# THERMAL BOUNDARY–LAYER INSTABILITIES IN POROUS MEDIA: A CRITICAL REVIEW

D. A. S. REES

Department of Mechanical Engineering, University of Bath,
Claverton Down, Bath, BA2 7AY, UK

## INTRODUCTION

The aims of this review are (i) to provide a comprehensive survey of the present knowledge of how thermal boundary layers in porous media are destabilised, (ii) to give a critical account of the present state–of–the–art, and (iii) to recommend strategies for the future development of this important topic. At present there are, to the author's knowledge, fewer than thirty publications which deal with the instability of thermal boundary layers in porous media, only two of which are concerned with the nonlinear development of instability. Therefore this topic can only be regarded as being in its infancy. By contrast, the study of the instability of the well–known Blasius boundary–layer flow of an isothermal clear fluid over a flat plate left the realms of linearised theory based on the parallel–flow approximation many decades ago. Weakly nonlinear theories have been developed, as have nonparallel analyses, and it is now possible to employ the techniques of Direct Numerical Simulation to determine in detail the temporal evolution of primary and secondary instabilities en route to fully developed turbulence. Thus our present topic is lagging well behind in its development, especially when it is realised that fully numerical simulations in porous media flows are no more difficult to obtain than are Navier–Stokes simulations, and are, in fact, easier should boundary effects in the form of the Brinkman terms be neglected. It is hoped, therefore, that the present review will serve as a stimulus for greatly increased work in this field.

We will also emphasize the sometimes subtle roles played by the various approximations which are used routinely in this topic, namely, the boundary–layer approximation and the parallel–flow approximation. The former serves to reduce the full, spatially elliptic, equations of motion to either a set of ordinary differential equations (thereby obtaining a self–similar flow), or, at worst, a parabolic set of partial differential equations (thereby obtaining a nonsimilar flow which is computed using a suitable marching scheme). The latter serves to simplify the linearised disturbance equations to ordinary differential form and therefore these solutions are relatively easy to obtain compared with the full partial differential form. Both approximations are discussed and we show that care must be taken in their use. It is also hoped that bringing these issues to the fore will allow the subject to progress in a more informed and effective manner.

In this review we will attempt to maintain a consistent notation throughout. Upper–case letters will usually refer to dimensional variables, and lower–case to dimensionless variables. The $w$–subscript refers to conditions at the heated surface, while the $\infty$–subscript refers to ambient conditions. The variable $\eta$ is the similarity or pseudo–similarity variable, and $\xi$ and $Ra_x$ are scaled streamwise coordinates; precise definitions of these will depend on the flow configuration in question. Coordinates used as subscripts denote derivatives with respect to that coordinate, while primes denote derivatives with respect to $\eta$. Occasionally these conventions will differ from those in some of the papers reviewed.

The references follow the usual numerical style with the following modification. References [1] to [25] comprise all the known work on the title topic; these are ordered according to their appearance. The remainder are concerned with those papers which have a strong bearing on the review, such as those dealing with basic boundary–layer flows in porous media, or those on the instability of clear fluids.

## THE NATURE OF THE BOUNDARY–LAYER APPROXIMATION

There are three methods commonly used to obtain the equations for thermal boundary–layer flows in porous media, and these will be illustrated using the near–horizontal free convection boundary–layer flow from a heated surface. Assuming the simplest possible situation, namely, that Darcy's law is valid, that the Boussinesq approximation applies, that the flow is steady, and that the medium is homogeneous, nondeformable and isotropic, the governing equations are,

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}+\frac{\partial W}{\partial Z}=0, \tag{1$a$}$$

$$U=-\frac{K}{\mu}\Big[\frac{\partial P}{\partial X}-\rho_\infty g\beta(T-T_\infty)\sin\phi\Big], \tag{1$b$}$$

$$V=-\frac{K}{\mu}\Big[\frac{\partial P}{\partial Y}-\rho_\infty g\beta(T-T_\infty)\cos\phi\Big], \tag{1$c$}$$

$$W=-\frac{K}{\mu}\frac{\partial P}{\partial Z}, \tag{1$c$}$$

$$U\frac{\partial T}{\partial X}+V\frac{\partial T}{\partial Y}+W\frac{\partial T}{\partial Z}=\alpha\Big[\frac{\partial^2 T}{\partial X^2}+\frac{\partial^2 T}{\partial Y^2}+\frac{\partial^2 T}{\partial Z^2}\Big]. \tag{1$e$}$$

