S_1' + (S_1')^2 = 0.$$

The **dominant balance** is given by

$$\pm\tfrac{1}{2}x^{-1/2} \sim \mp 2x^{1/2}S_1',$$

where the $=$ sign has been replaced by the $\sim$ sign in this approximation. Hence $S_1' \sim -\frac{1}{4}x^{-1}$ and therefore $S_1 \sim -\frac{1}{4}\ln x$. So far, then, we have found that

$$S \sim \pm\tfrac{2}{3}x^{3/2} - \tfrac{1}{4}\ln x + c \qquad \Rightarrow \qquad y(x) \sim Ax^{-1/4}\exp[\pm\tfrac{2}{3}x^{3/2}].$$

Note that this problem and many others may also be solved to this point by assuming a solution of the form,

$$y(x) \sim Ax^a e^{bx^c}$$

at the outset, where a, b and c are to be found. An advantage of this substitution is that it shows directly that further terms are composed of a power series. It is worth attempting the solution of Airy's equation this way to see how the method works.

We may continue to find further terms by means of the substitution,

$$y = Ax^{-1/4}\exp[\pm\tfrac{2}{3}x^{3/2}]w(x)$$

where $\lim_{x\to\infty} w(x) = 1$. After some lengthy algebra we find that $w(x)$ satisfies the equation

$$w'' + \left(\pm 2x^{1/2} - \frac{1}{2x}\right)w' + \frac{5}{16x^2}w = 0.$$

This equation is closely related to those for which a Frobenius series is required, but here we require the leading term to be 1. Therefore we use the following modification to a Taylor series:

$$w = \sum_{n=0}^{\infty} a_n x^{n\alpha} \tag{6.4}$$

where $a_0 = 1$ and $\alpha < 0$ must be found as part of the analysis. Direct substitution gives

$$\sum_{n=1}^{\infty} n\alpha(n\alpha-1)a_n x^{n\alpha-2} \pm \sum_{n=1}^{\infty} 2n\alpha\, a_n x^{n\alpha-\frac{1}{2}} - \tfrac{1}{2}\sum_{n=1}^{\infty} n\alpha\, a_n x^{n\alpha-2} + \tfrac{5}{16}\sum_{n=0}^{\infty} a_n x^{n\alpha-2} = 0. \tag{6.5}$$

The highest powers of x represented here are the first term of the second and fourth sums: $\pm 2\alpha a_1 x^{\alpha-1/2}$ and $\frac{5}{16}a_0 x^{-2}$. These must balance, and therefore $\alpha = -\frac{3}{2}$, confirming the fact that α is negative. We also find that

$$a_1 = \pm\tfrac{5}{48}a_0.$$

Using (6.5) we may derive the recurrence relation

$$a_{n+1} = \pm\frac{\frac{3}{4}(n+\frac{5}{6})(n+\frac{1}{6})}{(n+1)}a_n.$$

It is possible to solve this recurrence relation in terms of gamma functions:

$$a_n = \frac{1}{2\pi}\left(\pm\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!}, \tag{6.6}$$

where use has been made of the result, $\Gamma(\frac{1}{6})\Gamma(\frac{5}{6}) = 2\pi$. It is interesting to note that this asymptotic series is divergent for all values of x — this is typical of asymptotic series.

The final solution is

$$y = Ax^{-1/4}\exp\left[\pm\tfrac{2}{3}x^{3/2}\right]\sum_{n=0}^{\infty}\frac{1}{2\pi}\left(\pm\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!}x^{-3/2}.$$

7. Regular and Singular Perturbation Theory.

In perturbation theory equations are solved by expanding the solutions in terms of either an asymptotically small or an asymptotically large parameter. In lectures 1 to 6 we concentrated on asymptotic expansions in terms of large or small values of x; these are coordinate expansions, as opposed to parameter expansions on which we will now concentrate. As with coordinate expansions we have no *a priori* guarantee that the series solutions obtained will converge. We will illustrate the difference between the two flavours of perturbation theory, regular and singular, by means of two algebraic equations. This will be followed by two relatively straightforward examples consisting of ordinary differential equations.

Example 7.1 Find all the roots of the quartic polynomial, $x^4 - 2x^2 + \epsilon^2 = 0$, in the limit as $\epsilon \to 0$.

Although this may be accomplished by solving directly the quadratic equation for x^2, we will employ a perturbation series in powers of ϵ. (Note that we have no guarantee at this stage that such a series might work. In fact the appropriate series might have proceeded in terms of ϵ^2 or $\epsilon^{1/2}$. Often a little bit of trial-and-error will suffice to find the appropriate series to use.)

Let $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$. We will substitute this into the quartic equation and set to zero the coefficient of each power of ϵ in turn. The equation becomes,

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^4 - 2(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^2 + \epsilon^2 = 0$$

from which we obtain

$$(x_0^4 - 2x_0^2) + \epsilon(4x_0^3 x_1 - 4x_0 x_1) + \epsilon^2(4x_0^3 x_2 + 6x_0^2 x_1^2 - 4x_0 x_2 - 2x_1^2 + 1) + \cdots = 0.$$

Here we see clearly that each term in brackets should be set to zero. At $O(1)$ we have

$$x_0^4 - 2x_0^2 = 0$$

from which we conclude that

$$x_0 = 0, \quad 0, \quad \sqrt{2}, \quad -\sqrt{2}.$$

At leading order we have four roots, which is not surprising as the algebraic equation is of fourth order.

At $O(\epsilon)$ we have $4x_0^3 x_1 - 4x_0 x_1 = 0$. When $x_0 = 0$ this equation is satisfied automatically and therefore x_1 cannot be determined at this order. But when $x_0 = \pm\sqrt{2}$ we obtain $x_1 = 0$.

At $O(\epsilon^2)$ the equation is $4x_0^3 x_2 + 6x_0^2 x_1^2 - 4x_0 x_2 - 2x_1^2 + 1 = 0$. When $x_0 = 0$ we find that $x_1 = \pm 1/\sqrt{2}$, whereas when $x_0 = \pm\sqrt{2}$ we have $x_2 = \mp 1/(4\sqrt{2})$.

The four solutions correct to $O(\epsilon^2)$ are

$$x \sim \pm\frac{1}{\sqrt{2}}\epsilon, \qquad x \sim \sqrt{2} - \frac{1}{4\sqrt{2}}\epsilon^2. \tag{7.1}$$

An alternative way of describing these solutions is

$$x = \pm\frac{1}{\sqrt{2}}\epsilon + O(\epsilon^2), \qquad x = \sqrt{2} - \frac{1}{4\sqrt{2}}\epsilon^2 + O(\epsilon^3). \tag{7.2}$$

In (7.1) no information is given about how large the next term in the asymptotic expansion is expected to be, whereas it is given in (7.2). Note also the difference in the sign which is used; we can use equality in (7.2) because the left and right hand sides are indeed equal even though we have not written the right hand sides in detail! Also of interest in this problem is that fact that at $O(\epsilon^n)$ the term which is found depends on which root is being considered. In the case where $x_0 = 0$, the $O(\epsilon^n)$ equation determines x_{n-1}, whereas it determines x_n in the other cases.

This was an example of a regular perturbation expansion. The next is a singular perturbation problem.

Example 7.2 Find the roots of $\epsilon^2 x^4 - 2x^2 + 1 = 0$.

If we attempt to use $x = x_0 + \epsilon x_1 + \epsilon x_2 + \cdots$ in this equation we find that

$$x = \pm\frac{1}{\sqrt{2}}\left(1 + \frac{\epsilon^2}{8} + \cdots\right).$$

But now we have only two roots, whereas the original equation has four. The reason for this is that such a regular perturbation expansion is not valid for all the roots of this equation, because we have in effect balanced the magnitudes of the x^2 and 1 terms, and this balance has reduced the order of the equation from 4 to 2. Such a reduction of order almost always marks the presence of a singular perturbation problem. So let us try to find an appropriate order of magnitude balance.

If we try to balance $\epsilon^2 x^4$ with 1, then we find that $x = O(\epsilon^{-1/2})$. But this means that the x^2 term which we have neglected is larger than the other two terms. This is not good, and therefore we have to balance the $\epsilon^2 x^4$ and x^2 terms. This yields $x = O(\epsilon^{-1})$. This size leads to a consistent approximation, and therefore we rescale x according to

$$y = \frac{x}{\epsilon}$$

and the quartic transforms to

$$y^4 - 2y^2 + \epsilon^2 = 0,$$

which is in fact the same as Example 7.1! Our transformation has changed a singular perturbation problem into a regular perturbation problem! Therefore the two $O(1)$ roots for y are

$$y \sim \pm\sqrt{2}\Big(1 - \frac{\epsilon^2}{8}\Big),$$

and so the final two roots for x are

$$x \sim \pm\frac{\sqrt{2}}{\epsilon}\Big(1 - \frac{\epsilon^2}{8}\Big).$$

Example 7.3 Solve the equation $y'' + \epsilon f(x)y = 0$ subject to $y(0) = 1$, $y'(0) = 0$.

We introduce a perturbation series of the form

$$y = \sum_{n=0}^{\infty} \epsilon^n y_n(x)$$

into the given equation. We obtain

$$\sum_{n=0}^{\infty}\Big(\epsilon^n y_n'' + \epsilon^{n+1} f(x) y_n\Big) = 0,$$

from which obtain the ordinary differential recurrence relation

$$y_n'' = -f(x)y_{n-1}, \qquad n = 1, 2, \cdots \qquad \text{with} \qquad y_0'' = 0.$$

In principle, this series of coefficient functions, y_0, y_1, y_2, etc., may be solved sequentially. If we take $f(x) = -x$ we obtain a scaled version of Airy's equation. In this case we find that

$$\begin{aligned}
y_0'' &= 0 && \Rightarrow && y_0 = 1 && \text{(this satisfies both initial conditions)}\\
y_1'' &= xy_0 && \Rightarrow && y_1 = \frac{x^3}{2.3}\\
y_2'' &= xy_1 && \Rightarrow && y_2 = \frac{x^6}{2.3.5.6}
\end{aligned}$$

and so on. Hence the regular perturbation series yields

$$y(x) = 1 + \sum_{n=1}^{\infty} \frac{\epsilon^n x^{3n}}{(2.5.8.\cdots.(3n-1)).(3.6.9.\cdots.(3n))} = \Gamma(-\tfrac{1}{3})\sum_{n=0}^{\infty}\frac{\epsilon^n x^{3n}}{3^{2n} n!\Gamma(n-\frac{1}{3})}.$$

This series is convergent for all values of x for all finite values of ϵ; this is not always the case as, strictly speaking, we have been undertaking an asymptotic expansion as $\epsilon \to 0$. But in the present case the value $\epsilon = 1$ gives us that Taylor series solution of Airy's equation which satisfies the initial conditions.

This was a regular perturbation problem. We were able to proceed to all algebraic orders in ϵ partly because the leading order equation could be solved analytically, and partly because the equation is linear. The usefulness of this approach is diminished only slightly if the leading order equation (and hence the remaining equations) has to be solved numerically.

Example 7.4 Solve the boundary value problem $\epsilon^2 y'' - y = -1$ subject to $y(0) = 0$ and $y(1) = 1$.

This is a singular perturbation problem, but we will attempt the same regular perturbative solution as we did for Example 7.3. This time the equation becomes

$$\sum_{n=0}^{\infty} (\epsilon^{n+2} y_n'' - \epsilon^n y_n) = -1,$$

from which we obtain

$$y_0 = 1, \qquad y_1 = 0, \qquad \text{and} \quad y_n = y_{n-2}'', \quad \text{for} \quad n = 2, 3, \cdots.$$

This recurrence relation gives $y_n = 0$ for all values of n greater than or equal to 2. Hence the series solution gives $y = 1$.

But we have not satisfied the boundary condition at $x = 0$. In fact, if the right hand side of the equation had been a different constant we would not have satisfied either boundary condition. Just as the algebraic equation in Example 7.2 lost its highest power in the limit $\epsilon \to 0$, so the present equation loses its highest derivative. Here the order of the O.D.E. is reduced from 2 to 0, and therefore we would expect not to satisfy either boundary condition in general. The resolution of this difficulty lies in the existence of a "boundary layer" in which the function varies rapidly.

For the present problem a boundary layer is present at $x = 0$. We can see this by considering the analytical solution,

$$y = 1 - \frac{\sinh(1-x)/\epsilon}{\sinh 1/\epsilon},$$

which is plotted in Figure 7.1 for various values of ϵ, and shows how the boundary layer develops as $\epsilon \to 0$.
Let us suppose that x variations take place over the lengthscale, δ, and that $y = O(1)$, given the solution we have obtained so far. Hence the equation and the orders of magnitude of its constituent terms are given by

$$\epsilon^2 \frac{d^2 y}{dx^2} - y = -1 \qquad \sim \qquad \left[\frac{\epsilon^2}{\delta^2}\right] \quad [1] \quad [1] \tag{7.3}$$

Therefore δ, and hence x, is of $O(\epsilon)$. We rescale x according to $x = \epsilon\xi$ and the equation reduces to

$$\frac{d^2 y}{d\xi^2} - y = -1,$$

subject to $y(0) = 0$ and $y(\xi) \to 0$ as $\xi \to \infty$.

Note that we are now concentrating on a solution in an asymptotically small region about $x = 0$. The solution in this region must match with that given by the regular expansion.

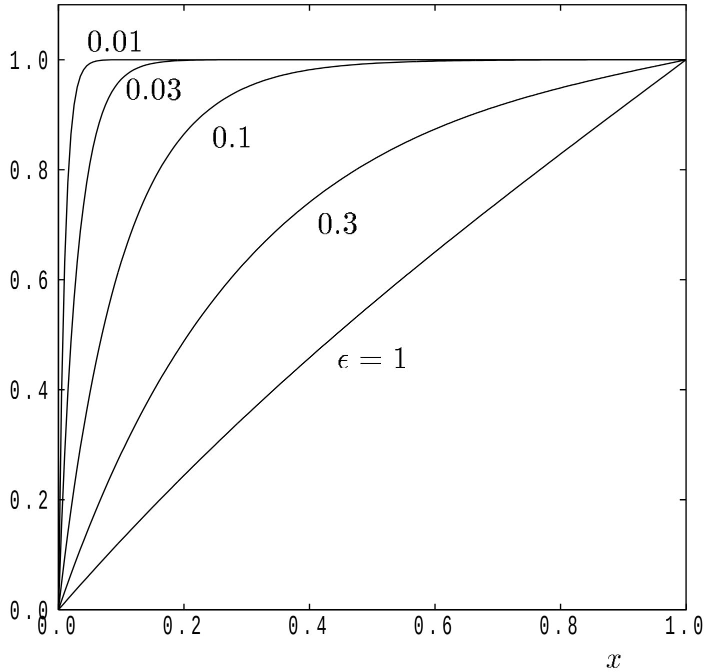

Figure 7.1. The solutions of $\epsilon^2 y'' - y = -1$ subject to $y(0) = 0$ and $y(1) = 1$ for various values of ϵ.

Given that the $x = O(1)$ solution is $y = 1$, we must have $y(\xi) \to 1$ as $\xi \to \infty$. This condition is known as the **matching condition**.

This "inner" equation has solution

$$y = 1 - e^{-\xi}.$$

If we check this against the exact solution written in terms of ξ we obtain,

$$y = 1 - \cosh \xi + \frac{\cosh(1/\epsilon)\sinh(x/\epsilon)}{\sinh(1/\epsilon)} \quad = \quad 1 - e^{-\xi} - \frac{2}{e^{2/\epsilon} - 1},$$

and therefore we see that the error is exponentially small. In fact, we can do no further asymptotic work on this problem unless we wish to determine these asymptotically small terms directly from the equations.

8. Boundary Layers I

The last example was of "boundary layer type" because a narrow layer develops near $x = 0$ as $\epsilon \to 0$. In some problems it is obvious from the differential equation itself, or from physical necessity where the boundary layer is situated. There are many possible cases: (i) at one end of the finite interval, (ii) at both ends of a finite interval, (iii) at an internal point. There is also the possibility of a global breakdown of the solution in the small-ϵ limit, but discussion of this type of case is deferred to later. It is also possible to have embedded boundary layers where, for example, one might have a boundary layer of thickness $O(\epsilon^2)$ embedded within another boundary layer of thickness $O(\epsilon)$.

A thorough and consistent analysis of these phenomena forms the theory of **matched asymptotic expansions**. The idea here is that since boundary layers are asymptotically narrow regions, the boundary layer solution (often termed the INNER solution) and the solution in the main region (termed the OUTER solution) must agree or match in some sense. The way in which such matching between the two asymptotic regimes is undertaken is called a **matching principle**.

If, as in Example 7.4, x is the outer variable, and ξ is the stretched coordinate in the inner region near $x = 0$, then we may write down **Prandtl's Matching Condition**:

$$\lim_{x\to 0} y^{\text{outer}}(x) \quad = \quad \lim_{\xi\to\infty} y^{\text{inner}}(\xi). \tag{8.1}$$

This condition, though plausible, cannot be applied in all cases. In particular it is difficult to apply when the inner solution grows linearly in order to match with the first derivative term in the Taylor series expansion of the outer solution.

Example 8.1 Solve the equation $\epsilon^2 y'' - y = -e^{-x}$ subject to $y(0) = 0$ and $y \to 0$ as $x \to \infty$.

Let the outer solution where $x = O(1)$ be expanded in a series in ϵ:

$$y = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots$$

At $O(1)$ we obtain immediately the solution

$$\boxed{y_0 = e^{-x}}$$

which automatically satisfies the large-x condition, but not the $x = 0$ condition. We will therefore assume that there is a boundary layer at $x = 0$. Given the close similarity between this problem and Example 7.4, we let $x = \epsilon\xi$ be the inner, stretched variable. We will also denote the inner solution by $Y(\xi)$ in order to avoid notational confusion. The equation for Y is

$$Y'' - Y = -e^{\epsilon\xi} = -1 + \epsilon\xi - \tfrac{1}{2}\epsilon^2\xi^2 + \tfrac{1}{6}\epsilon^3\xi^3 + \cdots.$$

Here we use primes to denote derivatives with respect to x when applied to y, and with respect to ξ when applied to Y. We expand $Y(\xi)$ according to

$$Y(\xi) = Y_0(\xi) + \epsilon Y_1(\xi) + \epsilon^2 Y_2(\xi) + \cdots.$$

At $O(1)$ we obtain the equation $Y_0'' - Y = -1$ which has solution

$$Y_0 = 1 + Ae^{\xi} + Be^{-\xi}.$$

Using Prandtl's matching condition we see that

$$\lim_{x\to 0} y(x;\epsilon) = 1$$

and therefore we must have $A = 0$ in order to avoid exponential growth. The application of the initial condition, $Y(0) = 0$ yields $B = -1$, and therefore we have

$$\boxed{Y_0 = 1 - e^{-\xi}}.$$

The $O(\epsilon)$ equations in the two regions are

$$y_1 = 0, \qquad Y_1'' - Y = \xi. \tag{8.2}$$

The solution for ξ which satisfies the initical condition, $Y_1(0) = 0$, is $Y_1 = -\xi + A(e^{\xi} - e^{-\xi})$, but we must have $A = 0$ in order to remove exponential growth, as before. Hence $Y_1 = -\xi$ and the overall solutions correct to $O(\epsilon)$ are

$$y = e^{-x} + O(\epsilon^2) \qquad Y = (1 - e^{-\xi}) - \epsilon\xi + O(\epsilon^2).$$

In one sense we have applied the Prantdl Matching Condition to remove the exponentially growing solution in Y_1. But we are forced to retain the linearly growing term. Indeed it may be recognised as corresponding to the second term in a Taylor series expansion of the outer solution. Therefore we must have a more robust form of the matching principle. This was stated by VanDyke:

$$\begin{aligned} &\text{The } m\text{–term inner expansion of (the } n\text{–term outer expansion)} \\ = \quad &\text{The } n\text{–term outer expansion of (the } m\text{–term inner expansion).} \end{aligned}$$

This is only a summary of the process, but the detailed application involves taking limits as $\epsilon \to 0$, as shown below. However asymptotic series sometimes proceed in different powers of ϵ in the two regions (e.g. an $\epsilon^{1/2}$ inner series and an ϵ outer series), or there exists gaps (where some powers of ϵ have zero coefficients), or there arises logarithmic terms (such $\epsilon^2 \ln\epsilon$ or $\epsilon\ln(\ln\epsilon)$). It is not always easy to define unambiguously what is meant by n terms, and therefore a better statement of the matching principle is

$$\begin{aligned} &\text{The } O(\Delta) \text{ inner expansion of (the } O(\delta) \text{ outer expansion)} \\ = \quad &\text{The } O(\delta) \text{ outer expansion of (the } O(\Delta) \text{ inner expansion).} \end{aligned} \tag{8.3}$$

Continuing with Example 8.1, we will employ this latter condition, (8.3), with $\Delta = \delta = \epsilon$:

$$\begin{array}{ll}
\text{Outer solution to } O(\epsilon): & y \sim e^{-x} \\
\text{In terms of the inner variable:} & y \sim e^{-\epsilon\xi} \\
\text{Expand for small } \epsilon: & y \sim 1 - \epsilon\xi \\
& \\
\text{Inner solution to } O(\epsilon): & Y \sim (1 - e^{-\xi}) - \epsilon\xi \\
\text{In terms of the outer variable:} & Y \sim 1 - e^{-x/\epsilon} - x \\
\text{Expand for small } \epsilon: & Y \sim 1 - x \\
\text{In terms of the inner variable:} & Y \sim 1 - \epsilon\xi
\end{array} \tag{8.4}$$

Note that we have neglected the term $e^{-x/\epsilon}$ which is transcendentally small as $\epsilon \to 0^+$; this term is smaller than any power of ϵ.

An alternative way of applying this principle is to employ a variable which is intermediate between x and ξ in terms of size. We may do this using $X = x/\epsilon^{1/2}$. Here we write both solutions in terms of X, let $\epsilon \to 0$, and compare the results.

$$\begin{array}{ll}
\text{Outer solution to } O(\epsilon): & y \sim e^{-x} \\
\text{In terms of } X: & y \sim e^{-\epsilon^{1/2}X} \\
\text{expand for small } \epsilon: & y \sim 1 - \epsilon^{1/2}X \\
& \\
\text{Inner solution to } O(\epsilon): & Y \sim (1 - e^{-\xi}) - \epsilon\xi \\
\text{In terms of } X: & Y \sim 1 - e^{-X/\epsilon^{1/2}} - \epsilon^{1/2}X \\
\text{expand for small } \epsilon: & Y \sim 1 - \epsilon^{1/2}X
\end{array} \tag{8.5}$$

Note that we could have used any intermediate regime defined by $x = \epsilon^a X$ where a satisfies $0 < a < 1$. In practice, it is sufficient to apply (8.4) safe in the knowledge that there is an intermediate regime where (8.5) may be used, but the application of the intermediate regime form of the matching principle often requires much more work, especially for higher order matching.

Now consider the $O(\epsilon^2)$ equations,

$$y_2 - y_0'' = 0 \qquad Y_2'' - Y_2 = -\tfrac{1}{2}\xi^2.$$

Hence $y_2 = e^{-x}$ which satisfies the large$-x$ condition. The inner solution is

$$Y_2 = \tfrac{1}{2}\xi^2 + 1 + Ae^{\xi} + Be^{-\xi}$$

which, on application of the boundary condition $Y_2(0) = 0$ gives

$$Y_2 = \tfrac{1}{2}\xi^2 + 1 - (1 + B)e^{\xi} + Be^{-\xi}.$$

Now let us apply the matching principle (8.4) with $\Delta = \delta = \epsilon^2$:

$$
\begin{array}{ll}
\text{Outer solution to } O(\epsilon^2): & y \sim e^{-x} + \epsilon^2 e^{-x} \\
\text{In terms of } \xi: & y \sim e^{-\epsilon\xi} + \epsilon^2 e^{-\epsilon\xi} \\
\text{expand for } \epsilon \ll 1 \text{ to } O(\epsilon^2): & y \sim 1 - \epsilon\xi + \epsilon^2(1 + \frac{1}{2}\xi^2)
\end{array}
$$

$$
\begin{array}{ll}
\text{Inner solution to } O(\epsilon^2): & Y \sim (1 - e^{-\xi}) - \epsilon\xi + \epsilon^2[\frac{1}{2}\xi^2 + 1 - (1+B)e^{\xi} + Be^{-\xi}] \\
\text{In terms of } x: & Y \sim (1 - e^{-x/\epsilon}) - x + \frac{1}{2}x^2 + \epsilon^2[1 - (1+B)e^{\xi/\epsilon}i + Be^{-\xi/\epsilon}] \\
\text{expand to } O(\epsilon^2): & Y \sim 1 - x + \frac{1}{2}x^2 + \epsilon^2 \\
 & \quad \text{where } B = -1 \text{ to avoid transcendental growth of } e^{\xi/\epsilon} \\
\text{In terms of } \xi: & Y \sim 1 - \epsilon\xi + \epsilon^2(1 + \frac{1}{2}\xi^2).
\end{array}
$$

This process is used for all types of asymptotic matching. It has been presented in a very formal way, but in practice asymptotic matching is often achieved by inspection, although the matching principles are not broken.

This was a straightforward problem in the sense that the equation solved is linear, and, apart from removing exponentially growing terms, the matching process served only to confirm that everything was proceeding correctly. In more complicated problems, such as those derived from fluid dynamics, the matching principle is used to obtain boundary conditions for the equations in the inner and outer regions. This aspect will be considered later.

Finally we may write down the overall solution as a "composite" of the inner and outer solutions. The additive form is

$$y_{\text{comp}} = y_{\text{inner}} + y_{\text{outer}} - y_{\text{match}}$$

where y_{match} is the part the solutions have in common. For the present problem we have

$$
\begin{aligned}
y_{\text{comp}} &= Y(\xi) + y(x) - [1 - \epsilon\xi + \epsilon^2(1 + \tfrac{1}{2}\xi^2)] \\
&= (e^{-x} - e^{-x/\epsilon})(1 + \epsilon^2).
\end{aligned}
$$

This composite solution should be compared with the exact solution

$$y = \frac{e^{-x} - e^{-x/\epsilon}}{1 - \epsilon^2}.$$

9. Boundary Layer Theory II.

In this lecture we will consider two boundary layers. The first is a case where we need to determine carefully how thick the boundary layer is. The second is an example of an internal boundary layer . These examples appear in a different notation in the book by Bush.

Example 9.1 Solve the equation $\epsilon y'' + x^2 y' - y = 0$ subject to $y(0) = 1$ and $y(1) = 2$ to leading order as $\epsilon \to 0$.

At leading order the outer expansion gives

$$x^2 y' - y \sim 0$$

from which we obtain $y \sim Ae^{-x^{-1}}$. The boundary condition at $x = 1$ gives $A = 2e$ and therefore

$$y \sim 2e^{1-x^{-1}}.$$

Suppose that the inner solution is denoted by $Y(\xi)$ where $x = \epsilon^a \xi$ defines the stretched variable ξ, and a is to be found. The equation for Y is now

$$\epsilon^{1-2a} Y'' + \epsilon^a Y' - Y = 0.$$

When $a = 0$ we recover the original equation. We now need to check how the relative orders of magnitude of the three terms in this equation vary as a increases. The final term remains $O(1)$, but when a increases the first term increases in size from $O(\epsilon)$, and the second decreases in size from $O(1)$. Clearly these two will be of the same order of magnitude when $a = \frac{1}{3}$, but at this point the third term is larger. When a rises to $\frac{1}{2}$ we then have the first and third terms of the same size, while the second may be neglected to leading order. So we take $a = \frac{1}{2}$, and the leading order equation for Y is

$$Y'' - Y = 0$$

subject to $Y(0) = 1$. The solution is

$$Y \sim Ae^{\xi} + (1 - A)e^{-\xi}$$

where A is to be found by matching. We do not have to go through the formal procedure in this case because it is enough to say that the exponentially growing solution must be discarded. Hence $A = 0$ and the inner solution is $Y \sim e^{-\xi}$.

Finally we can form a composite solution:

$$y \sim e^{-\xi} + 2e^{(1-x^{-1})}$$

where the part which the inner and outer solutions have in common is zero at this order in ϵ.

In more complicated equations it will be necessary to following the type of reasoning used here to obtain the thickness of the boundary layer.

Example 9.2 Solve the equation $\epsilon y'' + xy' + xy = 0$, subject to $y(-1) = e$ and $y(1) = 2e^{-1}$ in the limit as $\epsilon \to 0$.

A detailed numerical solution (see Figure 9.1) shows that an internal boundary layer arises at $x = 0$. Indeed we might have suspected that from the form of the equation at $x = 0$ where the coefficients of all the terms except the highest derivative are zero there. There will be three asymptotic regions in this case. Two main regions left and right of the origin and a boundary layer near $x = 0$. In terms of increasing x we will denote y by $y^-(x)$, $Y(\xi)$ and $y^+(x)$, where ξ is the stretched coordinate or boundary layer variable.

In the positive main region the leading order equation is

$$\frac{dy^+}{dx} + y^+ = 0, \qquad y^+(1) = 2e^{-1}$$

which has solution

$$y^+ = 2e^{-x}.$$

In the negative main region the leading order equation is

$$\frac{dy^-}{dx} + y^- = 0, \qquad y^- = e$$

which has solution

$$y^- = e^{-x}.$$

Now we assume that the internal boundary layer has thickness $O(\epsilon^a)$ where a is to be found. On setting $x = \epsilon^a \xi$ the equation in the boundary layer becomes

$$\epsilon^{1-2a} Y'' + \xi Y' + \epsilon^a \xi Y = 0.$$

The third term is clearly larger than the second term. And therefore we must balance the magnitudes of the first and third terms. This leads to $a = \frac{1}{2}$. Hence the leading order boundary layer equation is

$$Y'' + \xi Y' = 0.$$

This equation has the general solution

$$Y = A_1 \int_0^{\xi} e^{-t^2/2}\, dt + A_2 = B_1 \mathrm{erf}(\xi/\sqrt{2}) + B_2$$

where

$$\mathrm{erf}\, \xi = \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-t^2}\, dt.$$

This function has the property that $\mathrm{erf}\, \xi \to \pm 1$ as $\xi \to \pm\infty$.

The matching conditions are given by

$$\lim_{\xi\to\infty} Y(\xi) = B_1 + B_2 = \lim_{x\to 0+} y^+(x) = 2,$$

and

$$\lim_{\xi\to-\infty} Y(\xi) = -B_1 + B_2 = \lim_{x\to 0^-} y^-(x) = 1.$$

Hence we have the leading order solutions,

$$y^- \sim e^{-x}, \qquad Y \sim \tfrac{1}{2}\mathrm{erf}(\xi/\sqrt{2}) + \tfrac{3}{2}, \qquad y^+ = 2e^{-x}.$$

A composite form of this solution cannot be easily written down as we have two main regions with essentially different solutions.

10. The W.K.B. Method

We have studied equations whose solutions are of boundary layer type. This phenomenon arises because the term involving the highest derivative is multiplied by the small parameter. A regular perturbation solution in powers of ϵ breaks down at the location of the boundary layer. However, singular perturbation problems do not have only this manner of breaking down. It is also possible for solutions to suffer a global, as opposed to local, breakdown in the limit as $\epsilon \to 0$. This is exemplified in Example 10.1.

Example 10.1 Solve the equation $\epsilon y'' + y = 0$ subject to $y(0) = 0$ and $y(1) = 1$.

The exact solution is

$$y(x) = \frac{\sin(x/\sqrt{\epsilon})}{\sin(1/\sqrt{\epsilon})} \qquad \text{where } \epsilon \neq (n\pi)^{-2}. \tag{10.1}$$

Here the breakdown is global because it occurs over the whole interval, rather than near one isolated narrow region. Note that the solution for *negative* values of ϵ is

$$y(x) = \frac{\sinh(x/\sqrt{-\epsilon})}{\sin(1/\sqrt{-\epsilon})}, \tag{10.2}$$

is of boundary layer type.

Solutions such as (10.1) are examples of dispersive phenomena, where solutions exhibit rapid oscillations with relatively slowly varying changes in amplitude or wavelength. This type of problem may be solved as an asymptotic series for small ϵ by using a substitution of the form,

$$y(x) = \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right], \tag{10.3}$$

where $\delta = \delta(\epsilon)$ and the δ^{-1} term accounts for the rapid variation seen in solutions such as (10.1). This method was named after **W**entzel, **K**ramers and **B**rillouin who popularised it. Occasionally **J**efferies is included in the list: WKBJ.

We have to note that some texts develop the theory based on a large parameter expansion. For example Liouville's equation, $y'' + \lambda^2 f(x)y = 0$, is frequently cited where $\lambda \gg 1$. Clearly this may be converted easily into an equivalent small-parameter problem.

Example 10.2 Solve the Schrödinger equation, $\epsilon y'' = Q(x)y$ in the limit as $\epsilon \to 0$.

If we assume the expansion (10.3) and differentiate we obtain

$$y'(x) = \frac{1}{\delta}\Big(\sum_{n=0}^{\infty} \delta^n S_n'(x)\Big) \exp\Big[\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\Big],$$

and

$$y''(x) = \Big[\frac{1}{\delta^2}\Big(\sum_{n=0}^{\infty} \delta^n S_n'(x)\Big)^2 + \frac{1}{\delta}\Big(\sum_{n=0}^{\infty} \delta^n S_n''(x)\Big)\Big] \exp\Big[\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\Big].$$

Substitution into the governing equations yields

$$\frac{\epsilon}{\delta^2}\Big(\sum_{n=0}^{\infty} \delta^n S_n'(x)\Big)^2 + \frac{\epsilon}{\delta}\Big(\sum_{n=0}^{\infty} \delta^n S_n''(x)\Big) = Q(x).$$

The dominant balance here is that $\delta = \epsilon^{1/2}$ in order to balance the largest left hand side term with the right hand side term. Hence the leading term is

$$(S_0')^2 = Q(x)$$

which may be solved analytically in many cases. When $Q(x) < 0$ then S_0 is purely imaginary, and this accounts for the highly oscillatory nature of solutions such as (10.1). When $Q(x) > 0$ then S_0 is real, and this corresponds to the boundary layer type of solution given by (10.2). Therefore WKB theory may also be used to solve boundary layer problems.

If we now take $Q(x) = -1$ we recover the equation of Example 10.1. We have $S_0' = \pm i \Rightarrow S_0 = \pm ix$. Hence

$$y \sim \exp\Big[\pm\frac{ix}{\epsilon^{1/2}}\Big] \qquad \text{or} \qquad y = A\cos(x/\epsilon^{1/2}) + B\sin(x/\epsilon^{1/2}).$$

In this case, if we apply the Example 10.1 boundary conditions we recover the solution (10.1).

Conversely, if we take $Q(x) = 1$, then $S_0' = \pm 1 \Rightarrow S_0 = \pm x$. Hence

$$y \sim \exp\Big[\pm\frac{x}{\epsilon^{1/2}}\Big] \qquad \text{or} \qquad y = A\cosh(x/\epsilon^{1/2}) + B\sinh(x/\epsilon^{1/2}).$$

Application of the Example 10.1 boundary conditions leads to

$$y(x) = \frac{\sinh(x/\sqrt{\epsilon})}{\sin(1/\sqrt{\epsilon})}, \tag{10.4}$$

which is equivalent to (10.2), given the slight difference in how the equations are written.

If we take $Q(x) = -(1+x)$, that is we solve $\epsilon y'' + (1+x)y = 0$, then

$$(S_0')^2 = -(1+x), \qquad \Rightarrow \qquad S_0' = \pm\sqrt{1+x} \qquad \Rightarrow \qquad S_0 = \pm\tfrac{2}{3}i(1+x)^{3/2}.$$

Hence

$$\begin{aligned} y &\sim \exp\left[\pm\tfrac{2}{3}i(1+x)^{2/3}/\epsilon^{1/2}\right] \\ \text{or} \quad &= A\cos\left[\tfrac{2}{3}(1+x)^{2/3}/\epsilon^{1/2}\right] + B\sin\left[\tfrac{2}{3}(1+x)^{2/3}/\epsilon^{1/2}\right]. \end{aligned}$$

Application of the Example 10.1 boundary conditions yields

$$y = \frac{\sin\left[\tfrac{2}{3}(1+x)^{2/3}/\epsilon^{1/2}\right]}{\sin\left[\tfrac{2}{3}2^{2/3}/\epsilon^{1/2}\right]} \qquad \epsilon \neq (\tfrac{2}{3})^2 2^{4/3}/(n^2\pi^2). \tag{10.5}$$

Let us now generalise the problem and proceed to the next term. At leading order we have

$$S_0(x) = \pm\int^x \sqrt{Q(t)}\,dt.$$

The next term in the expansion of (10.3) is

$$2S_0'S_1' + S_0'' = 0.$$

This may be solved to get

$$S_1 = -\tfrac{1}{4}\ln Q(x).$$

Therefore a general solution may be written in the form

$$y = AQ^{-1/4}(x)\exp\left[\epsilon^{-1/2}\int_\alpha^x \sqrt{Q(t)}\,dt\right] + BQ^{-1/4}(x)\exp\left[-\epsilon^{-1/2}\int_\alpha^x \sqrt{Q(t)}\,dt\right].$$

The solutions given in (10.1) and (10.4) remain the same even with this extra term because they are already exact solutions of their respective equations. However when $Q(x) = -(1+x)$ we need to replace (10.5) by

$$y \sim \frac{2^{1/4}}{(1+x)^{1/4}}\frac{\sin\left[\tfrac{2}{3}(1+x)^{2/3}/\epsilon^{1/2}\right]}{\sin\left[\tfrac{2}{3}2^{2/3}/\epsilon^{1/2}\right]} \qquad \epsilon \neq (\tfrac{2}{3})^2 2^{4/3}/(n^2\pi^2). \tag{10.6}$$

From this latest solution we see that higher rates of oscillation are accompanied by lower amplitudes. The solution (10.6) is shown in Figure 10.1 for $\epsilon = (\tfrac{2}{3})^2 2^{4/3}/(121\pi/2)^2$.

WKB solutions taken as far as the S_1 term are known as the **physical optics** solution.

Further terms in the asymptotic expansions may be computed and these provide small corrections to the amplitude and phase of the oscillations.

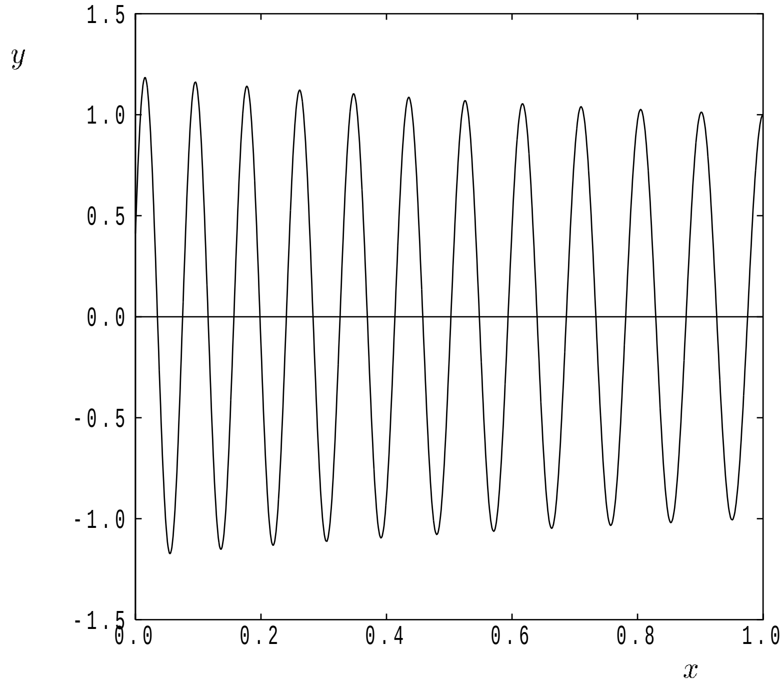

Figure 10.1. The 'physical optics' solution (10.6). Note the gradual change in amplitude of the solution.

We have concentrated on only those cases for which $Q(x)$ is single signed, either always positive or always negative. Of especial interest are those cases where $Q(x)$ changes sign in the interval being considered. Such points where $Q(x) = 0$ are called turning points. The solution of equations with a turning point involve obtaining two WKB formulae which are valid either side of the turning point, an examination of the detailed solution close to the turning point, which often involves Airy functions (which have oscillatory properties one side of the origin, and exponentially or superexponentially varying properties on the other side), and determining matching conditions between the two outer regions. Such conditions are called **connection formulae**; a study of these is outside the scope of this course, but different examples are given in the books by Bender & Orszag and by Bush.

11. The Method of Multiple Scales I.

In many physical problems there is often more than one characteristic lengthscale or timescale associated with them. Examples include the response of a relatively slowly varying nonlinear dynamical system to fast vibrations, systems displaying self-excited oscillations, the response to sudden pressure surges in hydraulic pipelines, and the effects of surface roughness or vibrations on fluid flows. The presence of certain small-amplitude terms in equations may have the effect of making what looks like a regular perturbation problem become nonuniformly convergent.

In the next two Examples we show the difference between regular perturbation problems which are and are not uniformly valid.

Example 11.1 Solve $y' + y = \epsilon y^2$ subject to $y(0) = 1$ as $\epsilon \to 0$.

We employ the regular perturbation series: $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$ and determine the equations which arise at each power of ϵ.

At $O(1)$: $$y_0' + y_0 = 0 \quad \Rightarrow \quad y_0 = e^{-x}$$

At $O(\epsilon)$: $$y_1' + y_1 = y_0^2 = e^{-2x} \quad \Rightarrow \quad y_1 = e^{-x} - e^{-2x}$$

At $O(\epsilon^2)$: $$y_2' + y_2 = 2y_0 y_1 = 2(e^{-2x} - e^{-3x}) \quad \Rightarrow \quad y_2 = e^{-x} - 2e^{-2x} + e^{-3x}$$

and so on. The solution is

$$y = e^{-x} + \epsilon(e^{-x} - e^{-2x}) + \epsilon^2(e^{-x} - 2e^{-2x} + e^{-3x}) + \epsilon(e^{-x} - 3e^{-2x} + 3e^{-3x} - e^{-4x}) + \cdots$$

or

$$y = e^{-x} \sum_{n=0}^{\infty} (1 - e^{-x})^n \epsilon^n, \tag{11.1}$$

a solution which is **uniformly valid**. By this is meant that the asymptotic sizes of each coefficient of ϵ remains of such a magnitude that each term in the sum (11.1) is asymptotically smaller than its preceding term. In general if the perturbation series takes the form

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \epsilon^3 y_3(x) + \cdots$$

then $\epsilon^n y_n(x) \ll \epsilon^{n-1} y_{n-1}(x)$ for all values of x as $\epsilon \to 0$. When this condition is violated the order of the terms in the series is altered and the original series is said to be **nonuniformly valid**. Example 11.2 shows such a case.

Example 11.2 Solve the equation $\dfrac{d^2y}{dt^2} + y + \epsilon y^3 = 0$ subject to $y(0) = 1$ and $y'(0) = 0$ as $\epsilon \to 0^+$.

The equation is known as Duffing's equation. When $\epsilon > 0$ all the solutions tend towards a stable periodic state. In fact it is quite straightforward to show that all solutions are bounded. If we multiply the equation by y' we get

$$y'y'' + yy' + \epsilon y^3 y' = 0$$

which, upon integration and application of the initial condition, give

$$\tfrac{1}{2}(y')^2 + \tfrac{1}{2}y^2 + \tfrac{1}{4}\epsilon y^4 = \tfrac{1}{2} + \tfrac{1}{4}\epsilon.$$

This implies that the maximum possible displacement is 1 and the maximum possible velocity is $\sqrt{1 + \frac{1}{2}\epsilon}$.

Now set $y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \epsilon^3 y_3(t) + \cdots$.

At $O(1)$:
$$y_0'' + y_0 = 0 \qquad \Rightarrow \qquad y_0 = \cos t$$

At $O(\epsilon)$:
$$y_1'' + y_1 = -y_0^3 = -\cos^3 t = -\tfrac{1}{4}(\cos 3t + 3\cos t)$$
$$\Rightarrow \qquad y_1 = \tfrac{1}{32}\cos 3t - \tfrac{3}{8}t\sin t.$$

Hence the solution is
$$y \sim \cos t + \epsilon\left[\tfrac{1}{32}\cos 3t - \tfrac{3}{8}t\sin t\right].$$

If we consider this solution on a finite domain such as $0 \le t \le T$ where T is fixed (and hence $t = \mathrm{Ord}(1)$ as $\epsilon \to 0$), then this series is uniformly valid. However, if we are considering an infinite domain, then the series is not uniformly valid. The nonuniformity becomes evident as soon as $t = O(\epsilon^{-1})$, in which case the term $\epsilon t \sin t$ is formally of the same order of magnitude as the $O(1)$ term. The troublesome term — $t\sin t$ in this case — is called a secular term. Various methods have been devised to try to overcome this difficulty and to determine a uniformly valid expansion; these include the method of averaging, the method of strained parameters and the method of multiple scales. We will consider only the method of multiple scales here, but the others generally require roughly the same amount of work to give the same solutions.

The method of multiple scales takes into account the breakdown of uniform validity when $t = O(\epsilon^{-1})$ in Example 11.2 by defining a second time scale, τ, according to $\tau = \epsilon t$. Thus the main oscillations take place over the timescale $t = O(1)$, whereas the nonuniformity corresponds to $\tau = O(1)$. The governing equation is written in terms of both t and τ as though they are independent variables. Hence we replace

$$\frac{d}{dt} \text{ by } \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial \tau} \qquad \text{and} \qquad \frac{d^2}{dt^2} \text{ by } \frac{\partial^2}{\partial t^2} + 2\epsilon\frac{\partial^2}{\partial t \partial \tau} + \epsilon^2\frac{\partial^2}{\partial \tau^2},$$

and we expand the solution according to

$$y = y^{(0)}(t,\tau) + \epsilon y^{(1)}(t,\tau) + \epsilon^2 y^{(2)}(t,\tau) + \cdots.$$

The full equation becomes

$$(y_{tt}^{(0)} + 2\epsilon y_{t\tau}^{(0)} + \epsilon^2 y_{\tau\tau}^{(0)}) + \epsilon(y_{tt}^{(1)} + 2\epsilon y_{t\tau}^{(1)} + \epsilon^2 y_{\tau\tau}^{(1)}) + +\epsilon^2(y_{tt}^{(2)} + 2\epsilon y_{t\tau}^{(2)} + \epsilon^2 y_{\tau\tau}^{(2)}) + \cdots$$
$$+ (y^{(0)} + \epsilon y^{(1)} + \cdots) + \epsilon(y^{(0)} + \epsilon y^{(1)} + \cdots)^3 = 0.$$

The overall guiding principle is to avoid secular terms.

At $O(1)$ we have $y_{tt}^{(0)} + y^{(0)} = 0$. We could write the solution in the form $y^{(0)} = A(\tau)\mathrm{cos}t$, where $A(\tau)$ is a slowly varying amplitude function, and where $A(0) = 0$. However, in many circumstances it is more convenient to use the complex form:

$$y^{(0)} = \tfrac{1}{2}\Big[A(\tau)e^{it} + \bar{A}(\tau)e^{-it}\Big],$$

where $A(\tau)$ is a complex amplitude which will be found further on in the expansion, and which satisfies $A(0) = 1$.

At $O(\epsilon)$ we have

$$\begin{aligned} y_{tt}^{(1)} + y^{(1)} &= -\,(y^{(0)})^3 - 2y_{t\tau}^{(0)} \\ &= -\,\tfrac{1}{8}[A^3e^{3it} + 3A^2\bar{A}e^{it} + 3A\bar{A}^2e^{-it} + \bar{A}^3e^{-3it}] \\ &\qquad - i\Big[A_\tau e^{it} - \bar{A}_\tau e^{-it}\Big]. \end{aligned}$$

If we want to avoid secular terms it is essential to remove the $\exp(\pm it)$ terms, which form the complementary function of the left hand side differential operator, and which therefore cause secularity. Therefore we must set

$$\tfrac{3}{8}A^2\bar{A} + iA_\tau = 0$$

in order to get rid of the resonant terms. Therefore we need to solve

$$A_\tau = \tfrac{3}{8}iA^2\bar{A}.$$

If we change A into polar form we are forced into recognising that it must have unit amplitude, given the initial condition. Therefore we set

$$A = e^{i\phi(\tau)};$$

we obtain

$$\phi_\tau = \tfrac{3}{8},$$

and hence $\phi = \frac{3}{8}\tau$, again insisting that $A(0) = 1$. Hence $A = \exp[\frac{3}{8}i\tau]$ and the leading order solution is now

$$y^{(0)} = \tfrac{1}{2}\left[e^{it}e^{3i\tau/8} + e^{-it}e^{-3i\tau/8}\right] = \cos(1 + \tfrac{3}{8}\epsilon)t.$$

Similarly we can find the $O(\epsilon)$ solution

$$y^{(1)} = \tfrac{1}{64}[A^3 e^{3it} + \bar{A}^3 e^{-3it}] = \tfrac{1}{32}\cos 3(1 + \tfrac{3}{8}\epsilon)t.$$

These solutions show that the wavenumber of the oscillation has changed from 1 to $1 + \frac{3}{8}\epsilon$; it is this very small change which caused the lack of uniform validity of the straightforward regular perturbation expansion.

$\cdots\cdots$but the story is not over$\cdots\cdots$ if we now attempt to find the $O(\epsilon^2)$ terms we again get a secular term. This means that the solution we have found so far is uniformly valid for $t = o(\epsilon^{-2})$. The appropriate analysis will involve the definition of a third timescale, and it is possible to show that the oscillation wavenumber is

$$1 + \tfrac{3}{8}\epsilon - \tfrac{69}{256}\epsilon^2 + O(\epsilon^3).$$

The theory of multiple scales is used frequently in weakly nonlinear analyses of fluid dynamical stability such as the Bénard problem and its porous medium analogue, the Darcy-Bénard problem. In these contexts it is usual for only two timescales to be required. Such applications will be considered later.