Here $X$, $Y$ and $Z$ are the dimensional streamwise, cross–stream and spanwise coordinates, respectively, $U$, $V$ and $W$ are the corresponding fluid flux velocities, $P$ is the pressure and $T$ the temperature. The material parameters, $K$, $\mu$, $g$, $\beta$ and $\alpha$ have their usual meanings, $T_\infty$ is the ambient temperature of the medium, and $\rho_\infty$ is the corresponding fluid density. The inclination above the horizontal of the upward–facing heated surface is $\phi$. The heated surface has a uniform surface temperature distribution: $T_w = T_\infty + \Delta T$ for $X > 0$, where $\Delta T$ is the temperature difference between the wall and the ambient medium. The resulting two–dimensional flow may be studied by introducing a streamfunction in the form,

$$U=\frac{\partial\Psi}{\partial Y},\qquad V=-\frac{\partial\Psi}{\partial X},\qquad W=0. \tag{2}$$

If, in addition, it is assumed that there are no $Z$–variations, equations (1) become,

$$\frac{\partial^2\Psi}{\partial X^2}+\frac{\partial^2\Psi}{\partial Y^2}=\frac{\rho_\infty g\beta K}{\mu\alpha}\left[\frac{\partial T}{\partial Y}\sin\phi-\frac{\partial T}{\partial X}\cos\phi\right] \tag{3a}$$

$$\frac{\partial^2 T}{\partial X^2}+\frac{\partial^2 T}{\partial Y^2}=\frac{\partial\Psi}{\partial Y}\frac{\partial T}{\partial X}-\frac{\partial\Psi}{\partial X}\frac{\partial T}{\partial Y}. \tag{3b}$$

The heated surface is assumed to be semi–infinite in extent and therefore there is no external physical lengthscale which may be used to nondimensionalise the equations. The three methods of analysing the boundary–layer flow treat this absence in different ways.

**Using the local Rayleigh number — method 1**

This is the most frequently used method for studying boundary–layer flows. It proceeds initially by assuming that $X \gg Y$, that is, the boundary–layer thickness is much smaller than the distance from the leading edge. It follows that the $X$–derivative diffusion terms on the left hand sides of equations (3a) and (3b) may be neglected in favour of the $Y$–derivatives. On following this procedure, equations (3) become parabolic and may be transformed into a more suitable form using the set of transformations,

$$\psi=\alpha\, Ra_x^{1/3} f(\xi,\eta), \qquad T=T_\infty+\Delta T\,\theta(\xi,\eta), \tag{4a,b}$$

where

$$\eta=\frac{Y}{X}Ra_x^{1/3}, \qquad \xi=Ra_x^{1/3}\tan\phi, \qquad Ra_x=\frac{\rho g\beta K\Delta T\,\cos\phi}{\mu\alpha}X. \tag{4c,d,e}$$

Finally, $f$ and $\theta$ may easily be shown to satisfy the pair of equations,

$$f''=\xi\theta'+\tfrac{2}{3}\eta\theta'-\tfrac{1}{3}\xi\theta_\xi, \qquad \theta''+\tfrac{1}{3}f\theta'=\tfrac{1}{3}\xi(f'\theta_\xi-\theta' f_\xi), \tag{5a,b}$$

subject to the boundary conditions

$$f(0)=0, \qquad \theta(0)=1 \qquad \text{and} \qquad f',\ \theta\to 0 \quad \text{as} \quad \eta\to\infty. \tag{5c,d,e,f}$$

Although the present notation is slightly different, the above analysis follows very closely that of Jang and Chang [3] who considered the more general case of a power–law surface temperature distribution.

When $\phi = 0$ the heated surface is horizontal, and, according to Jang and Chang [5] this corresponds to the $\xi = 0$ solution of equations (5), which was first presented by Chang and Cheng [26]; such a flow is termed self–similar. When $\phi$ is positive, equations (5) are solved using a marching scheme, such as the Keller–box method (see Keller and Cebeci [27]). The flow and temperature profiles change as $\xi$ increases; such a flow is termed non–similar.