12. The Method of Multiple Scales II.

Example 12.1 Solve van der Pol's equation $y'' + y = \epsilon(1 - y^2)y'$ as $\epsilon \to 0^+$.

This is an example of a self-excited oscillator and is a very good prototype for many convective stability problems where there is a balance between an unstable temperature gradient and stabilising viscous forces. In this equation the damping coefficient is negative when the amplitude is negative ($y'' - \epsilon y' + y \sim 0$) when y is small), but it becomes positive when y is large. Therefore we expect small disturbances to grow, but large disturbances to decay. The final solution is known as a **limit cycle**.

A straightforward regular perturbation expansion gives

$$y = a\mathrm{cos}t - \tfrac{1}{32}a[4(a^2 - 4)t\mathrm{cos}t + a^2\mathrm{sin}3t]\epsilon + O(\epsilon^2).$$

Again a secular term appears and the expansion is no longer uniformly valid when $t = \mathrm{Ord}(\epsilon^{-1})$. Therefore we will define the slow timescale, τ, according to $\tau = \epsilon t$ and introduce the multiple scales expansion:

$$y = y^{(0)}(t, \tau) + \epsilon y^{(1)}(t, \tau) + \epsilon^2 y^{(2)}(t, \tau) + \cdots.$$

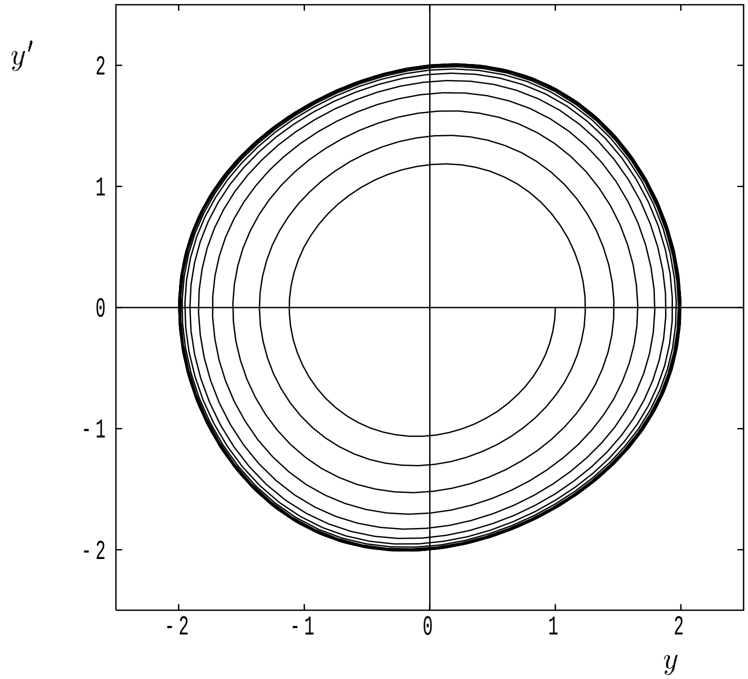

Figure 12.1. The solution of van der Pol's equation for $\epsilon = 0.1$. Here t varies between 0 and 100. The approach to the limit cycle becomes slower as $\epsilon \to 0^+$.

The first equations for $y^{(0)}$ and $y^{(1)}$ are

$$y^{(0)}_{tt} + y^{(0)} = 0 \tag{12.1}$$

$$y^{(1)}_{tt} + y^{(1)} = (1 - y^{(0)} y^{(0)}) y^{(0)}_t - 2 y^{(0)}_{t\tau} \tag{12.2}$$

Let the solution of (12.1) be

$$y^{(0)} = A(\tau) e^{it} + \bar{A}(\tau) e^{-it}.$$

Then equation (12.2) becomes

$$\begin{aligned} y^{(1)}_{tt} + y^{(1)} &= -2i[A_\tau e^{it} - \bar{A}_\tau e^{-it}] + i[Ae^{it} - \bar{A}e^{-it}]\,[1 - (Ae^{it} + \bar{A}e^{-it})^2] \\ &= (-iA^3)e^{3it} + (-2iA_\tau + iA - iA^2\bar{A})e^{it} + \text{ complex conjugates} \end{aligned}$$

We may eliminate secularity by simply setting

$$2A_\tau = A - A^2 \bar{A}.$$

If we turn A into polar form — $A = r(\tau)e^{i\phi(\tau)}$ — then we obtain $\phi' = 0$ indicating that the phase is constant; we may set it to zero. We also find that

$$r = \left[1 + \left(\frac{1}{a^2} - 1\right) e^{-\tau}\right]^{-1/2}$$

where $y = a$ at $t = 0$. Hence the leading order, uniformly valid, solution is given by

$$y = \frac{2}{\left[1 + \left(\frac{1}{a^2} - 1\right)e^{-\tau}\right]^{1/2}} \cos t.$$

Here we see how the multiple scales theory is able to give us how the overall amplitude of the oscillation varies with time. Further terms will provide corrections to this and provide small changes in the period and/or the phase of the motion. Indeed, had ϕ been time-dependent, then we would have determined these small variations in the period and/or phase at this order.

Example 12.2 Solve Mathieu's equation $y'' + (a + 2\epsilon \cos t)y = 0$.

This is an example of what are known as Hill's equations which are linear equations with periodic coefficients. The general theory of such equations is known as Floquet theory. It is possible to show that Hill's equations have solutions of the form $y = e^{\mu t}\phi(t)$ where $\phi(t + 2\pi) = \phi(t)$. Whenever $\mathrm{Re}(\mu) > 0$ the solution is said to be unstable, and is stable when $\mathrm{Re}(\mu) < 0$. Of interest are those solutions where $\mathrm{Re}(\mu) = 0$, in which case the solutions neither grow nor decay, although they may not be periodic! For Mathieu's equation the values of $\mu(a, \epsilon)$ are always real, and therefore the $\mu(a, \epsilon) = 0$ cases are always 2π-periodic. The curves in the a-ϵ plane where $\mu = 0$ are given in Figure 12.2. The curves were computed by choosing a value for ϵ, discretising Mathieu's equation using central differences and applying periodic boundary conditions, rewriting the resulting linear system in the form of a matrix eigenvalue problem for a, and using a library subroutine to determine the eigenvalues.

Let us perform a regular perturbation analysis for small values of ϵ. Let $y = \sum_{n=0}^{\infty} y_n(t)\epsilon^n$. At $O(1)$, $O(\epsilon)$ and $O(\epsilon^2)$ we obtain

$$\begin{aligned}
\frac{d^2 y_0}{dt^2} + a y_0 &= 0, \\
\frac{d^2 y_1}{dt^2} + a y_1 &= -\, 2 y_0 \cos t, \\
\frac{d^2 y_2}{dt^2} + a y_2 &= -\, 2 y_1 \cos t,
\end{aligned}$$

and so on. We get linear (i.e. secular) growth of the $O(1)$ solution only when $a = 0$. When $a > 0$ we have

$$y_0 = A e^{i\sqrt{a}t} + \bar{A} e^{-i\sqrt{a}t}.$$

Using this in the $O(\epsilon)$ equation gives

$$\frac{d^2 y_1}{dt^2} + a y_1 = -A e^{i(\sqrt{a}+1)t} - A e^{i(\sqrt{a}-1)t} + \text{ c.c.}$$

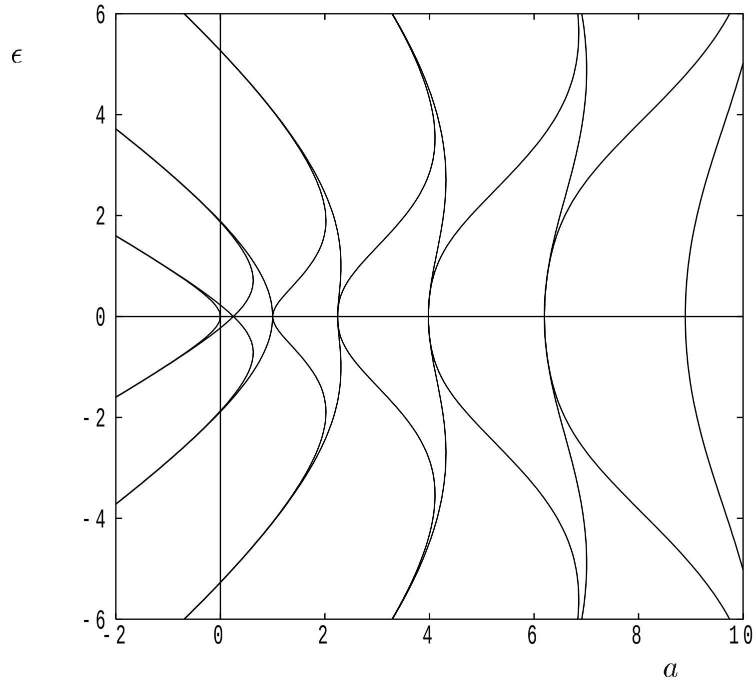

Figure 12.2. The neutral curves of Mathieu's equation showing the loci where solutions neither grow nor decay, but are quasi-periodic.

These forcing terms resonate with the differential operator only when $\sqrt{a}\pm 1 = \pm\sqrt{a}$. This occurs when $a = \frac{1}{4}$. If we were to continue to higher orders, then successive terms would yield, in turn, the following values of a at which secularity occurs: 1, $\frac{9}{4}$, 4, $\frac{25}{4}$ and so on.

When $a \neq \frac{1}{4}n^2$, where n is an integer, we can solve Mathieu's equation to all orders and obtain

$$y(t) = e^{i\sqrt{a}t}\phi(t) + \text{ c.c.} \qquad \text{where} \qquad \phi(t) = \sum_{n=0}^{\infty} \epsilon^n A_n e^{int}.$$

Let us use Multiple Scales theory to determine the stability boundary near $a = \frac{1}{4}$. The numerical evidence of Figure 12.2 suggests that we should set

$$a = \tfrac{1}{4} + a_1\epsilon + \cdots$$

and therefore Mathieu's equation becomes

$$\frac{d^2y}{dt^2} + \left[\tfrac{1}{4} + (a_1 + 2\cos t)\epsilon + \cdots\right]y = 0.$$

Now introduce the multiple scales expansion,

$$y = y_0(t,\tau) + \epsilon y_1(t,\tau) + \cdots$$

where $\tau = \epsilon t$. At $O(1)$ and $O(\epsilon)$ we get

$$\frac{\partial^2 y_0}{\partial t^2} + \tfrac{1}{4} y_0 = 0,$$

and

$$\frac{\partial^2 y_1}{\partial t^2} + \tfrac{1}{4} y_1 = -(a_1 + 2\text{cos}t) y_0 - 2\frac{\partial^2 y_0}{\partial t \partial \tau}.$$

The $O(1)$ solution is clearly

$$y_0 = A(\tau) e^{i\tau/2} + \bar{A}(\tau) e^{-i\tau/2}.$$

The $O(\epsilon)$ equation becomes

$$\frac{\partial^2 y_1}{\partial t^2} + \tfrac{1}{4} y_1 = -[a_1 A + iA_\tau + \bar{A}]e^{it} - Ae^{3it} + \text{ c.c..}$$

Secularity is eliminated by setting

$$iA_\tau = -a_1 A - \bar{A}.$$

This is most easily solved by setting $A = B + iC$ in which case we get

$$B_\tau = (1 - a_1)C \qquad C_\tau = (1 + a_1)B,$$

which yields

$$B'' = (1 - a_1^2)B,$$

and hence

$$B \propto \exp\left[\pm\sqrt{1 - a_1^2}\tau\right].$$

Instability now corresponds to when B grows exponentially, i.e. when $|a_1| < 1$, and we have stability when $|a_1| > 1$. Therefore the stability boundary is given by

$$a = \tfrac{1}{4} \pm \epsilon. \tag{12.1}$$

These curves intersect the axis at $45°$ angles which confirms the behaviour seen in Figure 12.2.

Higher order corrections to this formula for the stability boundary may be undertaken, but the degree of complexity rises. It may be shown that a more accurate version of (12.1) is

$$a = \tfrac{1}{4} \pm \epsilon - \tfrac{1}{2}\epsilon^2 + O(\epsilon^3).$$

The leading behaviour of the stability curve near $a = 1$ is resolved at $O(\epsilon^2)$. In fact the curves exhibit an increasing order of contact and the stability curves near $a = \frac{1}{4}n^2$ require an analysis to at least $O(\epsilon^n)$.

13. Solvability Conditions I.

In the method of multiple scales, secular terms were removed by setting to zero the coefficients of the resonant terms, thereby forming an equation for the unknown amplitude $A(\tau)$. A similar scenario arises in more complicated sets of equations where resonant inhomogeneous terms appear in ordinary or partial differential systems of equations. In these cases we cannot simply set to zero the coefficients of the resonant terms in each equation as there is usually only one free parameter in the system. But rather we have to derive a solvability condition.

Example 13.1 Solve the equation $y'' + \pi^2 y = \lambda \sin \pi x + x - x^2$ subject to $y(0) = 0$ and $y(1) = 0$.

Here it has to assumed that λ is unknown, but will be obtained as part of the solution procedure. This seems reasonable as the term it multiplies satisfies the inhomogeneous form of the equation and the boundary conditions.

This is an excellent prototype for solving similar systems of equations even though it is only one equation. Solutions do not exist for arbitrary values of λ, but there is only one value for which we may find solutions. We may find λ using the standard complementary function and particular integral method, then insist that the boundary conditions are satisfied and hence λ is found. But it is also possible to find λ without first solving the equation — this forms a solvability condition.

The direct solution method yields

$$y = \frac{x - x^2}{\pi^2} + \frac{2}{\pi^4} + A\sin\pi x + \frac{4}{\pi^4}(x - \tfrac{1}{2})\cos\pi x$$

where $\lambda = -8/\pi^3$, and A is arbitrary. The alternative approach requires us to denote by $Y(x)$ a solution of the inhomogeneous version of the given system:

$$y'' + \pi^2 y = 0, \qquad y(0) = 0, \quad y(1) = 0.$$

We may take $Y(x) = \sin\pi x$ although any nonzero multiple of this also works. Now we multiply the original equation by Y and integrate over the given interval.

$$\begin{aligned}\int_0^1 Y\left[\lambda\sin\pi x + x - x^2\right] dx &= \int_0^1 Y\left[y'' + \pi^2 y\right] dx \\ &= \int_0^1 y\left[Y'' + \pi^2 Y\right] dx = 0.\end{aligned}$$

This final integral we set to zero by definition of Y. Hence we get

$$\begin{aligned}\lambda &= -\frac{\int_0^1 (x-x^2)Y\,dx}{\int_0^1 (\sin\pi x)Y\,dx} \\ &= -\frac{\int_0^1 (x-x^2)\sin\pi x\,dx}{\int_0^1 \sin^2\pi x\,dx} = -\frac{8}{\pi^3}.\end{aligned} \tag{13.1}$$

Here (13.1) is the solvability condition (or orthogonality condition), and its derivation has not involved solving the equation. We have merely insisted that the equation must have a solution, and therefore we may now proceed to find it should we wish to do so! Note that the value of λ we have obtained is minus the Fourier Sine Series coefficient of $\sin\pi x$ in $x - x^2$; therefore there is now no coefficient of $\sin\pi x$ in the inhomogeneous term.

More generally, if the equation has the form, $\mathcal{L}y = \mathcal{R}$, where $\mathcal{L}$ is a differential operator, and $\mathcal{R}$ is the inhomogeneous term, and if Y satisfies both $\mathcal{L}Y = 0$ and the boundary conditions imposed on y, then in many circumstances we may show that

$$\int \mathcal{R}Y\,dx = \int Y\mathcal{L}y\,dx = \int y\mathcal{L}Y\,dx = 0,$$

thereby obtaining the solvability condition,

$$\int \mathcal{R}Y\,dx = 0.$$

Systems where this is valid are known as self-adjoint; they are characterised by the fact that the integration by parts procedure yields the same differential operator. Sometimes it is necessary to multiply an equation by an appropriate factor to make it self adjoint. An example of this is

$$\mathcal{L} = \frac{d^2}{dx^2{}^2} + \frac{1}{x}\frac{d}{dx} + 1;$$

here we need to multiply by x to obtain

$$\mathcal{L} = x\frac{d^2}{dx^2} + \frac{d}{dx} + x = \frac{d}{dx}\left(x\frac{d}{dx}\right) + x,$$

which is self adjoint. In other circumstances the equation cannot be made self-adjoint. If we have the system

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \mathcal{R},$$

then multiply by Y and integrate by parts assuming that Y satisfies suitable boundary conditions. We obtain

$$\int Y\Big(\frac{d^2y}{dx^2} + \frac{dy}{dx} + y\Big)\,dx = \int y\Big(\frac{d^2Y}{dx^2} - \frac{dY}{dx} + Y\big)\,dx.$$

Therefore the solvability condition for this non-adjoint system is

$$\int Y\mathcal{R}\,dx = 0 \quad \text{where } Y \text{ satisfies } \quad \frac{d^2Y}{dx^2} - \frac{dY}{dx} + Y = 0.$$

The differential operator

$$\mathcal{L}^* = \frac{d^2}{dx^2} - \frac{d}{dx} + 1$$

is called the adjoint operator to

$$\mathcal{L} = \frac{d^2}{dx^2} + \frac{d}{dx} + 1.$$

Now we turn to a system of equations and apply the same ideas.

Example 13.2 Solve the system of equations,

$$y'' + z' + 2y = \lambda \text{cos}t,$$

$$z'' - y' + 2z = \text{sin}t,$$

subject to the condition that the solution must have period 2π.

First it is essential to find all the possible solutions of the equivalent inhomogeneous system,

$$Y'' + Z' + 2Y = 0,$$

$$Z'' - Y' + 2Z = 0,$$

by setting $Y = y^* e^{\sigma t}$ and $Z = z^* e^{\sigma t}$. We get

$$\begin{pmatrix} \sigma^2 + 2 & \sigma \\ -\sigma & \sigma^2 + 2 \end{pmatrix} \begin{pmatrix} y^* \\ z^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{13.2}$$

Nonzero solutions correspond to the matrix having a zero determinant. Hence

$$0 = (\sigma^2 + 2)^2 + \sigma^2 = \sigma^4 + 5\sigma^2 + 4 = (\sigma^2 + 4)(\sigma^2 + 1),$$

and therefore $\sigma = \pm i, \pm 2i$. The important ones are $\pm i$. Given the form of the inhomogeneous terms we will set $Y = \text{cos}t$. Given equations (13.2) we therefore need $Z = -\text{sin}t$.

Now we will form the integrals

$$\int Y\Big(\lambda\text{cos}t\Big)\,dt = \int Y\Big(y'' + z' + 2y\Big)\,dt = \int \Big[y\Big(Y'' + 2Y\Big) - z\Big(Y'\Big)\Big]\,dt,$$

$$\int Z\left(\sin t\right) dt = \int Z\left(z'' - y' + 2z\right) dt = \int \left[z\left(Z'' + 2Z\right) + y\left(Z'\right)\right] dt.$$

Now add these two equations to get

$$\int \left(\lambda Y \cos t + Z \sin t\right) dt = \int y\left(Y'' + Z' + 2Y\right) dt + \int z\left(Z'' - Y' + 2Z\right) dt = 0.$$

Hence

$$\lambda = -\frac{\int Z \sin t \, dt}{\int Y \cos t} = 1.$$

Therefore we may obtain solutions only when $\lambda = 1$. In this case the solution is

$$\begin{pmatrix} y \\ z \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + A\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix},$$

where A is arbitrary.

14. Solvability Conditions II

We will now apply these ideas and extend them to more complicated systems.

Example 14.1 Determine the solvability condition for $y'' + \frac{1}{4}y = f(x)$ subject to $y'(0) = a$ and $y(\pi) = b$.

The complication here is that the boundary conditions are nonzero and one of them involves a derivative. Suppose $Y(x)$ is the adjoint function, although we presently know nothing about it. We follow the same procedure as above:

$$\begin{aligned}
\int_0^\pi Y(x)f(x)\,dx &= \int_0^\pi Y(y'' + \tfrac{1}{4}y)\,dx \\
&= \Big[Yy' - Y'y\Big]_0^\pi + \int_0^\pi y(Y'' + \tfrac{1}{4}Y)\,dx \\
&= Y(\pi)y'(\pi) - Y(0)a - Y'(\pi)b + Y'(0)y(0) + \int_0^\pi y(Y'' + \tfrac{1}{4}Y) \\
&= Y(\pi)y'(\pi) - Y(0)a - Y'(\pi)b + Y'(0)y(0),
\end{aligned}$$

on assuming that Y satisfies $Y'' + \frac{1}{4}Y = 0$.

We now insist that the coefficients of $y'(\pi)$ and $y(0)$ vanish, i.e. that $Y'(0) = 0$ and $Y(\pi) = 0$. Hence the solvability condition is that

$$\int_0^\pi Y(x)f(x)\,dx = -aY(0) - bY'(\pi).$$

Therefore we may take $Y = \cos\frac{1}{2}x$ and we obtain the solvability condition

$$\int_0^\pi f(x)\cos\tfrac{1}{2}x = \tfrac{1}{2}b - a.$$

Example 14.2 Find the solvability condition for $y'' + y'/x + \lambda y = f(x)$ subject to $y(x_1) = c_1$ and $y(x_2) = c_2$ where λ is the eigenvalue of the corresponding homogeneous system.

This last piece of information tells us that the equation $y'' + y'/x + \lambda y = 0$ subject to $y = 0$ at both $x = x_1$ and $x = x_2$ has a nonzero solution.

The differential operator corresponding to the left hand side of the given equation is not self-adjoint, but may be made so by multiplying by x. Therefore we obtain the equation

$$(xy')' + xy = xf(x).$$

Now we multiply this by $Y(x)$, the adjoint function, and integrate:

$$\begin{aligned}\int_{x_1}^{x_2} xY(x)f(x)\,dx &= \int_{x_1}^{x_2} Y\Big[(xy')' + xy\Big]\,dx \\ &= \Big[(xy')Y - (xY')y\Big]_{x_1}^{x_2} + \int_{x_1}^{x_2} y\Big[(xY')' + xY\Big]\,dx \\ &= x_2Y(x_2)y'(x_2) - x_1Y(x_1)y'(x_1) + x_1c_1Y'(x_1) - x_2c_2Y'(x_2),\end{aligned}$$

where it is assumed that $(xY')' + xY = xY'' + Y' + xY = 0$. We have to choose the boundary conditions, $Y(x_1) = Y(x_2) = 0$, in order that the solvability condition is independent of y. Finally we have

$$\int_{x_1}^{x_2} xY(x)f(x)\,dx = x_1c_1Y'(x_1) - x_2c_2Y'(x_2).$$

Example 14.3 Find the solvability condition for

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \pi^2\Big(\frac{n^2}{a^2} + \frac{m^2}{b^2}\Big)\psi = f(x,y),$$

subject to the boundary conditions $\psi = 0$ on $x = 0, a$ and $y = 0, b$.

This is a self-adjoint problem, a fact which I have assumed for now, but which is justified when we try to find the solvability condition. The homogeneous form of the problem has the solution

$$\Psi = \sin(n\pi x/a)\sin(m\pi y/b),$$

which we treat as the adjoint function. Now multiply the equation by Ψ and integrate over the whole rectangle:

$$\begin{aligned}\int_0^b\int_0^a f(x,y)\Psi\,dx,dy &= \int_0^b\int_0^a \Psi\Big[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \pi^2\Big(\frac{n^2}{a^2} + \frac{m^2}{b^2}\Big)\psi\Big]\,dx\,dy \\ &= \int_0^b\int_0^a \psi\Big[\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \pi^2\Big(\frac{n^2}{a^2} + \frac{m^2}{b^2}\Big)\Psi\Big]\,dx\,dy \\ &= 0.\end{aligned}$$

Note that we needed to interchange the order of the integration when dealing with the ψ_{yy}-term:

$$\begin{aligned}\int_0^b\int_0^a \psi_{yy}\Psi\,dx\,dy &= \int_0^a\int_0^b \psi_{yy}\Psi\,dy\,dx\\ &= \int_0^a \left[\Big(\psi_y\Psi-\psi\Psi_y\Big)_0^b + \int_0^b \psi\Psi_{yy}\,dy\right]dx\\ &= \int_0^a\int_0^b \psi\Psi_{yy}\,dy\,dx \quad = \quad \int_0^b\int_0^a \psi\Psi_{yy}\,dx\,dy.\end{aligned}$$

Example 14.4 The free response of a two-degree-of-freedom system is governed by

$$\begin{aligned}y'' + \tfrac{1}{2}z' + (\tfrac{1}{2}+\epsilon\sigma)y &= \epsilon yz\\ z'' - \tfrac{1}{2}y' + \tfrac{1}{2}z &= \epsilon y^2\end{aligned}$$

where $\epsilon \ll 1$. Determine the equations describing the amplitudes and phases. (after Nayfeh)

This problem uses the method of multiple scales in conjunction with solvability conditions in order to achieve a solution. Although it involves two variables, y and z, the form is very similar to Example 12.1, and therefore we expect the slow time scale to be of $O(\epsilon^{-1})$. Therefore we solve this systems by taking an expansion of the form

$$y = y_0(t,\tau)+\epsilon y_1(t,\tau)+\cdots, \qquad z = z_0(t,\tau)+\epsilon z_1(t,\tau)+\cdots,$$

where $\tau = \epsilon t$. At leading order we obtain

$$\begin{aligned}y_0'' + \tfrac{1}{2}z_0' + \tfrac{1}{2}y_0 &= 0,\\ z_0'' - \tfrac{1}{2}y_0' + \tfrac{1}{2}z_0 &= 0.\end{aligned}$$

These equations admit solutions which are proportional to both $e^{it/2}$ and e^{it}. After a little more analysis we take as the leading order solutions,

$$\begin{aligned}y_0 &= A(\tau)e^{it/2} + B(\tau)e^{it} + \text{complex conjugates}\\ z_0 &= iA(\tau)e^{it/2} - iB(\tau)e^{it} + \text{complex conjugates}\end{aligned} \tag{14.1}$$

At $O(\epsilon)$ we obtain the somewhat more complicated equations,

$$\begin{aligned}y_1'' + \tfrac{1}{2}z_1' + \tfrac{1}{2}y_1 &= -\sigma y_0 - 2y_{0\tau}' - \tfrac{1}{2}z_0' + y_0 z_0,\\ z_1'' - \tfrac{1}{2}y_1' + \tfrac{1}{2}z_1 &= -2z_{0\tau}' + \tfrac{1}{2}y_0 + y_0^2.\end{aligned}$$

The right hand side terms involve terms which are resonant, i.e. will cause secular solutions, and others which aren't, such as $e^{3it/2}$. Here primes attached to y and z denote derivatives with respect to t. We obtain

$$y_1'' + \tfrac{1}{2}z_1' + \tfrac{1}{2}y_1 = [-\sigma A - \tfrac{3}{2}iA' - 2iB\bar{A}]e^{it/2} + [-\sigma B - \tfrac{3}{2}iB' + iA^2]e^{it} + \text{c.c.} + \text{N.S.T.}$$
$$z_1'' - \tfrac{1}{2}y_1' + \tfrac{1}{2}z_1 = [\tfrac{3}{2}A' + 2B\bar{A}]e^{it/2} + [-\tfrac{3}{2}B' + A^2]e^{it} + \text{c.c.} + \text{N.S.T.},$$

where c.c. stands for "complex conjugates" and N.S.T. for "non secular terms".

In this problem we need to apply solvability conditions for both modes, $e^{it/2}$ and e^{it}. Given that the resonant terms take the form of complex exponential we need to multiply the equations by $e^{-it/2}$ and by e^{-it} before integrating to get the solvability conditions. The system of equations is self-adjoint and, although the detailed analysis is a little lengthy, it nevertheless follows the same procedure as in previous examples, such as Example 13.2. We eventually obtain the following two solvability conditions which correspond respectively to the $e^{it/2}$ and e^{it} resonances:

$$A' = \tfrac{1}{3}i\sigma A - \tfrac{4}{3}B\bar{A}, \qquad B' = \tfrac{1}{3}i\sigma B + \tfrac{2}{3}A^2.$$

These complex equations may be solved numerically to determine how the the amplitude functions defined in (14.1) vary with τ.

15. Convection in a Porous Layer inclined at a Small Angle to the Horizontal

In this study we will consider the onset of convection in a porous layer heated from below. The standard Darcy equations are assumed for the fluid flow within the solid medium forming the porous matrix. Beginning with the nondimensionalised equations in two dimensions, we determine the criterion for the onset of convection in a horizontal layer in terms of the value of R, the Darcy-Rayleigh number, which will depend on k, the wavenumber of the disturbance. Then we perturb this solution by inclining the layer slightly away from the horizontal. The expansion is shown not to be uniformly convergent for large values of k, and we undertake an analysis which is appropriate for the region where the straightforward expansion breaks down.

This lecture is based on "The onset of Darcy-Bénard convection in an inclined layer heat from below" by Rees and Bassom which has been accepted for publication in *Acta Mechanica.*

The equations of motion are

$$u_x + v_y = 0, \tag{15.1}$$

$$u = -p_x + R\theta\sin\alpha, \tag{15.2}$$

$$v = -p_y + R\theta\cos\alpha, \tag{15.3}$$

$$\theta_t + u\theta_x + v\theta_y = \theta_{xx} + \theta_{yy}, \tag{15.4}$$

in which the Darcy–Rayleigh number is defined as

$$R = \frac{\rho g \beta K d(T_h - T_c)}{\mu\kappa}. \tag{15.5}$$

The introduction of the streamfunction, ψ, according to

$$u = -\psi_y, \qquad v = \psi_x, \tag{15.6}$$

simplifies the equations further and what remains is the coupled system

$$\psi_{xx} + \psi_{yy} = R(\theta_x \cos\alpha - \theta_y \sin\alpha), \tag{15.7}$$

$$\theta_{xx} + \theta_{yy} = \psi_x\theta_y - \psi_y\theta_x + \theta_t, \tag{15.8}$$

which has to be solved subject to the boundary conditions that $\psi = 0$ on both $y = 0$ and $y = 1$, and $\theta = 1$ on $y = 0$ and $\theta = 0$ on $y = 1$.

Here α is the angle the layer makes with the horizontal, and x is the coordinate along the surface. There is a basic solution which depends only on y:

$$\psi = \psi_b = -\tfrac{1}{2}y(1-y)R\sin\alpha, \qquad \theta = \theta_b = 1 - y. \tag{15.9}$$

This is valid for all values of R and α, although it is not necessarily the only solution. If we perturb the solutions of the full system, (15.7) and (15.8), about this solution using $\psi = \psi_b + \Psi$ and $\theta = \theta_b + \Theta$, then assume that both Ψ and Θ are so small in magnitude that we may neglect quadratic terms, then equations (15.7) and (15.8) reduce to,

$$\nabla^2\Psi = R[\Theta_x\cos\alpha - \Theta_y\sin\alpha], \qquad \Theta_t = \nabla^2\Theta + \Psi_x + (R\sin\alpha)(y - \tfrac{1}{2})\Theta_x. \tag{15.10, 15.11}$$

The layer is assumed to be of infinite extent in the x-direction and therefore we may Fourier-decompose the x-dependent part. Therefore we set

$$\Psi = if(y)\exp[ikx + \lambda t], \qquad \Theta = g(y)\exp[ikx + \lambda t], \tag{15.12, 15.13}$$

where the 'i' coefficient for f is used for convenience. Hence f and g satisfy the equations,

$$f'' - k^2 f = (kR\cos\alpha)g + i(R\sin\alpha)g', \tag{15.14}$$

$$g'' - k^2 g = kf - ik(R\sin\alpha)(y - \tfrac{1}{2})g + \lambda g, \tag{15.15}$$

subject to $f(0) = f(1) = g(0) = g(1) = 0$. This is a complex eigenvalue problem. We wish to find values of R as a function of k for which the real part of λ is zero; i.e. for which the disturbance is neutrally stable — it neither grows nor decays. This is an eigenvalue problem for both R and $\mathrm{Im}(\lambda)$, i.e. given a value of k we wish to find the values of R and $\mathrm{Im}(\lambda)$. We therefore need to apply a normalising condition which we take to be $g'(0) = \pi$

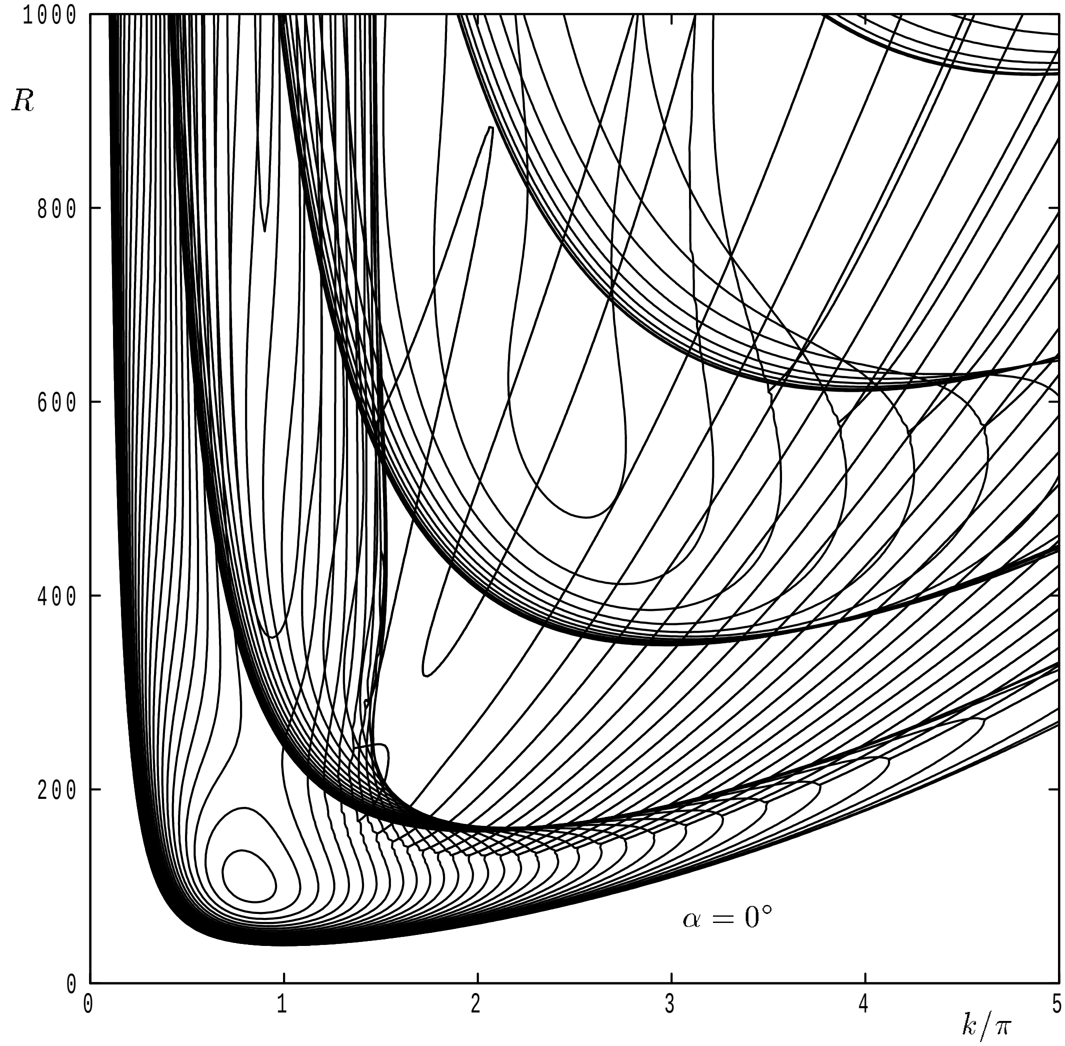

Figure 15.1. The neutral curves for the onset of convection in an inclined layer. The curves represent different modes and different angles from the horizontal. The smallest closed loop represents 31°

for convenience. There is no analytical solution of this system except for when $\alpha = 0$, when the layer is horizontal. The full numerical solution is given in Figure 15.1 given on the next page. The neutral curves (also known as neutral stability curves or marginal stability curves) were computed by reducing the ODEs to a matrix eigenvalue problem using a central difference approximation. But our aim here is to determine how the $\alpha = 0$ case is modified when the layer is at small angles to the horizontal.

When $\alpha = 0$ equations (15.14) and (15.15) have the solutions,

$$f = -\frac{\pi^2 + k^2}{k}\sin\pi y, \qquad g = \sin\pi y, \qquad R = \frac{(\pi^2 + k^2)^2}{k^2}, \quad \lambda = 0. \tag{15.16}$$

The $R(k)$ curve for $\alpha = 0$ is shown in Figure 15.1. If we were instead to fix the value of R and determine λ then we would find that λ is negative when it is less than the value given in (15.16), implying that disturbances decay, but is positive when it is greater than this value.

Now we will determine what happens when the α is very small. Given the presence of terms involving $\sin\alpha$ we use a power series in α:

$$\begin{pmatrix} f \\ g \\ R \\ \lambda \end{pmatrix} = \begin{pmatrix} f_0 \\ g_0 \\ R_0 \\ \lambda_0 \end{pmatrix} + \alpha \begin{pmatrix} f_0 \\ g_0 \\ R_0 \\ \lambda_0 \end{pmatrix} + \alpha^2 \begin{pmatrix} f_0 \\ g_0 \\ R_0 \\ \lambda_0 \end{pmatrix} + + \cdots, \tag{15.17}$$

On performing this regular perturbation expansion up to $O(\alpha^2)$ for $O(1)$ values of k, we obtain the following three systems of equations

$$f_0'' - k^2 f_0 = R_0 k g_0, \tag{15.18}$$
$$g_0'' - k^2 g_0 = k_0 f_0 + \lambda_0 g_0, \tag{15.19}$$

$$f_1'' - k^2 f_1 = R_0(kg_1 + ig_0') + R_1 k g_0, \tag{15.20}$$
$$g_1'' - k^2 g_1 = k f_1 - iR_0 k(y - \tfrac{1}{2})g_0 + \lambda_0 g_1 + \lambda_1 g_0, \tag{15.21}$$

$$f_2'' - k^2 f_2 = R_0(kg_2 + ig_1' - \tfrac{1}{2}kg_0) + R_1(kg_1 + ig_0') + R_2 k g_0 \tag{15.22}$$
$$g_2'' - k^2 g_2 = k_0 f_2 - iR_0 k(y - \tfrac{1}{2})g_1 - ikR_1(y - \tfrac{1}{2})g_0 + \lambda_0 g_2 + \lambda_1 g_1 + \lambda_2 g_0, \tag{15.23}$$

subject to the boundary conditions that each f_n and g_n $(n = 0, 1, 2)$ vanishes at both $y = 0$ and $y = 1$.

The solutions for f_0, g_0, R_0 and λ_0 are given by (15.16).

At $O(\epsilon)$ we have two free parameters multiplying the only resonant terms in the two equations, (20) and (21), and we may set them both to zero in order to ensure that there is a solution. Therefore we have $R_1 = \lambda_1 = 0$. The solutions for f_1 and g_1 are

$$f_1 = \frac{(\pi^2 + k^2)^3}{8\pi k^2} i(y - y^2)\cos\pi y + \frac{(\pi^2 + k^2)^2(k^2 + 5\pi^2)}{8\pi^2 k^2} i(y - \tfrac{1}{2})\sin\pi y, \tag{15.24}$$
$$g_1 = -\frac{(\pi^2 + k^2)^2}{8\pi k} i(y - y^2)\cos\pi y - \frac{(\pi^2 + k^2)^2}{8\pi^2 k} i(y - \tfrac{1}{2})\sin\pi y. \tag{15.25}$$

We note that the other term involving g_0 in (15.21) is multiplied by $(y-\frac{1}{2})$ which changes its symmetry. Thus this term is not resonant, although it increases substantially the analytical complexity of the algebra.

At $O(\alpha^2)$ the equations are

$$f_2'' - k^2 f_2 - R_0 k g_2 = R_0\left(ig_1' - \tfrac{1}{2}kg_0\right) + R_2 k g_0 \equiv \mathcal{R}_1 \tag{15.26}$$
$$g_2'' - k^2 g_2 - k f_2 = -iR_0 k\left(y-\tfrac{1}{2}\right)g_1 + \lambda_2 g_0 \equiv \mathcal{R}_2 \tag{15.27}$$

where these equations define $\mathcal{R}_1$ and $\mathcal{R}_2$. These equations contain resonant terms, but they also contain the parameters R_2 and $\mathrm{Im}\lambda_2$. In fact all the terms on the right hand sides of (15.26) and (15.27) are resonant and we therefore need to derive a solvability condition. The differential operators on the left hand sides are self-adjoint although we need to multiply (15.27) by R_0 to derive the solvability condition. Therefore we have

$$\begin{aligned}\int_0^1 (\mathrm{R}_1 f_0 + \mathrm{R}_2 g_0 R_0)\,dy &= \int_0^1 [f_2'' - k^2 f_2 - R_0 k g_2] f_0\,dy + \int_0^1 [g_2'' - k^2 g_2 - k f_2] g_0\,dy \\ &= \int_0^1 [f_0'' - k^2 f_0 - R_0 k g_0] f_2\,dy + \int_0^1 [g_0'' - k^2 g_0 - k f_0] g_2\,dy \\ &= 0 \qquad \text{using the definition of } f_0 \text{ and } g_0.\end{aligned} \tag{15.28}$$

Clearly the evaluation of this condition is very lengthy and the final expression becomes

$$R_2 = \frac{(\pi^2+k^2)^2}{32\pi^4 k^4}(\pi^6 + 23\pi^4 k^2 + 11\pi^2 k^4 + 5k^6) + \frac{(\pi^2+k^2)^4(\pi^2-k^2)}{96\pi^2 k^4}, \tag{15.29}$$

where $\mathrm{Im}(\lambda_2)=0$. For a given value of k this gives the $O(\alpha^2)$ correction to the value given in (15.16). Figure 15.1 shows solutions for the first four modes. We have considered only the first mode, but the value of R_2 for the n^{th} mode is given by replacing all appearances of π by $n\pi$ in (15.29).

It is clear from looking at Figure 15.1 that when α is small, the curves corresponding to the first two modes eventually coalesce as k becomes large. The value of k at which this happens becomes large as $\alpha \to 0^+$. The aim now is to try to describe this behaviour.

If we consider the small–α, large–k limits of the neutral curve, then we need to expand both (15.16) and (15.29) for large values of k. We get

$$\text{Mode 1}: \qquad R \sim (k^2 + 2\pi^2 + \ldots) + \alpha^2\left(\frac{k^6(15-\pi^2)}{96\pi^4} + \ldots\right) + O(\alpha^4) \tag{15.30}$$

$$\text{Mode 2}: \qquad R \sim (k^2 + 8\pi^2 + \ldots) + \alpha^2\left(\frac{k^6(15-4\pi^2)}{1536\pi^4} + \ldots\right) + O(\alpha^4). \tag{15.31}$$

Firstly we see that the $O(\alpha^2)$ correction is positive for the first mode, but negative for the second — this is in qualitative agreement with Figure 15.1. But we also see that the expansion is not uniformly valid for all k. In fact the leading part of the $O(\alpha^2)$ term becomes of the same order of magnitude as the second part of the $O(1)$ term when $\alpha^2 k^6 = O(1)$, i.e. when $k = O(\alpha^{-1/3})$.

We will use this information to rescale equations (15.14) and (15.15). We set

$$\alpha = \beta/k^3, \qquad R = k^2 S_0 + S_2 + \cdots, \qquad \lambda = i\sigma + O(k^{-2}), \tag{15.32$a-c$}$$

$$f = kF_0 + k^{-1}F_2 + \cdots, \qquad g = G_0 + k^{-2}G_2 + \cdots, \tag{15.32d,e}$$

in (15.14) and (15.15). The analysis is lengthy and is omitted, but at leading order we obtain

$$F_0 + S_0 G_0 = 0, \qquad F_0 + G_0 = 0, \tag{15.33}$$

from which we deduce that $S_0 = 1$. At the next order the equations are

$$F_2 + G_2 = F_0'' - S_2 G_0, \qquad F_2 + G_2 = G_0'' + i\beta(y - \tfrac{1}{2})G_0. \tag{15.34}$$

Using $G_0 = -F_0$ and equating the right hand sides of (15.34) we obtain

$$F_0'' + \tfrac{1}{2}[i(y - \tfrac{1}{2})\beta - i\sigma + S_2]F_0 \tag{15.35}$$

which is to be solved subject to $F_0(0) = F_0(1) = 0$. This is a complex version of Airy's equation and may be solved easily only when $\beta = 0$, the small-inclination limit. In this case we get

$$F_0 \propto \sin n\pi y, \qquad S_2 = 2n^2\pi^2, \tag{15.36}$$

the value of S_2 corresponds to the second term in equations (15.30) and (15.31).

Equation (15.35) was solved numerically for the first two modes and it was found that they coalesce when $S_2 \simeq 57.00$ and $\beta \simeq 196.997$. Therefore we get coalescence of the two modes when

$$R \sim k^2 + 57.00 \qquad \text{where} \qquad \alpha = 196.997/k^3 \quad \text{as } k \to \infty. \tag{15.37}$$

Alternatively we could write that

$$R \sim \frac{(196.997)^{2/3}}{\alpha^{2/3}} + 57.00 \tag{15.38}$$

as $\alpha \to 0$, at which point the wavenumber is given by

$$k \sim \frac{(196.997)^{1/3}}{\alpha^{1/3}}. \tag{15.39}$$

16. Weakly Nonlinear Stability Theory for Porous Layers

In this lecture we will concentrate on the horizontal porous layer and extend the linear stability theory which formed the focus of the last lecture into the weakly nonlinear regime. The marginal stability curve for the horizontal layer is given by

$$R = \frac{(\pi^2 + k^2)^2}{k^2} \tag{16.1}$$

and it achieves a minimum value at $k = \pi$ at which point $R = 4\pi^2$. If we expand the expression (16.1) about $k = \pi$ using $k = \pi + \epsilon$ then we obtain,

$$R \sim 4\pi^2 + 4\epsilon^2 - \frac{4\epsilon^3}{\pi} + \frac{5\epsilon^4}{\pi^2}. \tag{16.2}$$

Therefore, when the Rayleigh number is $O(\epsilon^2)$ above the critical value, there is a band of wavenumbers of width $O(\epsilon)$ about $k = \pi$ which are all unstable, or which correspond to growing disturbances to the no-flow basic state. As we will see, not all of these possible flows are stable, and an aim of weakly nonlinear theory is to determine which wavenumbers correspond to stable solutions. This is the regime in which weakly nonlinear theory is valid and it forms an asymptotic analysis as $\epsilon \to 0$. We use the method of multiple scales as part of this in order to allow small changes in the critical wavenumber.

In two dimensions the full governing equations are

$$\nabla^2\psi = R\theta_x \qquad \nabla^2\theta + \psi_x = \theta_y\psi_x - \psi_y\theta_x + \theta_t \tag{16.3a,b}$$

where we have subtracted out the basic state. Changes of $O(\epsilon)$ to the wavenumber may be accommodated by using a slow-x scale, and it is also convenient to use a slow timescale. Therefore we define

$$x = \epsilon X, \qquad t = \tfrac{1}{2}\epsilon^2\tau \tag{16.4a,b}$$

where the new t coordinate is merely a rescaling, whereas the new x coordinate is a stretched coordinate. Equations (16.3) become

$$\nabla^2\psi - R\theta_x = -2\epsilon\psi_{xX} - \epsilon^2\psi_{XX} + \epsilon R\theta_X, \tag{16.5a}$$

$$\nabla^2\theta + \psi_x = -2\epsilon\theta_{xX} - \epsilon^2\theta_{XX} + (\theta_y\psi_x - \psi_y\theta_x) + \epsilon(\theta_y\psi_X - \psi_y\theta_X) + \tfrac{1}{2}\epsilon^2\theta_\tau. \tag{16.5b}$$

The solution is expanded in a power series in ϵ:

$$\begin{pmatrix}\psi\\ \theta\\ R\end{pmatrix} = \begin{pmatrix}0\\ 0\\ R_0\end{pmatrix} + \epsilon\begin{pmatrix}\psi_1\\ \theta_1\\ R_1\end{pmatrix} + \epsilon^2\begin{pmatrix}\psi_2\\ \theta_2\\ R_2\end{pmatrix} + \cdots, \tag{16.6}$$

where we expect to get $R_1 = 0$ because of the fact that we are investigating $O(\epsilon^2)$ changes to R. We also have $R_0 = 4\pi^2$.

At $O(\epsilon)$ we obtain the equations

$$\nabla^2\psi_1 - R_0\theta_{1x} = 0, \qquad \nabla^2\theta_1 + \psi_x = 0, \tag{16.7}$$

which are precisely the linear stability equations we derived in Lecture 15. The solutions are the same, of course, but we are concentrating on the case $k = \pi$ at the bottom of the neutral curve. In line with previous cases of multiple scales theory we employ the complex form of the eigensolution

$$\psi_1 = -i\pi[Ae^{i\pi x} - \bar{A}e^{-i\pi x}]\sin\pi y, \qquad \theta_1 = \tfrac{1}{2}[Ae^{i\pi x} + \bar{A}e^{-i\pi x}], \tag{16.8}$$

where $A = A(X, \tau)$ is the complex amplitude.