The problem of the absence of a physical lengthscale has been circumvented by the definitions contained in equation (4) where $Ra_x$ plays the role of a stretched streamwise coordinate and $\eta$ is the pseudo–similarity variable. It is interesting to note that the governing partial differential equations have not been nondimensionalized using this formulation, at least not in the usual way. It is straightforward to show that $X \gg Y$ using the definition of $\eta$ when $Ra_x \gg 1$, and therefore a very large local Rayleigh number is a necessary condition for the boundary–layer approximation to be valid. Most workers in the general field of boundary–layer flows use such an approach, but it is sufficient to cite the very first two papers dealing with the basic flows in porous media, Cheng and Chang [26] and Cheng and Minkowycz [28], and to mention that most of the linear stability papers cited in [1]–[25] also do so. Stability analyses undertaken using this formulation of the basic flow result in *finite* values for $Ra_x$ which correspond to neutral stability.

**Using a fictitious lengthscale — method 2**

An alternative approach is to define a fictitious lengthscale denoted by $L$. The idea here is that $L$ is used to define a global (rather than a local) Rayleigh number. Therefore if attention is fixed at some station downstream of the leading edge then this could be defined as the dimensional lengthscale and therefore the equivalent nondimensional distance from the leading edge is precisely unity. Further, when the Rayleigh number is much greater than 1, then the boundary–layer is fully developed.

Beginning with the full dimensional equations (3) the following scalings are defined:

$$(X,Y) = L(x,y), \qquad T = T_\infty + \Delta T\,\theta, \qquad \Psi = \alpha\psi, \tag{6}$$

where $\Psi$ is defined in equation (2), and therefore equations (3) become

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = Ra\left[\frac{\partial\theta}{\partial y}\sin\phi - \frac{\partial\theta}{\partial x}\cos\phi\right], \tag{7a}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}. \tag{7b}$$

Here

$$Ra = \frac{\rho_\infty g\beta K L\Delta T}{\mu\alpha} \tag{8}$$

is the Rayleigh number (or Darcy–Rayleigh number) based on the lengthscale $L$. Given the definition of $L$ as the streamwise lengthscale of interest, $x$ should be set to be $O(1)$ as $Ra \to \infty$ in a formal asymptotic analysis.

As interest is focused on how inclinations of the surface from the horizontal modify the horizontal boundary–layer flow, a scale analysis is undertaken which balances the magnitude of the $y$–derivative term on the left hand side of equation (7a) with the $x$–derivative term on the right. Such an analysis also balances the $y$–diffusion term in equation (7b) with the right hand side terms. The resulting scalings follow: $\psi = O(Ra^{1/3})$ and $y = O(Ra^{-1/3})$. Thus, when $Ra \to \infty$ then $y \ll x$ in magnitude and the $x$–derivative diffusion terms in equation (7) may be formally neglected. Indeed, within this asymptotic framework the size of the neglected diffusion terms are easily seen to be $O(Ra^{-2/3})$ relative to the retained terms. The governing parabolic equations may now be obtained using the substitution,

$$\psi = Ra^{1/3}x^{1/3}f(\xi,\eta), \qquad \theta = \theta(\xi,\eta), \tag{9a,b}$$

where $\eta$ and $\xi$ now have the following definitions,

$$\eta = \frac{Ra^{1/3}y}{x^{2/3}}, \qquad \xi = x^{1/3}, \tag{10a, b}$$

and hence

$$f'' = Ra^{1/3}\xi\theta' \sin\phi + [\tfrac{2}{3}\eta\theta' - \tfrac{1}{3}\xi\theta_\xi] \cos\phi, \qquad \theta'' + \tfrac{1}{3}f\theta' = \tfrac{1}{3}\xi[f'\theta_\xi - \theta' f_\xi], \tag{11a, b}$$

are to be solved subject to boundary conditions (5c–f).

The presence of the $Ra$ term in equation (11a) signals a difficulty as this term is asymptotically larger than the remaining terms. However, this difficulty may be avoided by introducing the scaling $\phi = O(Ra^{-1/3})$, which is equivalent to having a heated surface which is very nearly horizontal. If $\phi = Ra^{-1/3}$ is set in (11a), then the governing equations become,

$$f'' = \xi\theta' + \tfrac{2}{3}\eta\theta' - \tfrac{1}{3}\xi\theta_\xi, \qquad \theta'' + \tfrac{1}{3}f\theta' = \tfrac{1}{3}\xi[f'\theta_\xi - \theta' f_\xi], \tag{12a, b}$$

which are identical to equations (5a,b). Other numerical multiples of $Ra^{-1/3}$ will result in equations identical to equations (12) if the definitions of $f$, $\xi$ and $\eta$ are modified slightly; see Rees and Riley [29] or Ingham et al. [30] for details.