At $O(\epsilon^2)$ the equations are

$$\begin{aligned}\nabla^2\psi_2 - R_0\theta_{1x} =& - 2\psi_{1xX} + R_1\theta_{1x} + R_0\theta_{1X}\\ =&R_1\theta_{1x},\end{aligned} \tag{16.9a}$$

$$\begin{aligned}\nabla^2\theta_2 + \psi_{2x} &= -2\theta_{1xX} - \psi_{1X} + \psi_{1x}\theta_{1y} - \psi_{1y}\theta_{1x}\\ &= \psi_{1x}\theta_{1y} - \psi_{1y}\theta_{1x}\\ &= \pi^3 A\bar{A}\sin 2\pi y.\end{aligned} \tag{16.9b}$$

All the terms involving X-derivatives disappear; this is a consequence of the fact that we are basing the analysis at the minimum of the neutral curve. The only term in these equations which involves a resonant term is that multiplying R_1, and hence we simply set $R_1 = 0$, without needing to invoke a more complicated solvability condition. The solution is

$$\psi_2 = 0, \qquad \theta_2 = -\tfrac{1}{4}\pi A\bar{A}\sin 2\pi y. \tag{16.10}$$

At $O(\epsilon^3)$ the equations are

$$\begin{aligned}\nabla^2\psi_3 - R_0\theta_{3x} =& - 2\psi_{2xX} - \psi_{1XX} + R_2\theta_{1x} + R_1\theta_{1X} + R_1\theta_{2x} + R_0\theta_{2X}\\ =& - \psi_{1XX} + R_2\theta_{1x} + \text{ nonresonant terms}\\ =&i\pi[A_{XX}e^{i\pi x} - \bar{A}e^{-i\pi x}]\sin\pi y\\ &+ \tfrac{1}{2}i\pi R_2[Ae^{i\pi x} - \bar{A}e^{-i\pi x}]\sin\pi y] + \text{ nonresonant terms}\end{aligned} \tag{16.11a}$$

$$\begin{aligned}\nabla^2\theta_3 + \psi_{3x} &= -2\theta_{2xX} - \theta_{1XX} - \psi_{2X} + (\psi_{2x}\theta_{1y} - \psi_{2y}\theta_{1x})\\ &\quad + (\psi_{1x}\theta_{2y} - \psi_{1y}\theta_{2x}) + (\psi_{1X}\theta_{1y} - \psi_{1y}\theta_{1X}) + \tfrac{1}{2}\theta_{1\tau}\\ &= -\theta_{1XX} + \psi_{1x}\theta_{2y} + \tfrac{1}{2}\theta_{1\tau} + \text{ nonresonant terms}\\ &= -\tfrac{1}{2}[A_{XX}e^{i\pi x} + \bar{A}_{XX}e^{-i\pi x}]\sin\pi y\\ &\quad + \tfrac{1}{4}\pi^4[A^2\bar{A}e^{i\pi x} + A\bar{A}^2e^{-i\pi x}]\,[\sin\pi y - \sin 3\pi y]\\ &\quad + \tfrac{1}{4}[A_\tau e^{i\pi x} + \bar{A}_\tau e^{-i\pi x}]\sin\pi y + \text{ nonresonant terms}\end{aligned} \tag{16.11b}$$

The right hand sides are complicated, but they fall into the same pattern as we have met before. All that is necessary now is to determine a solvability condition. In fact, if we define the right hand sides of (16.11a,b) to be R_1 and R_2, respectively, and the appropriate eigensolutions to be Ψ and Θ, then the solvability condition is that

$$\int_0^1 \int_0^2 \left[\Psi \mathcal{R}_1 + R_0 \Theta \mathcal{R}_2 \right] dx\, dy = 0. \tag{16.12}$$

Here we have integrated across the layer ($0 \leq y \leq 1$) and over one period of the basic disturbance given in (16.8). Application of this solvability condition gives the following equation for the complex amplitude,

$$A_\tau = R_2 A + 4A_{XX} - \pi^4 A^2 \bar{A}. \tag{16.13}$$

The form of this equation is such that it admits solution which are proportional to $\exp(iKX)$. Therefore, if we set

$$A = B(\tau) e^{iKX} \tag{16.14}$$

in (16.13) where B is real, we obtain

$$B_\tau = (R_2 - 4K^2) B - \pi^4 B^3 \tag{16.15}$$

for which the steady solutions are $B = 0$ (for all R_2) and $B = \pm (R_2 - 4K^2)^{1/2}/\pi^2$ (for $R_2 > 4K^2$). If we consider perturbations to the $B = 0$ solution by setting $B = \delta B$ into (16.15) and linearising, we get

$$\delta B_\tau = (R_2 - 4K^2) \delta B \tag{16.16}$$

which shows that such disturbances grow when $R_2 > 4K^2$, and decay otherwise. Thus, in this weakly nonlinear regime the neutral curve is $R_2 = 4K^2$ — compare this with (16.2) noting that the deviation from the critical wavenumber is ϵK, rather than ϵ as given in that equation. (To see this, consider the result of substituting (16.14) into (16.8b):

$$\theta_1 = \tfrac{1}{2}[B e^{i\pi x + iKX} + \bar{B} e^{-i\pi x - iKX}] \sin \pi y = B \cos(\pi + \epsilon K) x \sin \pi y. \tag{16.17}$$

Here we see that the "iKX" exponent in (16.14) is equivalent to an ϵK change in the wavenumber of the flow away from the critical value.)

Therefore, for a disturbance with wavenumber $\pi + \epsilon K$, it decays when $R_2 < 4K^2$ and hence the zero solution is recovered. But when $R_2 > 4K^2$ that disturbance grows and the zero solution is unstable. In fact the phase of this solution is entirely arbitrary since we are in an infinite layer. Hence we could replace (16.14) by $A = B(\tau) e^{i(KX + \phi)}$ where ϕ is an arbitrary phase.

But is the solution given by

$$A = \pi^{-2} (R_2 - 4K^2)^{1/2} e^{iKX} \tag{16.18}$$

stable to disturbances of other wavenumbers? This must be investigated by setting

$$A = \Big[A_0 + Be^{iLX} + Ce^{-iLX}\Big]e^{iKX} \tag{16.19}$$

into equation (16.13), where $A_0 = \pi^{-2}(R_2 - 4K^2)^{1/2}$. We must take this form of disturbance in order to obtain a self-contained disturbance. Given the form of the nonlinearity in (16.13), a disturbance with wavenumber $(K+L)$ induces a response with response with wavenumber $(K-L)$ and vice versa. We obtain the following linearised system of equations for B and C:

$$B_\tau = R_2 B - 4(K+L)^2 B - \pi^4 A_0^2 \bar{C} - 2\pi^4 A_0^2 B, \tag{16.20a}$$

$$C_\tau = R_2 C - 4(K-L)^2 C - \pi^4 A_0^2 \bar{B} - 2\pi^4 A_0^2 C, \tag{16.20b}$$

If we now set both B and $\bar{C}$ to be proportional to $\exp \lambda\tau$ then (16.20a) and the complex conjugate of (16.20b) lead to the following determinantal equation

$$\begin{aligned}
0 &= \begin{vmatrix} R_2 - 4(K+L)^2 - \lambda - 2\pi^4 A_0^2 & -\pi^4 A_0^2 \\ -\pi^4 A_0^2 & R_2 - 4(K+L)^2 - \lambda - 2\pi^4 A_0^2 \end{vmatrix} \\
&= \begin{vmatrix} R_2 - 4K^2 + 8KL + 4L^2 + \lambda & R_2 - 4K^2 \\ R_2 - 4K^2 & R_2 - 4K^2 - 8KL + 4L^2 + \lambda \end{vmatrix} \\
&= (R_2 - 4K^2 + 4L^2 + \lambda)^2 - (8KL)^2 - (R_2 - 4K^2)^2 \\
&= 2(R_2 - 4K^2)(4L^2 + \lambda) + (4L^2 + \lambda)^2 - (8KL)^2.
\end{aligned} \tag{16.21}$$

As the determinant is symmetric we may set λ to zero to determine neutral stability, although this also follows from solving (16.21) for λ in terms of R_2, K and L. Given $\lambda = 0$ we may solve (16.21) to get

$$R_2 = 12K^2 - 2L^2, \tag{16.22}$$

where, for a given value of L, values of R_2 which are less than $12K^2 - 2L^2$ give instability (i.e. $\lambda > 0$). Thus (16.22) gives the lower bound for stability of the basic flow. This region of stability is minimised as $L \to 0$, that is, for disturbances whose wavenumbers are very close to and on either side of that of the basic flow. The region of stability is, therefore,

$$R_2 \geq 12K^2 \tag{16.23}$$

which should be compared with $R_2 \geq 4K^2$ which is the region of existence of rolls. Therefore, for a given value of R_2, rolls exist with values of K lying in the range $|K| < (\frac{1}{4}R_2)^{1/2}$, whereas only those lying in the range $|K| < (\frac{1}{12}R_2)^{1/2}$ are stable.

There are many other aspects which may be considered within this type of framework. The most obvious one is to determine numerically the evolution of disturbances when the basic flow is unstable. This would be undertaken using an expansion of the form

$$A = e^{iKX} \sum_{n=-N}^{M} A_n e^{iLX}$$

where a system of first order equations for the coefficient functions $A_n(\tau)$ may be solved as an initial value problem. The most dangerous disturbance will, in general be the $e^{i(K-L)X}$ mode when both K and L are positive. But although this mode may grow, its wavenumber may not lie within the stable regime, and it, in turn, will be destabilised by the $e^{i(K-2L)X}$ mode, and so on until a stable wavenumber is achieved. This instability is named either the **sideband** or the **Eckhaus** instability.

We have not considered three-dimensional disturbances. The first analysis of this type was provided by Newell & Whitehead (1969) for the Bénard problem, and with simple numerical scalings their analysis also applies here. Such an analysis shows that there are two other instability mechanisms, the zigzag instability and the cross-roll instability. In the zigzag instability the flow is destabilised by a pair of modes which are aligned at equal and opposite $O(\epsilon^{1/2})$ angles to the basic roll pattern. This instability operates only when $K < 0$, indicating that all rolls are unstable when the wavenumber is less than the critical value. The cross-roll instability is the result of considering roll patterns at finite angles to the basic roll. For the Bénard and Darcy-Bénard problems the most dangerous disturbance is of the form of a roll of exactly the critical wavelength orientated at right angles to the original roll. Stability to this disturbance is ensured when $R_2 > 40K^2/3$, which shows that this instability mechanism is more important than the sideband instability.

Further information on this type of analysis in the context of porous layers may be found in some of my papers: Rees & Riley (1986, 1989a, 1989b, 1990), Rees (1990, 1996). A comprehensive review of Darcy-Bénard convection which includes many other modifications to Darcy's law and the effects of inertia, thermal dispersion, internal heating, layering etc. may be found in my review article Rees (2000).

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17. Higher Order Boundary Layer Theory — Matched Asymptotic Expansions

In this lecture I will develop boundary layer theory for free convection flow from an inclined heated surface and determine how a second cooled or insulated bounding surface may affect the boundary layer flow. In free convective boundary layer flows buoyancy forces drive the fluid motion directly (unless the heated surface is horizontal, in which case the flow is driven by a hydrostatically induced pressure gradient). Most boundary layer analyses consider only the leading order problem. Strictly speaking, boundary layer theory is an asymptotic theory, the asymptotically large parameter being the Darcy-Rayleigh number in the context of free convection in porous media. Therefore any results which are used at a finite fixed value of this Rayleigh number are equivalent to a one-term truncated asymptotic expansion.

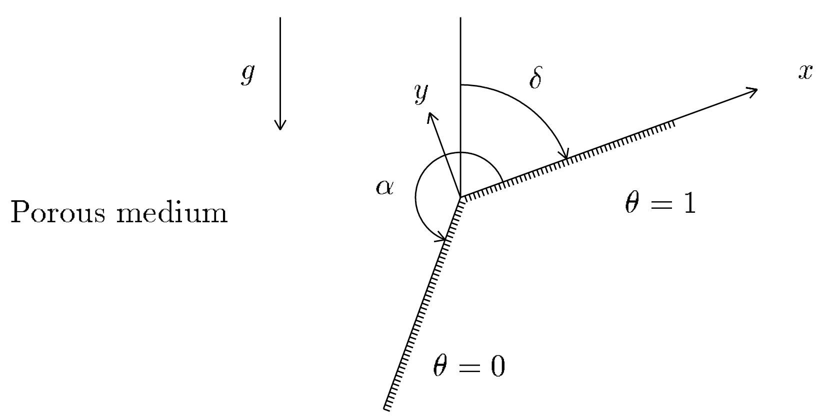

Figure 17.1. Definition of coordinate system and configuration.

Here we will consider a wedge-shaped porous medium where the heated surface is held at a constant temperature and is inclined such that its leading edge is lower than the rest of it. The second bounding surface lies at a finite angle away from the first, as shown in Figure 17.1. We will assume that Darcy's law holds and the simplest form of the steady energy equation also applies. The dimensional equations are as given in Riley & Rees (1985) but may be nondimensionalised to give

$$u_x + v_y = 0$$

$$u = -p_x + \theta\cos\delta, \qquad v = -p_y + \theta\sin\delta$$

$$\theta_{xx} + \theta_{yy} = u\theta_x + v\theta_y.$$

There is no natural lengthscale in this problem and therefore the Darcy–Rayleigh number, defined as

$$R = \frac{\rho g K \beta (T_{\max} - T_{\min}) L}{\mu\kappa},$$

has been set equal to 1, and this defines a natural lengthscale, L, in terms of all the other constants which are known properties: ρ is a reference density; g is gravity; K is permeability; β is the coefficient of cubical expansion of the fluid; $(T_{\max} - T_{\min})$ is the reference temperature drop in the system; μ is viscosity; and κ is thermal diffusivity.

Given that we have set the Rayleigh number equal to 1, we will therefore perform an asymptotic analysis for large values of x, the coordinate along the heated surface. The process is identical to that using $R \gg 1$ and $x = O(1)$, and the results may easily be converted from one form to the other.

Our intent here is to determine the leading order boundary layer flow and to find out how this causes a flow in the rest of the porous medium. The outer flow, in turn, affects the boundary layer at higher orders, and so on. Unfortunately this process cannot be continued indefinitely because eventually, at third order in fact, the homogeneous form of the boundary layer equations admits eigensolutions. This means that the third order equations cannot be solved unless an associated logarithmic term is introduced into the expansion, and even then those equations do not have a unique solution. Further terms may, of course, be computed, but their usefulness is in doubt due to the unknown amplitude of the third order eigensolution.

Leading order solution.

We introduce the streamfunction, ψ, such that $u = \psi_y$ and $v = -\psi_x$, which satisfies the equation of continuity. The remaining equations become

$$\nabla^2\psi = \theta_y \cos\delta - \theta_x \sin\delta, \qquad \nabla^2\theta = \psi_y\theta_x - \psi_x\theta_y, \tag{17.1}$$

subject to the boundary conditions, $\psi = 0$, $\theta = 1$ on $y = 0$ ($x \geq 0$), and $\psi = 0$, $\theta = 0$ on the other surface. We may, alternatively, set the normal derivative of θ to be zero on that surface.

We now invoke the boundary layer approximation by concentrating on that regime where $x \gg y$. At leading order equations (17.1) may be approximated by

$$\psi_{yy} = \theta_y \cos\delta, \qquad \theta_{yy} = \psi_y\theta_x - \psi_x\theta_y, \tag{17.2}$$

which may be reduced to ordinary differential (or **self–similar**) form using

$$\psi = x^{1/2} f(\eta) \qquad \theta = g(\eta), \qquad (\eta = y/x^{1/2}) \tag{17.3}$$

where f and g satisfy the equations,

$$f'' = g' \cos\delta, \qquad g'' + \tfrac{1}{2} f g' = 0, \tag{17.4}$$

subject to $f(0) = 0$, $g(0) = 1$ and $f', g \to 0$ as $\eta \to \infty$. These equations (with $\delta = 0$) were solved first by Cheng & Minkowycz (1977). It is possible to show that $g'(0) = -0.444(\cos\delta)^{1/2}$.

Leading order outer solution.

Now we consider how this flow affects the region outside the thin boundary layer. As we come out of the boundary, that is, as $\eta \to \infty$, f tends towards $1.616(\cos\delta)^{1/2}$, which we will call a_0. The boundary layer is thin, and from the point of view of the porous wedge, the streamfunction takes the value $a_0 x^{1/2}$ on the x-axis and 0 on the other surface. The temperature field, on the other hand, decays exponentially and becomes proportional to $\exp[-\frac{1}{2}a_0\eta]$, and therefore we may neglect the temperature in this outer regime. As the domain is wedge-shaped, it is best to solve the momentum equation in polar coordinates given by

$$x = r\cos\phi \qquad y = r\sin\phi. \tag{17.5}$$

Therefore we need to solve

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} = 0 \tag{17.6}$$

subject to $\psi = a_0 r^{1/2}$ on $\phi = 0$ and $\psi = 0$ on $\phi = \alpha$. This is done easily by means of the substitution

$$\psi = r^{1/2} F(\phi) \tag{17.7}$$

where $F'' + \frac{1}{4}F = 0$ with $F(0) = a_0$ and $F(\alpha) = 0$. The solution is

$$F = \frac{\sin\frac{1}{2}(\alpha - \phi)}{\sin\frac{1}{2}\alpha} a_0. \tag{17.8}$$

This solution is finite unless $\alpha = 0$ (which is nonsense, since it implies that the wedge has zero angle), or when $\alpha = 2\pi$. We will continue by assuming that $\alpha \neq 2\pi$, although the resolution of this difficulty lies in introducing a logarithmic term.

- You may like to attempt this exercise: when $\alpha = 2\pi$, let $\psi = r^{1/2}[\tilde{F}(\phi)\ln r + F(\phi)]$ and show that $\tilde{F} = (a_0/2\pi)\sin\frac{1}{2}\phi$.

Second inner solution.

In the outer region we have

$$\begin{aligned}
\psi &= r^{1/2}\frac{\sin\frac{1}{2}(\alpha - \phi)}{\sin\frac{1}{2}\alpha} a_0 \\
&= r^{1/2}\frac{(\sin\frac{1}{2}\alpha\cos\frac{1}{2}\phi - \cos\frac{1}{2}\alpha\sin\frac{1}{2}\phi)}{\sin\frac{1}{2}\alpha} a_0 \\
&= a_0 r^{1/2}\cos\tfrac{1}{2}\phi - a_0\cot\tfrac{1}{2}\alpha r^{1/2}\sin\tfrac{1}{2}\phi \\
&\sim a_0 r^{1/2} - (\tfrac{1}{2}a_0\cot\tfrac{1}{2}\alpha) r^{1/2}\phi \qquad \text{for small } \phi
\end{aligned} \tag{17.9}$$

It is clearly quite a complicated matter trying to match between a similarity variable and polar coordinates. However,

$$\eta = y/x^{1/2} = r^{1/2}\sin\phi/(\cos\phi)^{1/2} \sim r^{1/2}\phi$$

for small values of ϕ, and this implies that the next term we need in the boundary layer must be $O(1)$. Therefore we take

$$\psi = x^{1/2}f_0(\eta) + f_1(\eta) + \cdots, \qquad \theta = g_0(\eta(+x^{-1/2}g_1(\eta) + \cdots.$$

The functions f_0 and g_0 satisfy equations (17.4). But f_1 and g_1 satisfy

$$f_1'' = g_1'\cos\delta + \tfrac{1}{2}\eta g_0'\sin\delta, \qquad g_1'' + \tfrac{1}{2}(f_0 g_1' + f_0' g_1) = 0. \tag{17.10a,b}$$

Equation (17.10b) may be integrated once and the constant of integration set to zero. The resulting linear equation may also be solved analytically and shown to be zero. The solution to equation (17.10a) may be written in analytical form, but it involves an integral. From its form, however, we can see that it must satisfy

$$f_1 \sim a_1 + b_1\eta \qquad \text{as} \quad \eta\to\infty. \tag{17.11}$$

Therefore, for large values of η, ψ satisfies

$$\begin{aligned}\psi &\sim x^{1/2}a_0 + a_1 + b_1\eta \\ &= (r\cos\phi)^{1/2}a_0 + a_1 + b_1\Big(\frac{r\cos\phi}{r^{1/2}\cos^{1/2}\phi}\Big) \\ \text{for small } \phi \qquad &= r^{1/2}(a_0 + b_1\phi) + a_1 + \cdots \end{aligned} \tag{17.12}$$

If we now compare (17.12) with (17.9) we obtain the matching condition

$$b_1 = -\tfrac{1}{2}a_0 \cot \tfrac{1}{2}\alpha. \tag{17.13}$$

Therefore we are able to solve (17.10a) for f_1 using the boundary condition $f_1(0) = 0$ and the matching condition, $f'(\infty) = b_1$. This numerical solution will give us the value of a_1 in (17.11), which, in turn, provides us with a matching condition for the next outer solution.

Second outer solution

In the outer region we set

$$\psi = r^{1/2}F_0(\phi) + F_1(\phi) + \cdots, \tag{17.14}$$

where $F_1'' = 0$ subject to $F_1(0) = a_1$ and $F_1(\alpha) = 0$. Hence

$$F_1 = a_1\Big(1 - \frac{\phi}{\alpha}\Big). \tag{17.15}$$

Third inner solution.

Now we proceed to the next inner term. We continue downwards in powers of $x^{-1/2}$: setting

$$\psi = x^{1/2}f_0 + f_1 + x^{-1/2}f_2 \qquad \theta = g_0 + x^{-1/2}g_1 + x^{-1}g_2 \tag{17.16}$$

in (17.1) yields

$$f_2'' = g_2' + \tfrac{1}{4}(f_0 - \eta f_0' - \eta^2 f_0'') \tag{17.17a}$$

$$g_2'' + \tfrac{1}{2}f_0 g_2' + f_0' g_2 - \tfrac{1}{2}g_0' f_2 = -\tfrac{3}{4}\eta g_0' - \tfrac{1}{4}\eta^2 g_0''. \tag{17.17b}$$

However the inhomogeneous form of these equations admits the eigensolutions

$$f_2^* = A(\eta f_0' - f_0) \qquad g_2^* = A\eta g_0' \tag{17.18}$$

where A is an arbitrary constant. Worse than that is the fact that the right hand sides are resonant. In this situation we invoke logarithms. Therefore we must modify our naive expansion (17.16) by setting

$$\psi = x^{1/2}f_0 + f_1 + x^{-1/2}\ln x\,\bar{f}_2 + x^{-1/2}f_2 \qquad \theta = g_0 + x^{-1/2}g_1 + x^{-1}\ln x\,\bar{g}_2 + x^{-1}g_2 \tag{17.19}$$

into equations (17.1). We get

$$\bar{f}_2'' = \bar{g}_2' \tag{17.20a}$$

$$\bar{g}_2'' + \tfrac{1}{2} f_0 \bar{g}_2' + f_0' \bar{g}_2 - \tfrac{1}{2} g_0' \bar{f}_2 = 0, \tag{17.20b}$$

and

$$f_2'' = g_2' + \tfrac{1}{4}(f_0 - \eta f_0' - \eta^2 f_0'') \tag{17.21a}$$

$$g_2'' + \tfrac{1}{2} f_0 g_2' + f_0' g_2 - \tfrac{1}{2} g_0' f_2 = -\tfrac{3}{4}\eta g_0' - \tfrac{1}{4}\eta^2 g_0'' + f_0' \bar{g}_2 - g_0' \bar{f}. \tag{17.21b}$$

Given the inhomogeneous terms of (17.21a) we expect that

$$f_2 \sim a_2 + b_2 \eta + \tfrac{1}{8} a_0 \eta^2$$

for large values of η. Matching with the solution for F_1 gives the matching condition,

$$b_2 = -\frac{a_1}{\alpha}. \tag{17.22}$$

If we take (17.18) to be the solution of the homogeneous equations (17.20), then straightforward numerical solution of (17.21) subject to the conditions $f_2(0) = 0$, $g_2(0) = 0$, $g_2'(0) =$any constant, $\lim_{\eta\to\infty} g_2 = 0$ and $\lim_{\eta\to\infty}(f_2' - \frac{1}{4}a_0\eta) = b_2$, will give us the value of A. Of course the solution for f_2 is not unique as it contains an arbitrary multiple of the eigensolution given in (17.18). Therefore the rate of heat transfer may only be computed to the third term:

$$\begin{aligned}\theta_y(y=0) &= x^{-1/2}[g_0'(0) + x^{-1/2} g_1'(0) + x^{-1}\ln x\, \bar{g}_2'(0) + \cdots] \\ &= -0.444(\cos\delta)^{1/2}[1 + Ax^{-1}\ln x + O(x^{-1})].\end{aligned} \tag{17.23}$$

Reference

D.S.Riley & D.A.S.Rees (1985) "Non-Darcy natural convection from arbitrarily inclined heated surfaces in saturated porous media" *Quarterly Journal of Mechanics and Applied Mathematics* **38**, 277–295.

18. Darcy-Brinkman Free Convection from a Horizontal Surface

In this lecture we consider a horizontal free convective boundary layer flow in a porous medium, but we also introduce what are known as "boundary effects". Modelled by the Brinkman term, these effects allow the no-slip conditions to be satisfied at an impermeable boundary of a porous medium. Without them, we may only impose a zero normal velocity condition. Such boundary effects are viscous in nature and arise because the microscopic flow field must satisfy the standard no-slip conditions. Given the relatively dense packing of most porous media it is to be expected that the fluid loses all knowledge of the presence of a boundary within a fairly small number of characteristic particle or pore lengths. Therefore this extra "viscous" term will be multiplied by a small parameter, and we expect a singular perturbation analysis to appear somewhere within our study.

We will see that the resulting flow may be viewed as modifying the original Chang & Cheng (1976) boundary layer flow where boundary effects were not included. The new Brinkman term causes the otherwise self-similar boundary layer flow to split into two asymptotic regions: the outer region comprises the classical boundary layer flow of Chang & Cheng, at least to leading order, while the inner region, which is of uniform thickness, is where the Brinkman term dominates. Unusually for a thermal boundary layer flow, there is a similar splitting into two near the leading edge. The inner region is where the thermal boundary layer resides, while the outer regime, of constant thickness, is where the momentum boundary layer adjusts to the far field conditions.

Equations of motion and boundary layer analysis

We assume that the flow is two–dimensional and steady. The nondimensional equations are

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0, \tag{18.1a}$$

$$\hat{u} = -\frac{\partial \hat{p}}{\partial \hat{x}} + D\nabla^2 \hat{u}, \qquad \hat{v} = -\frac{\partial \hat{p}}{\partial \hat{y}} + D\nabla^2 \hat{v} + R\hat{\theta}, \tag{18.1b,c}$$

$$\hat{u}\frac{\partial \hat{\theta}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{\theta}}{\partial \hat{y}} = \nabla^2 \hat{\theta}. \tag{18.1d}$$

Here $\hat{x}$ and $\hat{y}$ are Cartesian coordinates oriented along and perpendicular to the heated surface (with $\hat{x} = 0$ corresponding to the leading edge), $\hat{u}$ and $\hat{v}$ are the corresponding fluid flux velocities, $\hat{p}$ is the pressure and $\hat{\theta}$ is the temperature of the saturated medium.

On the heated surface, which is situated at $\hat{y} = 0$ and $\hat{x} \geq 0$, the temperature is $\hat{\theta} = 1$, whereas $\hat{\theta} \to 0$ far from the surface.

The two nondimensional parameters which appear in equation (18.1) are the Darcy and Rayleigh numbers:

$$D = \frac{K}{L^2\phi}, \qquad \text{and} \qquad R = \frac{\rho\hat{g}\beta K L \Delta T}{\mu\kappa}. \tag{18.2}$$

where K is the permeability; L is a macroscopic lengthscale; ϕ is the porosity; ρ sa reference fluid density; $\hat{g}$ is gravitational acceleration; β is the volumetric expansion coefficient; ΔT is the temperature drop between the heated surface and the ambient fluid; μ is the viscosity; and κ is the thermal diffusivity. The value of D is typically very small.

A streamfunction may be introduced in the usual way: $\hat{u} = \hat{\psi}_{\hat{y}}$, $\hat{v} = -\hat{\psi}_{\hat{x}}$, and we obtain the following pair of equations,

$$\nabla^2\hat{\psi} = -R\hat{\theta}_{\hat{x}} + D\nabla^4\hat{\psi}, \qquad \nabla^2\hat{\theta} = \hat{\psi}_{\hat{y}}\hat{\theta}_{\hat{x}} - \hat{\psi}_{\hat{x}}\hat{\theta}_{\hat{y}}, \tag{18.3a, b}$$

which are to be solved subject to the boundary conditions,

$$\hat{\psi} = \hat{\psi}_{\hat{y}} = 0, \quad \hat{\theta} = 1, \quad \text{on} \quad \hat{y} = 0, \quad (\hat{x} \geq 0), \qquad \psi_{\hat{y}},\, \psi_{\hat{y}\hat{y}},\, \hat{\theta} \to 0 \quad \text{as} \quad \hat{y} \to \infty. \tag{18.4}$$

Both R and D appear in equations (18.3), and although it is possible to eliminate both parameters at this stage by means of suitable rescalings, it is preferable to set the following analysis into the context of the well–known Darcy–flow study of Cheng and Chang (1976) where boundary frictional effects were assumed to be absent. Thus we use the traditional scalings for horizontal free convection boundary–layer flow in a porous medium and set

$$\hat{\psi} = R^{1/3}\bar{\psi}, \qquad \hat{\theta} = \bar{\theta}, \qquad \hat{x} = \bar{x}, \qquad \hat{y} = R^{-1/3}\bar{y}, \tag{18.5}$$

where we also take R to be asymptotically large. Substitution of equation (18.5) into equation (18.3) yields the following equations

$$\bar{\psi}_{\bar{y}\bar{y}} = -\bar{\theta}_{\bar{x}} + \alpha\bar{\psi}_{\bar{y}\bar{y}\bar{y}\bar{y}}, \qquad \bar{\theta}_{\bar{y}\bar{y}} = \bar{\psi}_{\bar{y}}\bar{\theta}_{\bar{x}} - \bar{\psi}_{\bar{x}}\bar{\theta}_{\bar{y}}, \tag{18.6a, b}$$

at leading order in R. Here we have α defined as

$$\alpha = DR^{2/3}, \tag{18.7}$$

and if we can assume at this stage that this quantity is of $O(1)$ as $R \to \infty$, then it implies that $D \ll 1$ in magnitude, which is physically realistic.

When $\alpha = 0$ in equation (18.6a), equations (18.6a) and (18.6b) comprise the boundary–layer equations for Darcy free convection. But the limit $\alpha \to 0$ gives us a singular perturbation problem. In this case we can rescale again and eliminate α from the equations. This is done by substituting

$$\bar{\psi} = \alpha^{1/4}\psi, \qquad \bar{\theta} = \theta, \qquad \bar{x} = \alpha^{3/4}x, \qquad \bar{y} = \alpha^{1/2}y, \tag{18.8}$$

into equation (18.6), and therefore we obtain

$$\psi_{yy} = -\theta_x + \psi_{yyyy}, \qquad \theta_{yy} = \psi_y\theta_x - \psi_x\theta_y. \tag{18.9a, b}$$

Finally, before the boundary–layer analysis is undertaken, it is essential to check that the boundary–layer approximation has not been violated by imposing the second transformation, (18.8). Thus, we require $\hat{x} \gg \hat{y}$ for the approximation to be valid, and this implies that $R \gg O(D^{-1/2})$.

Asymptotic analysis for small values of x

A straightforward scale analysis using $\psi = x^a F(\zeta)$ and $\theta = G(\zeta)$ where $\zeta = y/x^b$ where a and b are to be found, shows that the appropriate similarity form close to the leading edge is obtained by using

$$\psi = x^{3/5} F(\zeta, X), \qquad \theta = G(\zeta, X), \qquad \zeta = y/x^{2/5}, \qquad X = x^{2/5}, \tag{18.10}$$

Hence equations (18.9) become

$$F_{\zeta\zeta\zeta\zeta} + \tfrac{2}{5}\zeta G_\zeta - X^2 F_{\zeta\zeta} = \tfrac{2}{5} X G_X, \tag{18.11a}$$

$$G_{\zeta\zeta} + \tfrac{3}{5} F G_\zeta = \tfrac{2}{5} X (F_\zeta G_X - F_X G_\zeta). \tag{18.11b}$$

The boundary conditions at $\zeta = 0$ are that $F = F_\zeta = 0$ and $G = 1$, and we expect that $G \to 0$ as $\zeta \to \infty$.

The surprise here is that the ODEs obtained by setting $X = 0$ in equations (18.11) cannot be solved using the boundary conditions, $F_\zeta, F_{\zeta\zeta} \to 0$ as $\zeta \to \infty$, which correspond to the large–$\hat{y}$ conditions on ψ which are given in equation (18.4). However, a solution is easily found numerically if the large–ζ conditions are replaced by $F_{\zeta\zeta}, F_{\zeta\zeta\zeta} \to 0$. In this case we find that F varies linearly at sufficiently large values of ζ, rather than being a constant. Thus the requirement that $F_\zeta \to 0$ is violated. We will see, however, that this is accounted for by invoking the presence of an outer, 'momentum–adjustment' layer.

This outer layer corresponds to a region where the temperature field is exponentially small. That this is true may be seen from equation (18.11b) where, if $F_\zeta \to a_0$, a constant, as $\zeta \to \infty$, we have $G \propto \exp(-\frac{3}{10} a_0 \zeta^2)$. Therefore the outer layer requires us to balance the magnitudes of the two terms involving F in (18.11a).

It is now clear that the outer layer must correspond to where $y = O(1)$ as $x \to 0$ and we can introduce the following transformation for ψ in the outer layer:

$$\psi = x^{1/5} \mathcal{F}(y, x). \tag{18.12}$$

Equation (18.9a) reduces to

$$\mathcal{F}_{yyyy} - \mathcal{F}_{yy} = 0, \tag{18.13}$$

where the matching conditions at $\zeta = 0$ and the boundary conditions as $\zeta \to \infty$ are

$$\mathcal{F} = 0, \quad \mathcal{F}_y = a_0, \quad \text{at} \quad y = 0, \qquad \mathcal{F}_y,\, \mathcal{F}_{yy} \to 0 \quad \text{as} \quad y \to \infty. \tag{18.14}$$

The solution for $\mathcal{F}$ is

$$\mathcal{F} = a_0 (1 - e^{-y}). \tag{18.15}$$

The solution (18.15) is the leading term in the outer layer solution for small values of x. It is possible to proceed to quite a large number of terms in both the outer and main layers with the application of suitable matching conditions between the layers. To summarise the result of this process, which is quite straightforward, we can expand F and G in (18.11) and $\mathcal{F}$ in (18.13) using the following series,

$$(F,G)=\sum_{n=0}^{\infty} X^n\big(F_n(\zeta),G_n(\zeta)\big), \qquad \text{and} \qquad \mathcal{F}=\sum_{n=0}^{\infty} X^n \mathcal{F}_n(y). \tag{18.16a,b}$$

The equations for the various coefficient functions are

$$\begin{aligned} F_0'''' + \tfrac{2}{5}\zeta G_0' &= 0,\\ G_0'' + \tfrac{3}{5}F_0 G_0' &= 0, \end{aligned} \tag{18.17}$$

$$\begin{aligned} F_1'''' + \tfrac{2}{5}(\zeta G_1' - G_1) &= 0,\\ G_1'' + \tfrac{3}{5}(F_0 G_1' + F_1 G_0') &= \tfrac{2}{5}(F_0' G_1 - F_1 G_0'), \end{aligned} \tag{18.18}$$

$$\begin{aligned} F_n'''' + \tfrac{2}{5}(\zeta G_n' - nG_n) &= F_{n-2}'',\\ G_n'' + \tfrac{3}{5}\sum_{i=0}^{n} F_i G_{n-i}' &= \tfrac{2}{5}\sum_{i=1}^{n}(F_{n-i}' G_i - F_i G_{n-i}'), \end{aligned} \qquad n=2,3,\ldots, \tag{18.19}$$

and

$$\mathcal{F}_n'''' - \mathcal{F}_n'' = 0, \qquad n=0,1,\ldots\,. \tag{18.20}$$

The large–ζ behavior of the F_n functions in the inner layer are

$$F_0 \sim (a_0\zeta + b_0) \tag{18.21a}$$

$$F_1 \sim -\tfrac{1}{2!}a_0\zeta^2 + (a_1\zeta + b_1), \tag{18.21b}$$

$$F_2 \sim \tfrac{1}{3!}a_0 - \tfrac{1}{2!}a_1\zeta^2 + (a_2\zeta + b_2), \tag{18.21c}$$

$$F_3 \sim -\tfrac{1}{4!}a_0 + \tfrac{1}{3!}a_1 - \tfrac{1}{2!}a_2\zeta^2 + (a_3\zeta + b_3), \tag{18.21d}$$

and so on, with the constants a_n and b_n being found when solving the appropriate equations for F_n and G_n.

The outer solutions are, simply,

$$\mathcal{F}_0 = a_0(1-e^{-y}), \qquad \mathcal{F}_n = b_{n-1} + a_n(1-e^{-y}), \quad n=1,2,\ldots\,. \tag{18.22}$$

The values of a_n and b_n which appear in (18.21) and (18.22) are given below in Table 1 for the first six terms in the series solution. They were calculated using a fourth order Runge–Kutta scheme linked to the shooting method, where ζ ranged between 0 and 20 with 1600 equally spaced intervals; these solutions are accurate as shown in the Table, and comparisons were made with coarser grids and larger values of $\zeta_{\max}$. The inner solutions obtained are presented in Figure 18.1 in terms of the various temperature profiles. The surface shear stress and rate of heat transfer may be found by using,

$$\frac{\partial^2 F}{\partial\zeta^2}\bigg|_{\zeta=0} = \sum_{n=0} X^n F_n''(0), \qquad \frac{\partial G}{\partial\zeta}\bigg|_{\zeta=0} = \sum_{n=0} X^n G_n'(0), \tag{18.23a}$$

together with the data in the final two columns of Table 18.1. The values of the surface shear stress and rate of heat transfer given in (18.23a) are plotted in Figure 18.2 together with the results of the full numerical simulation obtained using the Keller-box method. This information is presented in two forms:

(i)
$$F_{\zeta\zeta}|_{\zeta=0}, \quad -G_{\zeta}|_{\zeta=0} \quad (x \le 1)$$
and
$$x^{-4/5} f_{\eta\eta}|_{\eta=0}, \quad -x^{-4/15} g_{\eta}|_{\eta=0} \quad (x \ge 1) \tag{18.23b}$$

which is suitable for small values of x where the quantities are non–zero at $x = 0$, and

(ii)
$$x^{2/15} F_{\zeta\zeta}|_{\zeta=0}, \quad -x^{4/15} G_{\zeta}|_{\zeta=0} \quad (x \le 1)$$
and
$$x^{-2/3} f_{\eta\eta}|_{\eta=0}, \quad -g_{\eta}|_{\eta=0} \quad (x \ge 1) \tag{18.23c}$$

n	a_n	b_n	$F''_n(0)$	$G'_n(0)$
0	1.1488	-1.0392	0.97534	-0.45619
1	1.7233	-2.1230	-0.15296	1.0291×10^{-1}
2	2.4621	-3.2583	6.5140×10^{-4}	-3.2358×10^{-3}
3	3.3729	-4.5410	2.1878×10^{-2}	-3.6892×10^{-4}
4	4.5293	-6.1284	1.4068×10^{-2}	-5.9935×10^{-4}
5	6.0474	-8.2238	5.5784×10^{-4}	-9.4625×10^{-4}

Table 18.1. Values of a_n, b_n, $F''_n(0)$ and $G'_n(0)$.

Asymptotic analysis for large values of x

Far from the leading edge we revert to the Darcy–flow similarity form and use

$$\psi = x^{1/3} f(\eta, \xi), \qquad \theta = g(\eta, \xi), \qquad \eta = y/x^{2/3}, \qquad \xi = x^{4/3} \tag{18.24a, b, c, d}$$

in equations (18.9) to obtain

$$f_{\eta\eta} - \tfrac{2}{3}\eta g_{\eta} - \xi^{-1} f_{\eta\eta\eta\eta} = -\tfrac{4}{3}\xi g_{\xi}, \qquad g_{\eta\eta} + \tfrac{1}{3} f g_{\eta} = \tfrac{4}{3}\xi(f_{\eta} g_{\xi} - f_{\xi} g_{\eta}), \tag{18.25a, b}$$

subject to

$$f = 0, \quad f_{\eta} = 0, \quad g = 1 \quad \text{at} \quad \eta = 0 \qquad \text{and} \qquad f_{\eta},\, f_{\eta\eta},\, g \to 0 \quad \text{as} \quad \eta \to \infty. \tag{18.26}$$

We expect this similarity form because the Brinkman term (ψ_{yyyy}) and the Darcy term (ψ_{yy}) only balance when $y = O(1)$. At large distances from the leading edge the main boundary layer will be very thick compared with $y = O(1)$, and hence we neglect the fourth derivative term, at least at first!

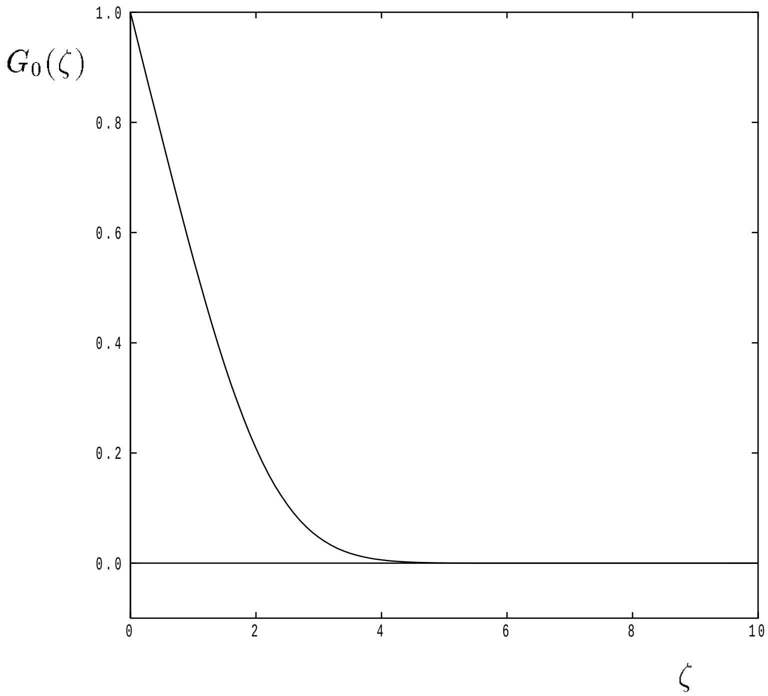

Figure 18.1a. $G_0(\zeta)$ as obtained by solving Eq. (18.17).

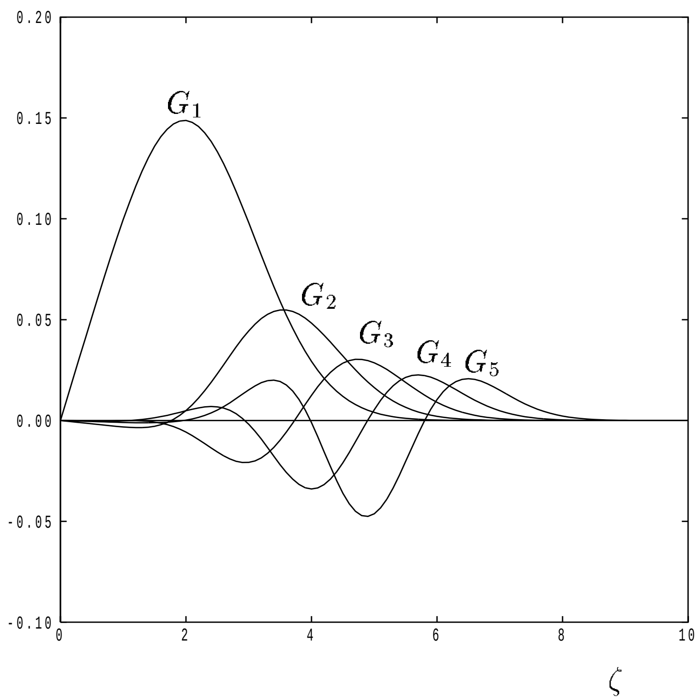

Figure 18.1b. $G_n(\zeta)$ $(n = 1, \ldots, 5)$ as obtained by solving Eqs. (18.18) and (18.19).

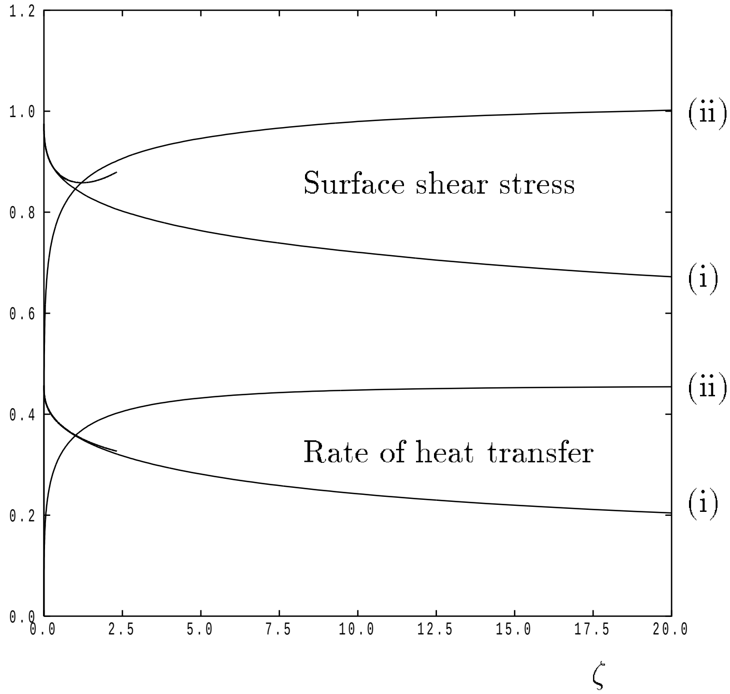

Figure 18.2. Surface shear stress and rate of heat transfer given by (i) Eqs. (18.23b) and (ii) Eqs. (18.23c). Also shown are the corresponding results obtained from the small–x asymptotic analysis.

A straightforward expansion of equations (18.25) in an inverse power series in ξ shows that the fourth derivative term in equation (18.25a) is negligible at leading order, and therefore the large–ξ analysis is a singular perturbation problem with two layers appearing again. Therefore we need to install a relatively thin layer near the surface where η is very small. This inner layer must have $y = O(1)$ in order to balance the momentum terms in equation (18.25a). Therefore we rewrite equations (18.25) using the transformations,

$$f(\eta,\xi) = \mathcal{F}(y,\xi), \qquad g(\eta,\xi) = \mathcal{G}(y,\xi), \tag{18.27}$$

and hence we obtain

$$\mathcal{F}_{yy} - \mathcal{F}_{yyyy} + \tfrac{4}{3}\mathcal{G}_\xi = 0, \qquad \mathcal{G}_{yy} + \tfrac{1}{3}\xi^{-1/2}\mathcal{F}\mathcal{G}_y = \tfrac{4}{3}\xi^{1/2}(\mathcal{F}_y\mathcal{G}_\xi - \mathcal{F}_\xi\mathcal{G}_y). \qquad (18.28a,b)$$

The presence of the $\xi^{1/2}$ terms in equations (18.28) suggests that the solutions of equations (18.25) and (18.28) proceed in inverse powers of $\xi^{1/2}$. However, the homogeneous form of the $O(\xi^{-1/2})$ equations in the expansion of equations (18.25) admits eigensolutions and are insoluble. Therefore logarithmic terms must be introduced: let

$$f = f_0(\eta) + f_{1L}(\eta)\xi^{-1/2}\ln\xi + f_1(\eta)\xi^{-1/2} + \dots, \qquad (18.29a)$$

$$g = g_0(\eta) + g_{1L}(\eta)\xi^{-1/2}\ln\xi + g_1(\eta)\xi^{-1/2} + \dots, \qquad (18.29b)$$

in equations (18.25) where

$$
\begin{aligned}
f_0'' - \tfrac{2}{3}\eta g_0' &= 0, \\
g_0'' + \tfrac{1}{3} f_0 g_0' &= 0,
\end{aligned}
\tag{18.30}
$$

$$
\begin{aligned}
f_{1L}'' - \tfrac{2}{3}(\eta g_{1L}' + g_{1L}) &= 0, \\
g_{1L}'' + \tfrac{1}{3}(f_0 g_{1L}' - f_{1L} g_0' + 2 f_0' g_{1L}) &= 0,
\end{aligned}
\tag{18.31}
$$

$$
\begin{aligned}
f_1'' - \tfrac{2}{3}(\eta g_1' + g_1) &= \tfrac{4}{3} g_{1L}, \\
g_1'' + \tfrac{1}{3}(f_0 g_1' - f_1 g_0' + 2 f_0' g_1) &= \tfrac{4}{3}(f_0' g_{1L} - f_{1L} g_0').
\end{aligned}
\tag{18.32}
$$

In the inner layer the expansion begins as follows,

$$
\mathcal{F} = \mathcal{F}_0(y) + \mathcal{F}_1(y)\xi^{-1/2} + \mathcal{F}_{2L}\xi^{-1}\ln\xi + \mathcal{F}_2\xi^{-1} + \ldots, \tag{18.33a}
$$

$$
\mathcal{G} = \mathcal{G}_0(y) + \mathcal{G}_1(y)\xi^{-1/2} + \mathcal{G}_{2L}\xi^{-1}\ln\xi + \mathcal{G}_2\xi^{-1} + \ldots, \tag{18.33b}
$$

and the boundary conditions are that

$$
\mathcal{F}_0(0) = \mathcal{F}_0'(0) = 0, \quad \mathcal{G}_0(0) = 1, \tag{18.34a}
$$

and

$$
\mathcal{F}_n(0) = \mathcal{F}_n'(0) = \mathcal{G}_n(0) = 0 \quad \text{for} \quad n = 1, 2, \ldots \tag{18.34b}
$$

with suitable matching conditions between the layers as $y \to \infty$ and $\eta \to 0$.