With this method an artificial lengthscale has been introduced. The chief advantage of such a device is to allow the boundary–layer to be studied using rigorous methods of asymptotic methods. Here, the Rayleigh number, based on $L$, is taken as the asymptotically large parameter. Definitive statements can be made about the size of neglected terms, something which is done with more difficulty when using method 1, and often is not done. The above–quoted papers, Rees and Riley [29] and Ingham et al. [30], are typical examples of the use of this method.

### Using a lengthscale suggested by the physical parameters — method 3

The definition of $Ra_x$ in expression (4e) may be written in the form,

$$Ra_x = \frac{\rho g\beta K\Delta T \cos\phi}{\mu\alpha} X \equiv X\cos\phi/L \equiv x\cos\phi, \tag{13}$$

and it may be shown that this quantity is dimensionless. Setting $Ra_x = x\cos\phi$ can therefore be thought of as defining a lengthscale, $L$, in terms of the properties of the fluid and the medium. Thus if $L$, which is given by

$$L = \frac{\mu\alpha}{\rho g\beta K\Delta T}, \tag{14}$$

is chosen as a lengthscale, then the value of $Ra$ defined for method 2 is precisely unity, and the value of $Ra_x$ used in method 1 is precisely $x\cos\phi$, where $x$ is nondimensional. The above analysis of method 2 carries through in the same way, except that $Ra = 1$ must be substituted into equations (9) to (11). It is difficult to translate the method 2 criterion for the asymptotic magnitude of $\phi$ into a satisfactory form for study, but method 3 yields no problems when studying the horizontal ($\phi = 0$) self-similar flow, or other self-similar flows.

Analyses undertaken using this method are asymptotic analyses with $x$ as the asymptotically large parameter. Neglected terms in the full elliptic equations have well-defined orders of magnitude, but these are now expressed in terms of $x$. A typical paper which uses this approach is the high-order, boundary-layer analysis of Riley and Rees [31]. Stability analyses result in *finite* values of $x$ beyond which the boundary-layer is deemed unstable.

## Discussion

All three of the above methods have been used to derive the boundary-layer equations for convection in a porous medium, but all three suffer from disadvantages which will now be discussed.

The first method is not often used as an asymptotic method, but rather as an approximation to the governing equations where streamwise diffusion has been neglected. Many of the users of this method, especially those working outside the field of porous medium convection, typically invoke what is termed the boundary-layer approximation by neglecting such derivatives, and then proceed with a set of substitutions analogous to expressions (4). In the interests of balance, it must be said that the higher-order analyses of Chang and Cheng [32] and Cheng and Hsu [33] show that method 1 may be formulated in terms of a rigorous asymptotic analysis. However, method 1 can very easily lead to incorrect results if not applied correctly, and this is especially true for nonsimilar flows. An example of this is the above (method 1) basic flow analysis taken from Jang and Chang [5] which is concerned with vortex instabilities of the inclined thermal boundary-layer flow. Although streamwise diffusion was neglected in favour of cross-stream diffusion, due to the thinness of the boundary-layer, both the temperature terms on the right hand side of equation (3a) were retained. There are good formal reasons for doing this when using method 2, and the inclination of the surface must then be sufficiently small. But there is no mention in the text of Jang and Chang [5] of restrictions on the inclination angle, and it is claimed there that the stability analysis is valid for a wide range of inclinations. However, the balancing of the two temperature terms at $O(1)$ inclinations implies necessarily that both the streamwise and cross-stream diffusion terms must be of the same order of magnitude, and hence that the flow is elliptic. Of course, this negates the original assumption that the boundary-layer approximation is valid and renders their results incorrect.