The solution of equations (18.30) form the solution first presented by Cheng and Chang (1976). The small–η behavior of f_0 and g_0 must be examined to provide matching conditions for the inner–layer solutions:

$$
f_0(\eta) = a\eta + \tfrac{1}{9} b\eta^3 + \ldots, \qquad g_0(\eta) = 1 + b\eta - \tfrac{1}{18} ab\eta^3 + \ldots, \tag{18.35}
$$

where $a = 1.055748$ and $b = -0.430213$ to six decimal places. Therefore we deduce that the first two terms in the inner–layer solution are

$$
\mathcal{F}_0(y) = 0, \qquad \mathcal{G}_0(y) = 1, \tag{18.36}
$$

and

$$
\mathcal{F}_1(y) = a(e^{-y} - 1 + y), \qquad \mathcal{G}_1(y) = by. \tag{18.37a,b}
$$

The "-1" term in equation (18.37a) provides a forcing term for the $O(\xi^{-1/2})$ outer layer solution via the matching conditions. Therefore equations (18.31) and (18.32) must be solved subject to the boundary conditions,

$$
f_{1L}(0) = 0, \quad g_{1L}(0) = 0, \qquad f_{1L}', g_{1L} \to 0 \quad \text{as} \quad \eta \to \infty, \tag{18.38}
$$

$$
f_1(0) = -a, \quad g_1(0) = 0, \qquad f_1', g_1 \to 0 \quad \text{as} \quad \eta \to \infty. \tag{18.39}
$$

The solution of equations (18.31) subject to (18.38) is the eigensolution

$$f_{1L} = \lambda(\eta f_0' - f_0), \qquad g_{1L} = \lambda \eta g_0', \tag{18.40}$$

where λ is presently unknown, but its value is determined by insisting that equations (18.32) have a solution. The solution to equations (18.32) may be written in the form,

$$f_1 = \sigma(\eta f_0' - f_0) - f_0' + b f_{1\dagger}, \qquad g_1 = \sigma \eta g_0' - g_0' + b g_{1\dagger}, \tag{18.41}$$

where $f_{1\dagger}$ and $g_{1\dagger}$ satisfy the equations,

$$f_{1\dagger}'' - \tfrac{2}{3}(\eta g_{1\dagger}' + g_{1\dagger}) = \tfrac{4}{3}\lambda \eta g_0', \tag{18.42a}$$

$$g_{1\dagger}'' + \tfrac{1}{3}(f_0 g_{1\dagger}' - f_{1\dagger} g_0' + 2 f_0' g_{1\dagger}) = \tfrac{4}{3}\lambda f_0 g_0', \tag{18.42b}$$

subject to the boundary conditions,

$$f_{1\dagger}(0) = 0, \quad g_{1\dagger}(0) = 1 \qquad \text{and} \qquad f_{1\dagger}, g_{1\dagger} \to 0 \quad \text{as} \quad \eta \to \infty. \tag{18.43}$$

We note that equations (18.42) cannot be solved with λ set to zero, i.e. in the absence of logarthmic terms in (18.29). The derivation of a solvability condition is not straightforward for such a problem with variable coefficients. Therefore we have opted for a direct numerical approach. As the complementary function part of the solution to (18.42) has arbitrary amplitude, we fix that amplitude by applying an extra normalising boundary condition, $g_{1\dagger}'(0) = 0$. This fifth boundary condition for equations (18.42) enables us to evaluate λ, but the choice of the fifth boundary condition does not affect the computed value of λ.

We find that

$$\lambda = -0.149516, \quad \text{and} \quad f_{1\dagger}'(0) = -0.469567. \tag{18.44}$$

The $O(\xi^{-1}\ln\xi)$ and $O(\xi^{-1})$ solutions in the inner layer are now,

$$\mathcal{F}_{2L} = 0, \qquad \mathcal{G}_{2L} = \lambda b y, \tag{18.45}$$

and

$$\mathcal{F}_2 = b f_{1\dagger}'(0) y, \qquad \mathcal{G}_2 = \sigma b y. \tag{18.46}$$

Therefore we may summarize these various results in terms of the surface shear stress and rate of heat transfer:

$$\mathcal{F}_{yy}|_{y=0} = a\xi^{-1/2} + o(\xi^{-1}), \tag{18.47a}$$

$$\mathcal{G}_y|_{y=0} = b\xi^{-1/2} + \lambda b \xi^{-1}\ln\xi + b\sigma\xi^{-1} + o(\xi^{-1}). \tag{18.47b}$$

In terms of f, g and η (see (18.24)) the above translates into

$$f_{\eta\eta}|_{\eta=0} = a\xi^{1/2} + o(1), \tag{18.48a}$$

$$g_\eta|_{\eta=0} = b + \lambda b \xi^{-1/2}\ln\xi + b\sigma\xi^{-1/2} + o(\xi^{-1/2}). \tag{18.48b}$$

Some relevant references

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19. Vertical Free Convection with Steady Surface Temperature Variations

In this lecture we examine how the steady free convective boundary layer flow induced by vertical heated surface is affected by the presence of sinusoidal surface temperature variations about a constant mean value which is above the ambient fluid temperature. The fully numerical techniques indicate that a near-wall layer develops as x becomes large. An asymptotic analysis is performed which shows that this inner layer has nonuniform thickness, unlike the corresponding porous medium boundary layer. The surface rate of heat transfer eventually alternates in sign with distance from the leading edge, but no separation occurs unless the amplitude of the thermal modulation is sufficiently high. The agreement between the numerical results and a two–term asymptotic analysis is excellent.

Governing equations and boundary layer analysis.

We consider the boundary–layer induced by a heated semi–infinite surface immersed in an incompressible Newtonian fluid. In particular, the heated surface is maintained at the steady scaled temperature,

$$\theta = 1 + a\sin(\pi x) \tag{19.1}$$

where $\theta = 0$ is the ambient temperature. The nondimensional steady two–dimensional equations of motion are given by

$$u_x + v_y = 0, \tag{19.2a}$$

$$uu_x + vu_y = -p_x + Gr^{-1/2}(u_{xx} + u_{yy}) + \theta, \tag{19.2b}$$

$$uv_x + vv_y = -p_y + Gr^{-1/2}(v_{xx} + v_{yy}), \tag{19.2c}$$

$$u\theta_x + v\theta_y = \sigma^{-1}Gr^{-1/2}(\theta_{xx} + \theta_{yy}), \tag{19.2d}$$

where Gr is the Grashof number and σ is the Prandtl number. In the derivation of equations (19.2) the Boussinesq approximation has been assumed. We note that the Grashof number has been based on d, half the *dimensional* wavelength of the thermal waves.

When the surface temperature is uniform and the Grashof number is very large, the resulting boundary–layer flow is self–similar. But the presence of sinusoidal surface temperature distributions, such as that given by (19.1), renders the boundary–layer flow nonsimilar. The boundary layer equations are obtained by introducing the scalings,

$$u = u^*, \quad v = Gr^{-1/4}v^*, \quad x = x^*, \quad y = Gr^{-1/4}y^*, \quad p = Gr^{-1/2}p^*, \quad \theta = \theta^* \tag{19.3}$$

into equations (19.2), formally letting Gr become asymptotically large, and retaining only the leading order terms. Thus we obtain,

$$u_x + v_y = 0, \tag{19.4a}$$

$$uu_x + vu_y = u_{yy} + \theta, \tag{19.4b}$$

$$uv_x + vv_y = -p_y + v_{yy}, \tag{19.4c}$$

$$u\theta_x + v\theta_y = \sigma^{-1}\theta_{yy}, \tag{19.4d}$$

where the asterisk superscripts have been omitted for clarity of presentation. Equation (19.4c) serves to define the pressure field in terms of the two velocity components and is decoupled from the other three equations. Therefore we shall not consider it further.

As the equations are two–dimensional we define a streamfunction, ψ, in the usual way,

$$u = \psi_y, \qquad v = -\psi_x, \tag{19.5}$$

and therefore (19.4a) is satisfied automatically. Guided by the familiar self–similar form corresponding to a uniform surface temperature, we use the substitution,

$$\psi = x^{3/4}f(\eta, x), \qquad \theta = g(\eta, x), \tag{19.6a, b}$$

where

$$\eta = y/x^{1/4} \tag{19.6c}$$

is the pseudo–similarity variable. Equations (19.4b) and (19.4d) reduce to

$$f''' + g + \tfrac{3}{4}ff'' - \tfrac{1}{2}f'f' + x(f_x f'' - f'_x f') = 0, \tag{19.7a}$$

$$\sigma^{-1}g'' + \tfrac{3}{4}fg' + x(f_x g' - f'g_x) = 0, \tag{19.7b}$$

and the boundary conditions are

$$f = 0, \quad f' = 0, \quad g = 1 + a\sin\pi x \quad \text{at} \quad \eta = 0, \quad \text{and} \quad f', g \to 0 \quad \text{as} \quad \eta \to \infty. \tag{19.7c}$$

In equations (19.7), primes denote derivatives with respect to η.

Numerical and Asymptotic Solutions

The parabolic system of equations, (19.7), is nonsimilar and its solution used the well-known Keller-box method. The results were obtained using uniform grids in both coordinate directions. There were 201 gridpoints lying between $\eta = 0$ and $\eta = 20$, and 401 between $x = 0$ and $x = 20$.

In Figure 19.1 isotherms in x-η–space are shown for $\sigma = 0.7$ for three diffeent values of the wave amplitude. It is clear that a near-wall layer develops as x becomes large. In fact it also appears that a mean x-independent flow occurs in the main part of the boundary layer. In Figure 19.2 we display the surface rate of heat transfer and shear stress for one parameter case. Here we see that the shear stress exhibits oscillations which decay as x increases, but that the rate of heat transfer, in terms of $g'(0)$, has growing oscillations. Indeed, for all values of a, the rate of heat transfer will always eventually change sign. The aim of an asymptotic analysis is to describe all these phenomena.

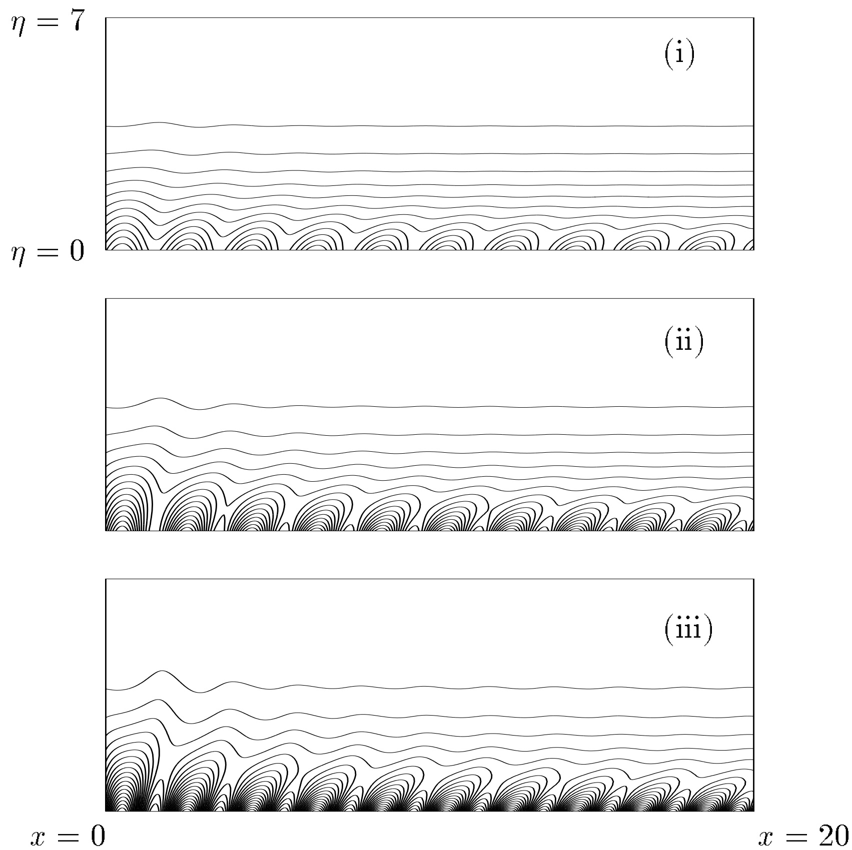

Figure 19.1. Isotherms for $\sigma = 0.7$ for (i) $a = 0.2$, (ii) $a = 0.5$, and (iii) $a = 1.0$.

The first task is to determine the thickness of the developing inner (near–wall) layer in

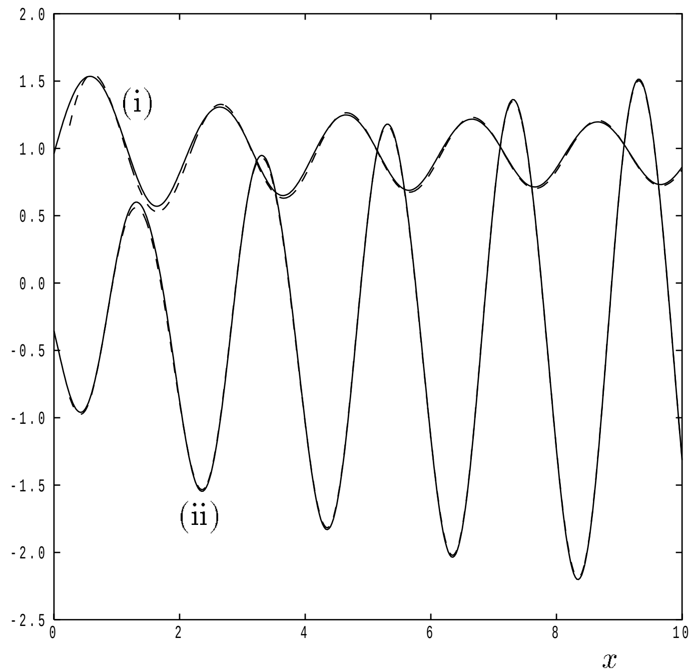

Figure 19.2. Comparisons between the fully numerical solution and the asymptotic solutions given by (19.29) and (19.30). Here $\sigma = 0.7$ and $a = 1.0$, and comparisons are made between (i) the surface shear stresses, $f''(x,0)$ and (ii) rate of heat transfer, $g'(x,0)$. The solid lines represent the numerical solutions and the dashed lines, the asymptotic solution.

terms of η. We begin by following the numerical evidence that the main boundary layer looks increasingly like the self–similar $a = 0$ case at large values of x. Thus we set $f \sim f_0(\eta)$ and $g \sim g_0(\eta)$ in (19.7) where f_0 and g_0 satisfy the equations,

$$f_0''' + g_0 + \tfrac{3}{4} f_0 f_0'' - \tfrac{1}{2} f_0' f_0' = 0, \tag{19.8a}$$

$$\sigma^{-1} g_0'' + \tfrac{3}{4} f_0 g_0' = 0, \tag{19.8b}$$

and are subject to the conditions,

$$f_0(0) = 0, \quad f_0'(0) = 0, \quad g_0(0) = 1, \quad \text{and} \quad f', g \to 0 \quad \text{as} \quad \eta \to \infty. \tag{19.8c}$$

For small values of η we can expand the solutions of (19.8) in the power series:

$$f_0 \sim \tfrac{1}{2} a_2 \eta^2 - \tfrac{1}{6}\eta^3 - \tfrac{1}{24} b_1 \eta^4 + \tfrac{1}{480} a_2^2 \eta^5, \tag{19.9a}$$

$$g_0 \sim 1 + b_1 \eta - \tfrac{1}{32}\sigma a_2 b_1 \eta^4 + \tfrac{1}{160}\sigma b_1 \eta^5, \tag{19.9b}$$

where $a_2 = f''(0)$ and $b_1 = g'(0)$ define the 'constants' a_2 and b_1, and we note that they are both functions of σ.

In the inner layer we must balance the orders of magnitude of the following terms in equation (19.7a): the buoyancy term, g, the highest derivative, f''', and the nonlinear terms involving x–derivatives, $x(f_x f'' - f'_x f')$. Given the form of the surface temperature variation, f_x values will vary over an $O(1)$ distance in the x direction even when $x \gg 1$. Thus the other nonlinear terms in (7a) are negligible.

Equation (19.9a) shows that $f'' = O(1)$ when η is small, and this leads us to the scalings, $\eta = O(x^{-1/3})$ and $f = O(x^{-1})$ when x is large. However, this size of η, taken together with the leading behaviour of f_0 in (19.9a), shows that $f = O(x^{-2/3})$ in the inner layer. This apparent error in the order–of–magnitude analysis simply indicates that the leading term in the inner layer passively transmits the main layer shear stress to the boundary, and it will be independent of x; this is confirmed below. Therefore the power series expansion in the inner layer must reflect this order of magnitude for f.

We shall denote f and g by F and G, respectively, in the inner layer, and define a new pseudo–similarity variable ζ, according to

$$\zeta = \eta x^{1/3}. \tag{19.10}$$

When ζ is rewritten in terms of x and y, $\zeta = yx^{1/12}$, we can deduce that the inner layer has a physical thickness which is $O(x^{-1/12})$ as x becomes large.

Equations (19.7) are transformed to

$$F''' = x^{-1/3}(\tfrac{3}{4}FF'' - \tfrac{5}{6}F'F') + x^{-1}G + x^{2/3}(F_x F'' - F'_x F') = 0, \tag{19.11a}$$

$$\sigma^{-1}G'' + \tfrac{3}{4}x^{-1/3}FG' + x^{2/3}(F_x G' - F'G_x) = 0, \tag{19.11b}$$

where primes denote derivatives with respect to ζ when used on inner variables. These equations are supplemented by the initial conditions,

$$F(0) = F'(0) = 0, \qquad G(0) = 1 + a\sin\pi x, \tag{19.11c}$$

but the boundary conditions as $\zeta \to \infty$ have to be obtained by matching with the outer flow solutions of (19.7a) and (19.7b).

Guided by the above scaling arguments we expand the solution of (19.11) in the form,

$$F = x^{-2/3}F_0 + x^{-1}F_1 + O(x^{-4/3}), \tag{19.12a}$$

$$G = G_0 + x^{-1/3}G_1 + O(x^{-2/3}), \tag{19.12b}$$

and the solution of (19.7a,b) in the form

$$f = f_0 + x^{-1/3}f_1 + O(x^{-2/3}), \tag{19.13a}$$

$$g = g_0 + x^{-1/3}g_1 + O(x^{-2/3}). \tag{19.13b}$$

The equations governing the leading order inner solutions are,

$$F_0''' = 0, \qquad \sigma^{-1}G_0'' + F_{0x}G_0' - F_0'G_{0x} = 0. \tag{19.14a, b}$$

The solution of (19.14a) which satisfies the appropriate matching condition obtained from (19.9a) is

$$F_0 = \tfrac{1}{2}a_2\zeta^2; \tag{19.15}$$

thus the surface shear stress term has indeed been transmitted unchanged from the outer region.

Given that $G_0 = 1 + a\sin\pi x$ at $\zeta = 0$, we need to substitute

$$G_0 = 1 + a\big[A_0(\zeta)\cos\pi x + B_0(\zeta)\sin\pi x\big] \tag{19.16}$$

into equation (19.14b) in order to find the leading order inner temperature field. Thus we obtain

$$A_0'' - (a_2\sigma\pi)\zeta B_0 = 0, \qquad B_0'' + (a_2\sigma\pi)\zeta A_0 = 0, \tag{19.17a, b}$$

subject to the boundary conditions,

$$A_0(0) = 0, \quad B_0(0) = 1, \quad \text{and} \quad A_0, B_0 \to 0 \quad \text{as} \quad \zeta \to \infty. \tag{19.17c}$$

We note that $C_0 = A_0 + iB_0$ may be regarded as a complex form of Airy's equation for which the only physically acceptable large–ζ asymptotic behaviour is

$$C_0 \quad \propto \quad \zeta^{-1/4}\exp\Big[\frac{2}{3}\Big(\frac{-1+i}{\sqrt{2}}\Big)\zeta^{3/2}(a_2\sigma\pi)^{1/2}\Big], \tag{19.18}$$

the other solution having superexponential growth. This result has been used to provide the large–ζ boundary conditions in (19.17c). We also note that it is possible to reduce equations (19.17a,b) to a canonical form using the the substitution $\tilde{\zeta} = (a_2\sigma\pi)^{1/3}\zeta$. Although no advantage is gained at higher order in the x expansion, it is easy to show that

$$A_0'(0) = -0.36451(a_2\sigma\pi)^{1/3}, \qquad B'(0) = -0.63135(a_2\sigma\pi)^{1/3}, \tag{19.19a, b}$$

by using the substitution together with the classical fourth order Runge–Kutta scheme allied to the shooting method. The solution of equations (19.17) is shown in Figure 19.3.

The solutions for A_0 and B_0, and hence G_0, see (19.16), decay superexponentially in ζ and therefore do not affect the main layer, at least to algebraic orders. Therefore we need now to consider the 2nd order inner equations,

$$F_1''' + G_0 = 0, \qquad \sigma^{-1}G_1'' + F_{1x}G_0' - F_0'G_{1x} - F_1'G_{0x} = 0, \tag{19.20a, b}$$

subject to

$$F_1 = F_1' = G_1 = 0 \quad \text{at} \quad \zeta = 0. \tag{19.20c}$$

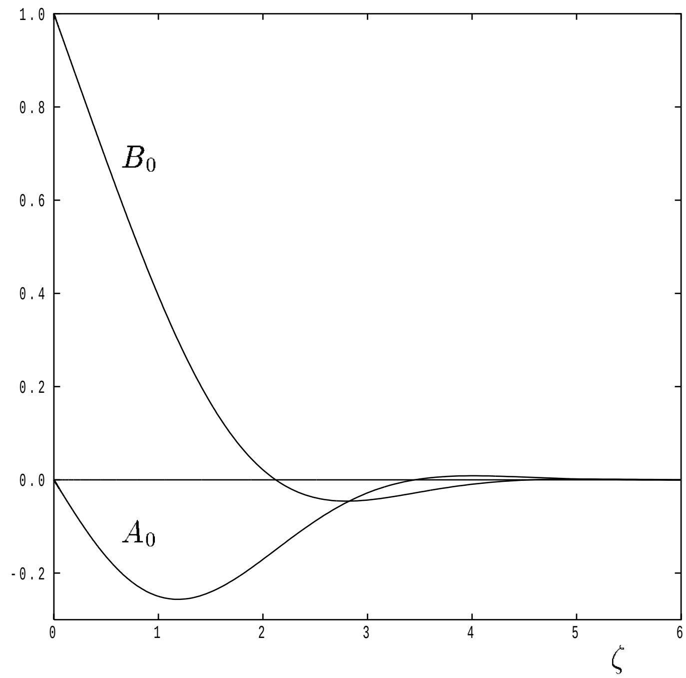

Figure 19.3. Solution of equations (19.17a) and (19.17b).

In view of (19.16) and the appropriate matching condition obtained from (19.9a) we need to set

$$F_1 = -\tfrac{1}{6}\zeta^3 - a\big[A_1(\zeta)\cos\pi x + B_1(\zeta)\sin\pi x\big], \qquad (19.21)$$

into (19.20a) to obtain,

$$A_1''' + a_2\pi(B_1 - \zeta B_1') = A_0, \qquad B_1''' - a_2\pi(A_1 - \zeta A_1') = B_0. \qquad (19.22a, b)$$

The boundary conditions at $\zeta = 0$ are

$$A_1 = B_1 = A_1' = B_1' = 0. \qquad (19.22c)$$

The complementary functions of (19.21), written in complex form have the following large–ζ asymptotic forms,

$$C_1 = A_1 + iB_1 \quad \propto \quad \zeta,$$

$$\zeta^{-3/4}\exp\Big[\frac{2}{3}\Big(\frac{-1+i}{\sqrt{2}}\Big)\zeta^{3/2}(a_2\pi)^{1/2}\Big],$$

and

$$\zeta^{-3/4}\exp\Big[\frac{2}{3}\Big(\frac{1-i}{\sqrt{2}}\Big)\zeta^{3/2}(a_2\pi)^{1/2}\Big], \qquad (19.22d)$$

Given that the superexponentially growing function is physically unacceptable, we see that the appropriate boundary conditions for A_1 and B_1 for large ζ are that

$$A_1'', B_1'' \to 0 \quad \text{as} \quad \zeta \to \infty. \tag{19.22e}$$

In general, then, these inner solutions exhibit linear growth as $\zeta \to \infty$, and this will affect and force further outer flow solutions. Indeed, given the form of (19.21), this effect will be x–dependent due to the presence of the $\sin\pi x$ and $\cos\pi x$ terms. If we denote by A_1^∞ and B_1^∞ the asymptotic values of A_1' and B_1' as $\zeta \to \infty$, then

$$F_1 \sim -\tfrac{1}{6}\zeta^3 - a\zeta(A_1^\infty\cos\pi x + B_1^\infty\sin\pi x) \quad \text{as} \quad \zeta \to \infty. \tag{19.23a}$$

Therefore the inner solution obtained so far yields,

$$F \sim \left[\tfrac{1}{2}a_2\zeta^2\right]x^{-2/3} + \left[-\zeta^3 - a\zeta(A_1^\infty\cos\pi x + B_1^\infty\sin\pi x)\right]x^{-1}, \tag{19.23b}$$

which, when written in outer variables, gives the following matching condition for small values of η,

$$f \sim (\tfrac{1}{2}\eta^2 - \tfrac{1}{6}\eta^3) + \left[-a\eta(A_1^\infty\cos\pi x + B_1^\infty\sin\pi x)\right]x^{-2/3}. \tag{19.24}$$

As this matching condition has no term at $O(x^{-1/3})$, and as the equations for f_1 and g_1 are homogeneous, we conclude that both f_1 and g_1 are zero. The equation for f_2 is now

$$f_{2x}f_0'' - f_{2x}'f_0' = 0, \tag{19.25}$$

which may be solved easily to obtain

$$f_2 = -(a/a_2)\left[A_1^\infty\cos\pi x + B_1^\infty\sin\pi x\right]f_0'(\eta). \tag{19.26}$$

Returning to the second order inner solution, the substitution of (19.15), (19.16) and (19.21) into (19.20b) yields

$$\begin{aligned}\sigma^{-1}G_1'' - a_2\zeta G_{1x} &+ \tfrac{1}{2}a^2\pi\left[B_1'A_0 - B_1A_0' + A_1'B_0 - A_1B_0'\right]\cos 2\pi x\\ &+ \tfrac{1}{2}a^2\pi\left[B_1'B_0 - B_1B_0' - A_1'A_0 + A_1A_0'\right]\sin 2\pi x\\ &+ \tfrac{1}{2}a^2\pi\left[(A_1B_0)' - (B_1A_0)'\right] + \tfrac{1}{2}a\pi\left[B_0\cos\pi x - A_0\sin\pi x\right]\zeta^2\\ &= 0.\end{aligned} \tag{19.27}$$

Solution of (19.27) is effected by first substituting the expression

$$G_1 = a^2[G_{10} + G_{12c}\cos 2\pi x + G_{12s}\sin 2\pi x] + a[G_{11c}\cos\pi x + G_{11s}\sin\pi x] + b_1\zeta, \tag{19.28}$$

but we omit the presentation of the equations for these five functions of ζ, and we note that the very last term in (19.28) is required in order to match with the second term of (19.9b).

The accurate numerical solution of equations (19.22) and those arising from the substitution of (19.28) into (19.27) prove to be quite difficult to perform using the shooting method due to the presence of unwanted complementary functions with superexponential growth. Thus, these sets of stiff equations were solved using a suitably modified form of the Keller box method. Although the Keller box method was devised originally for solving parabolic marching problems, it is equally well–suited for solving sets of ordinary differential equations. In fact, if the 'streamwise' variable used in the code is taken to be the Prandtl number, then straightforward modification is all that is required to obtain solutions over a wide range of values of σ by taking a parameter sweep. A nonuniform grid of 101 points lying between $\zeta = 0$ and $\zeta = 100.$ was used for these computations. Richardson's Extrapolation was used on results obtained by successive interval–halving of the basic grid to obtain highly accurate solutions.

The shear stresses and rates of heat transfer are of physical interest, but we are also interested in the cross–validation between the asymptotic analysis and the numerical results. Comparison between the numerical and asymptotic requires that the expressions for the asymptotic shear stress and heat transfer are expressed in terms of η–derivatives. Hence the surface shear stress is

$$\begin{aligned}\left.\frac{\partial^2 f}{\partial \eta^2}\right|_{\eta=0} &= F_0''(\zeta=0) \quad + \quad x^{-1/3}F_1''(\zeta=0) \quad + \quad O(x^{-2/3}),\\ &= a_2 \quad - \quad x^{-1/3}a\big[A_1''(0)\cos\pi x + B_1''(0)\sin\pi x\big] \quad + \quad O(x^{-2/3}),\end{aligned} \tag{19.29}$$

and the rate of heat transfer is,

$$\begin{aligned}\left.\frac{\partial g}{\partial \eta}\right|_{\eta=0} &= x^{1/3}G_0'(\zeta=0) \quad + \quad G_1'(\zeta=0) \quad + \quad O(x^{-1/3}),\\ &= x^{1/3}a\big[A_0'(0)\cos\pi x + B_0'(0)\sin\pi x\big]\\ &\quad + \quad a^2\big[G_{10}'(0) + G_{12c}'(0)\cos 2\pi x + G_{12s}'(0)\sin 2\pi x\big]\\ &\quad + \quad a\big[G_{11c}'(0)\cos\pi x + G_{11s}'(0)\sin\pi x\big] \quad + b_1 + \quad O(x^{-1/3}).\end{aligned} \tag{19.30}$$

Numerically, we find that $G_{10}'(0) = 0$, $G_{12c}'(0) = 0$ and $G_{11s}'(0) = 0$ for all values of σ, which simplifies (19.30) slightly, although the functions G_{10}, G_{12c} and G_{11s} are not identically zero.

In Figure 19.2 we show comparisons between the above expressions, (19.29) and (19.30), and the fully numerical results. Here we take $\sigma = 0.7$ and choose $a = 1$, a very large surface temperature wave amplitude. However, despite the size of a, we see that the curves are in virtually perfect agreement with excellent agreement even for values of x as low as 0.5.

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20. Free Convection through a Layered Porous Medium

Here we will consider free convection induced by a uniform temperature heated vertical surface in a layered porous medium. The interfaces between the layers are horizontal, and each layer has its own thermal conductivity, diffusivity and permeability. The chief novelty of this flow is that the boundary layer, which is normally described by parabolic partial differential equations, has to reduced to elliptic form very close to the interface between layers. In fact this happens twice yielding a boundary layer within a boundary layer at the interface. All of this takes place within what is usually called a boundary layer flow!

We assume that all variables have been nondimensionalised and that the Darcy-Rayleigh number has been defined using the properties of the first layer, which contains the leading edge. Hence we take

$$R = \frac{\rho g \beta K_1 (T_{\max} - T_{\min}) L}{\mu \alpha_1},$$

the numerical subscripts referring to layer 1. The value L is a typical lengthscale associated with the width of the layers. We will indicate the layer to which any variable belongs by a subscript involving the letter, i.

The steady two-dimensional equations are

$$\nabla^2 \psi_i = R(K_i/K_1)\frac{\partial \theta_i}{\partial y} \tag{20.1a}$$

$$(\alpha_i/\alpha_1)\nabla^2 \theta_i = \frac{\partial \psi_i}{\partial y}\frac{\partial \theta_i}{\partial x} - \frac{\partial \psi_i}{\partial x}\frac{\partial \theta_i}{\partial y}, \tag{20.1b}$$

where

$$y = 0: \qquad \psi_i = 0, \qquad \theta_i = 1 \tag{20.1c}$$

$$y \to \infty: \qquad \frac{\partial \psi_i}{\partial y} \to 0, \qquad \theta_i \to 0. \tag{20.1d}$$

We need to apply interface conditions in order to advance solutions from one layer to the next. Therefore we have continuity of temperature, heat flux, vertical velocity and pressure:

$$\theta_i = \theta_{i+1}, \tag{20.2a}$$

$$\alpha_i \frac{\partial \theta_i}{\partial x} = k_{i+1}\frac{\partial \theta_{i+1}}{\partial x}, \tag{20.2b}$$

$$u_i = u_{i+1} \quad \Rightarrow \quad \frac{\partial \psi_i}{\partial y} = \frac{\partial \psi_{i+1}}{\partial y} \quad \Rightarrow \quad \psi_i = \psi_{i+1}, \tag{20.2c}$$

$$p_i = p_{i+1} \quad \Rightarrow \frac{v_i}{K_i} = \frac{v_{i+1}}{K_{i+1}} \quad \Rightarrow \quad K_{i+1}\frac{\partial \psi_i}{\partial x} = K_i \frac{\partial \psi_{i+1}}{\partial x}. \tag{20.2d}$$

Now we transform equations (20.1) using the standard boundary layer variables defined by

$$\psi = R^{1/2}x^{1/2}f(\eta,x), \qquad \theta = g(\eta,x), \qquad \eta = R^{1/2}y/x^{1/2}, \tag{20.3}$$

but we will retain all terms in the resulting equations, even though we will be performing a large–R analysis. Hence

$$f_i'' - \Big(\frac{K_i}{K_1}\Big)g_i' = R^{-1}\Big[-xf_{i,xx} + \eta f_{i,x}' - \tfrac{1}{4}\eta^2x^{-1}f_i'' - f_{i,x} + \tfrac{1}{4}x^{-1}(f_i - \eta f_i')\Big], \tag{20.4a}$$

$$\Big(\frac{\alpha_i}{\alpha_1}\Big)g_i'' + \tfrac{1}{2}f_ig_i + x(f_{i,x}g_i' - f_i'g_{i,x}) = \Big(\frac{\alpha_i}{\alpha_1}\Big)R^{-1}\Big[-xg_{i,xx} + \eta g_{i,x}' - \tfrac{1}{4}\eta^2x^{-1}g_i'' - \tfrac{3}{4}x^{-1}\eta g_i')\Big]. \tag{20.4b}$$

As $R\to\infty$ we obtain, at leading order,

$$f_i'' - \Big(\frac{K_i}{K_1}\Big)g_i' = 0 \tag{20.5a}$$

$$\Big(\frac{\alpha_i}{\alpha_1}\Big)g_i'' + \tfrac{1}{2}f_ig_i + x(f_{i,x}g_i' - f_i'g_{i,x}) = 0. \tag{20.5b}$$

We may solve these equations in any layer beginning with the inteface values of f_i and g_i. In layer 1 these solutions will be self-similar, i.e. independent of x. We can integrate as far as the next interface, but we cannot proceed further because our equations are parabolic, whereas there are four interface conditions to be satisfied:

$$f_{i+1} = f_i \quad g_{i+1} = g_i \quad K_i\frac{\partial f_{i+1}}{\partial x} = K_{i+1}\frac{\partial f_i}{\partial x} \quad \alpha_{i+1}\frac{\partial g_{i+1}}{\partial x} = \alpha_i\frac{\partial g_i}{\partial x}. \tag{20.6a, b, c, d}$$

Although the first derivatives with respect to x of both f and g appear in the equations, we have lost the streamwise diffusion terms, and this poses a problem which can only be resolved if we install a thin layer near the interface.

Intermediate region.

We will now change notation to one where we centre on a typical interface which will be at $x = x_i$. But values of f and g at smaller values of x will be denoted as f^- and g^-, while the values on the other side of the interface are f^+ and g^+. The absence of such a superscript denotes either variable.

The balancing of the f'' and $R^{-1}f_{xx}$ terms in (20.4a) requires us to define the stretched varaible, X according to

$$x = x_i + R^{-1/2}X.$$

Substitution of this into equations (20.4) yields

$$F_{XX} + F'' - KG' = 0, \qquad F_XG' - F'G_X = 0, \tag{20.7a, b}$$

where K now denotes the permeability ratio shown explicitly in (20.4a), and we denote by F and G the values of f and g in this asymptotic regime. The equation for F is elliptic as it is second order in X, and therefore we may now apply both the interface conditions:

$$F^+ = F^- \qquad K^- F_X^+ = K^+ F_X^- \qquad \text{at} \quad X = 0. \tag{20.8a, b}$$

The matching condition for large negative values of X is

$$F^-(X \to -\infty, \eta) = f(x \to x_i^-, \eta), \tag{20.9a}$$

and this may be applied directly in the numerical solution. If we use $F_X^+(X\to\infty = 0$ as the large-X boundary condition, we will obtain the profile which may be used to continue the solution in the outer region:

$$f^+(x \to x_i^+, \eta) = F^+(X \to \infty, \eta). \tag{20.9b}$$

However, the equation for G remains parabolic. This means that this regime is only an intermediate regime and that there must exist a thinner boundary layer embedded within it.

Inner region.

If we balance the terms

$$x f_x g' \qquad \text{with} \qquad R^{-1} x g_{xx}$$

we see that the inner stretched variable is defined according to

$$x = x_i + R^{-1}\xi.$$

Hence equations (20.4) reduce to

$$\mathcal{F}_{\xi\xi} = O(R^{-1}), \qquad \mathcal{G}_{\xi\xi} + \mathcal{F}_\xi \mathcal{G}' - \mathcal{F}' \mathcal{G}_\xi = O(R^{-1}), \tag{20.10a, b}$$

where $\mathcal{F}$ and $\mathcal{G}$ denote f and g in this inner regime. It is interesting to note that $O(1)$ variations in ξ take place over an asymptotically smaller lengthscale than do $O(1)$ variations in η; i.e. this inner layer is thinner than the main boundary layer itself.

Before solving these equations it is necessary to determine the large-ξ matching conditions by considering the small-X behaviour of the intermediate regime solutions. Near the interface F and G may be expanded in a Taylor Series:

$$F(X, \eta) = F_0(0, \eta) + X F_1(0, \eta) + \cdots \tag{20.11a}$$

$$G(X, \eta) = G_0(0, \eta) + X G_1(0, \eta) + \cdots \tag{20.11b}$$

where

$$F_1^{\pm} = \lim_{x\to 0^{\pm}} \frac{\partial F^{\pm}}{\partial x} \qquad \text{and} \qquad G_1^{\pm} = \lim_{x\to 0^{\pm}} \frac{\partial G^{\pm}}{\partial x}. \tag{20.11c}$$

Now the F_1^+ and F_1^- values are related by the fact that the interface conditions have been satisfied, i.e. that $K^-F_1^+ = K^+F_1^-$; this is because the equation for F is elliptic. However G_1^+ and G_1^- are not related by the appropriate derivative interface condition (20.6d). And therefore $\alpha^+G_1^+ \neq \alpha^-G_1^-$ in general.

In terms of ξ, the Taylor Series in (20.11) give the matching conditions,

$$\mathcal{F}(\xi,\eta) \longrightarrow F_0(0,\eta) + R^{-1/2}\xi F_1^+(0,\eta) \qquad \text{as } \xi\to\infty, \tag{20.12a}$$

$$\mathcal{G}(\xi,\eta) \longrightarrow G_0(0,\eta) + R^{-1/2}\xi G_1^+(0,\eta) \qquad \text{as } \xi\to\infty. \tag{20.12b}$$

$$\mathcal{F}(\xi,\eta) \longrightarrow F_0(0,\eta) + R^{-1/2}\xi F_1^-(0,\eta) \qquad \text{as } \xi\to-\infty, \tag{20.12c}$$

$$\mathcal{G}(\xi,\eta) \longrightarrow G_0(0,\eta) + R^{-1/2}\xi G_1^-(0,\eta) \qquad \text{as } \xi\to-\infty. \tag{20.12d}$$

Therefore we must set $(\mathcal{F},\mathcal{G}) = (\mathcal{F}_0,\mathcal{G}_0) + R^{-1/2}(\mathcal{F}_1,\mathcal{G}_1) + \cdots$.

The functions $\mathcal{F}$ and $\mathcal{G}$ satsify

$$\mathcal{F}_{0,\xi\xi} = 0, \qquad \mathcal{G}_{0,\xi\xi} + \mathcal{F}_{0,\xi}\mathcal{G}_0' - \mathcal{F}_0'\mathcal{G}_{0,\xi} = 0. \tag{20.13a, b}$$

In view of the $O(1)$ matching conditions in (20.12) we must have

$$\mathcal{F}(\xi,\eta) = F_0(X=0,\eta), \qquad \mathcal{G}_0(\xi,\eta) = G_0(X=0,\eta). \tag{20.14a, b}$$

At $O(R^{-1/2})$ the equations are

$$\mathcal{F}_{1,\xi\xi} = 0, \qquad \alpha^{\pm}\mathcal{G}_{1,\xi\xi} + \mathcal{F}_{0,\xi}\mathcal{G}_1' - \mathcal{F}_0'\mathcal{G}_{1,\xi} + \mathcal{F}_{1,\xi}\mathcal{G}_0' - \mathcal{F}_1'\mathcal{G}_{0,\xi} = 0. \tag{20.15a, b}$$

Clearly we must have

$$\mathcal{F}_1 = F_1^+(X=0,\eta)\xi \qquad \text{for } \xi > 0, \tag{20.16a}$$

and

$$\mathcal{F}_1 = F_1^-(X=0,\eta)\xi \qquad \text{for } \xi < 0, \tag{20.16b}$$

where there may also be a “constant” of integration which would be a function of η. These solutions indicate that the behaviour of $\mathcal{F}$ is passive.

Using (20.14) and (20.16), equations (20.15b) becomes,

$$\alpha^{\pm}\mathcal{G}_{1,\xi\xi}^{\pm} + F_1^{\pm}G_0' - F_0'\mathcal{G}_{1,\xi}^{\pm} = 0. \tag{20.17}$$

This formidable equation is only a linear ordinary differential equation where η is a parameter. After some thought, it becomes clear that the solution of (20.17) is

$$\mathcal{G}_1^- = \left(\frac{F_1^- G_0'}{F_0'}\right)\xi + Ae^{(F_0'/\alpha^-)\xi} - A + B, \tag{20.18a}$$

$$\mathcal{G}_1^+ = \left(\frac{F_1^+ G_0'}{F_0'}\right)\xi + B, \tag{20.18b}$$

where $A = A(\eta)$ is to be found, and $B = B(\eta)$ is arbitrary. Note that the coefficients of ξ in (20.18a) and (20.18b) may easily be shown to be equal to G_1^- and G_1^+, respectively. There cannot be an exponential term in (20.18b) as we would expect it to be a growing solution since there should be a positive fluid velocity upwards, and $F_0' > 0$. Continuity of $\mathcal{G}$ has already been applied to get (20.18), but now we apply continuity of heat flux: $\alpha^+\mathcal{G}_{1,\xi}^+ = \alpha^-\mathcal{G}_{1,\xi}^-$. Therefore we get

$$A(\eta) = \frac{\alpha^+ G_1^+ - \alpha^- G_1^-}{(F_0'/\alpha^-)}.$$

Note that $A(0) = 0$, and therefore both $\mathcal{G}_1^+$ and $\mathcal{G}_1^-$ satisfy $\mathcal{G}(\eta = 0) = 0$. We have to assume that the arbitrary function also satisfies $B(0) = 0$.

In conclusion, we have obtained a boundary layer within a boundary layer within a boundary layer! In practice we solve (20.5) in layer 1. This solution is self-similar, but this is of no great consequence. This solution at the first interface provides us with matching conditions via (20.9a) and its thermal counterpart so that we may be able to solve numerically the intermediate regime equations (20.7). We need not be concerned about the inner region as we have obtained an analytical solution in terms of the intermediate regime numerical solutions. Finally, the intermediate regime provides us with matching conditions from which we may continue the outer regime parabolic simulation into the second layer. Our analysis has been fairly general, and this procedure may be used at any interface where $x = O(1)$. (We note that when $x = O(R)$, we enter a completely new asymptotic situation for this problem. It may be shown that the intermediate and inner regimes presented here become the outer and inner regimes of a two-region analysis. The methodology is very similar indeed to all of the above analysis.)

This work is also described in:

N. Banu, "Convection in fluid-saturated porous media", Ph.D. thesis (University, of Bath, in preparation).

Some sample questions.

I have compiled a few questions which are relevant to the first 14 lectures. Most of these have worked solutions.

1. Ordinary points. Find the Taylor series solutions about $x = 0$ of the following equations.

(i) $y'' - y = 0$ **(ii)** $y''' - y = 0$ **(iii)** $y' + 2xy = 0$

(iv) $y'' - 2xy' + \lambda x = 0$ Hermite's equation. In this case show that one of the resulting pair of generally infinite series becomes finite in length when λ takes positive even integer values.

(v) $y'' + x^2 y = 0$ **(vi)** $(1 - x)y' - y = 0$.

2. Singular points. Classify all the singular points of the following equations.

(i) $y'' + y = 0$ **(ii)** $xy'' + y = 0$ **(iii)** $(\sin x)y' + y = 0$

(iv) $y'' + e^x y' - y = 0$ **(v)** $x^3 y' = y$ **(vi)** $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$ (Legendre's equation)

(vii) $(\sin x)y'' + xy' + x^3 y$ **(viii)** $x^2(1 - x^2)^3 y'' + 2xy' + 4y = 0$ **(ix)** $(x + 1)^3 y''' + x^2 y'' - y = 0$.

3. Frobenius I.
(i) Solve Bessel's equation of order $\frac{1}{2}$: $x^2 y'' + xy' + (x^2 - \frac{1}{4})y = 0$.
(ii) Solve the modified Bessel's equation of order $\frac{1}{2}$: $x^2 y'' + xy' - (x^2 + \frac{1}{4})y = 0$.
(iii) Solve $2xy'' + y' - 2y = 0$.
(iv) Solve $2x^2 y'' + 3xy' - (1 + x)y = 0$. Use the method of variation of parameters to find the analytical solution.
(v) Consider the equation $y'' + [p(x)/x]y' + [q(x)/x^2]y = 0$ where both $p(x)$ and $q(x)$ have Taylor series expansions about $x = 0$. Show that the indicial equation is $c^2 + c(p_0 - 1) + q_0 = 0$.

4. Frobenius II Solve the following equations

(i) $xy'' + y' - y = 0$ (ii) $x^2 y'' - 3xy' + (4 - x^2)y = 0$ (iii) $x^2 y''' + 3xy'' + y' - y = 0$

5. Frobenius III Solve the following equations

(i) $xy'' + y - (x + 1)y = 0$, (ii) $x^2 y''' + 2xy'' - y = 0$, (iii) $x^2 y''' + 4xy'' + 2y' - y = 0$.

Irregular singular points

6. Find the leading behaviour as $x \to 0$ of the equation, $y'' - x^{1/2}y = 0$, by employing the substitution,

$$y = \sum_{n=0}^{\infty} a_n x^{n\alpha},$$

where $\alpha > 0$ is to be found.

7. Find the leading behaviour of the equation $x^4y'' - y = 0$ as $x \to 0$. Continue the analysis to try to find a series correction to this leading behaviour. Check by direct substitution into the equation the surprising result you should get

8. Find the leading behaviour of the equation $x^4y'' - 2x^2y' + y = 0$.

9. Find the leading behaviour of $y'' - x^2y = 0$ as $x \to \infty$. What about the equation $y'''' - x^2y = 0$?

10. Find the the full asymptotic solution as $x \to \infty$ of $xy'' + y' + xy = 0$, the zero order Bessel's equation.

Regular and Singular Perturbations

11. Obtain two-term expressions for the solutions of the following equations:

(i) $(x-1)(x-2)(x-3) + \epsilon = 0$
(ii) $\epsilon x^3 + x^2 + 3x + 2 = 0$
(iii) $\epsilon x^3 + x^2 + 2x + 1 = 0$
(iv) $\epsilon x^4 - x^2 + 3x - 2 + \epsilon = 0$.

12. Rework Example 7.4 with the boundary conditions $y(-1) = 0$ and $y(1) = 0$. What is the analytical solution?

13. Solve the equation $\epsilon y'''' - y'' = 1$ subject to $y(\pm 1) = y'(\pm 1) = 0$ to one term in the boundary layers and to two terms in the interior.

WKB Problems

14. Obtain the WKB approximation to the equation $\epsilon^2 y'' + x^{1/2}y = 0$ for $\epsilon \to 0$ in the range $x > 0$.

15. Use WKB theory to obtain the "physical optics" solution to $\epsilon y'' + a(x)y' + b(x)y = 0$ for $a(x) > 0$. Make sure you obtain two independent solutions.

16. Obtain the WKB approximation to the equation $\epsilon^4 y'''' = Q(x)y$ for $\epsilon \to 0$. Find the solution to the equation $\epsilon^4 y'''' + x^2 y = 0$.

17. Show that the eigenvectors of the Sturm-Liouville problem

$$y'' + \lambda^2 Q(x) y = 0 \qquad y(0) = 0 \quad y(1) = 0$$

are orthogonal with respect to the weight function $Q(x) > 0$ in the interval $0 \le x \le 1$. In other words, if $y_n(x) \neq 0$ satisfies $y_n'' + \lambda_n^2 Q(x) y_n = 0$, show that

$$\int_0^1 Q(x) y_n(x) y_m(x)\, dx = 0 \qquad \text{when } n \neq m.$$

18. Use the WKB approximation to find an expression for the large eigenvalues, λ, of the system

$$y'' + \lambda^2 e^{2x} y = 0, \qquad y(0) = y(1) = 0.$$

Multiple Scales Theory

19. Solve Mathieu's equation near to the value $a = 1$ on the ϵ-axis using the method of multiple scales in order to determine how the stability boundary behaves near that point.

20. Solve the equation $y'' + y + \epsilon y (y')^2 = 0$ using a regular perturbation expansion. Show that secular terms arise when $t = O(\epsilon^{-1})$. Use a Multiple Scales analysis to make this two-term expansion uniformly convergent.

21. Consider the equation $y'' + y = \epsilon y^2$ as $\epsilon \to 0$ subject to $y(0) = 2$ and $y'(0) = 0$. At what order in terms of ϵ does the first secular term appear in a regular perturbation solution? Find $y(t)$ correct to that order. Introduce a suitable long time scale to render this solution uniformly convergent.