Method 2 does not suffer from the danger of misapplication as it is a rigorous asymptotic analysis. But, as will be seen below, there are difficulties in presenting a similarly rigorous stability analysis within the same framework. This may be illustrated best by considering briefly the vortex stability of a generally inclined heated surface with $0 < \phi < \frac{\pi}{2}$. If the nondimensionalisation (6) is used, with appropriate scalings for $Z$, $U$, $V$, $W$ and $P$, then equations (1) become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{15a}$$

$$u = -\frac{\partial p}{\partial x} + Ra\,\theta \sin\phi, \qquad v = -\frac{\partial p}{\partial y} + Ra\,\theta\cos\phi, \qquad w = -\frac{\partial p}{\partial z}, \tag{15b, c, d}$$

$$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}+w\frac{\partial\theta}{\partial z}=\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}. \tag{15e}$$

The basic flow is given by $\psi = Ra^{1/2}x^{1/2}f(\eta)$, $\theta = \theta(\eta)$ where $u = \psi_y$, $v = -\psi_x$, $w = 0$ and $\eta$ is now defined according to

$$\eta = Ra^{1/2}y/x^{1/2}. \tag{16}$$

The functions $f(\eta)$ and $\theta(\eta)$ satisfy the equations,

$$f' = g\sin\phi, \qquad g'' + \tfrac{1}{2}fg' = 0, \qquad f(0) = 0, \quad g(0) = 1, \qquad g \to 0 \quad \text{as} \quad \eta \to \infty. \tag{17}$$

Denoting the basic flow by a "$B$" superscript, the linearised equations for perturbations to this flow are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{18a}$$

$$u=-\frac{\partial p}{\partial x}+\theta\sin\phi, \qquad v=-\frac{\partial p}{\partial y}+\theta\cos\phi, \qquad w=-\frac{\partial p}{\partial z}, \tag{18b,c,d}$$

$$u\frac{\partial\theta^B}{\partial x}+v\frac{\partial\theta^B}{\partial y}+u^B\frac{\partial\theta}{\partial x}+v^B\frac{\partial\theta}{\partial y}=\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}. \tag{18e}$$

Assuming that $z = O(y) = O(Ra^{-1/2})$ when $x = O(1)$, see expression (16), and taking $\theta$ to be $O(1)$, since this is a homogeneous system, we now proceed to find scalings for the other variables. Clearly equation (18a) indicates that that the magnitude of $u$ is at most $O(v\,Ra^{1/2})$ where the magnitude of $v$ is to be found. As $w$ and $p_z$ balance in equation (18d), then so do $v$ and $p_y$ in equation (18c). However, from equation (18c), $v$ must be at least of magnitude $O(Ra)$, given the size of the temperature term. In turn, this means that the magnitude of the $v\theta_y^B$ term in equation (18e) is $O(Ra^{3/2})$, which is asymptotically larger than all the other terms in that equation. Apart from the streamwise diffusion term, which is $O(1)$, the others are $O(Ra)$ in size. Thus this formulation would seem to lead to a contradiction as no single term can be asymptotically larger than all the others in an equation. There is a resolution to this problem which involves introducing a thin sublayer within the main boundary–layer, but this will be discussed later.

Why, then, does method 1 seem to succeed in giving what look like good stability analyses when the rigorous method 2 (when applied naively) quite obviously does not? The straightforward answer to this is that method 1 computes a value for $Ra_x$ and therefore the *necessary* requirement for the validity of the boundary–layer approximation, that $Ra_x \gg 1$, is over–ruled. Indeed, the strict imbalance in the sizes of the terms in equation (18e) is ignored in Jang and Chang [5], where the term in their analysis which is equivalent to the present troublesome term is found to be multiplied explicitly by $Ra_x^{1/2}$ (see equation (42) in [5]). Thus, if method 1 is treated as a rigorous asymptotic analysis as $Ra_x \to \infty$, then presentation of results in the form of finite values of $Ra_x$ means that the method is inconsistent.

Stability analyses using method 3 are prone to the same difficulties as method 1. The use of a coordinate, $x$, as the arbitrarily large parameter makes it difficult to formulate this type of analysis, but finite values of $x$ are still computed. In method 3 it is easier to 'hide' the fact that $x$ is asymptotically large. For self–similar flows these values will be the same as those of $Ra_x \cos\phi$ using method 1. However, method 3 is the best way to attempt numerical simulations of the full nonlinear equations where the boundary–layer approximation has not been invoked.

So far, then, the picture that has been painted is a somewhat depressing one. None of the methods used for deriving both the boundary–layer and linearised perturbation equations fare particularly well. Method 2 is the most reliable method for computing the basic flow, but a straightforward linear stability analysis, as expounded above, is seemingly inconsistent. Methods 1 and 3 can give good results for the basic flow, but care must be taken over their implementation. However, in stability analyses, orders of magnitude often are not consistently applied with method 1, and method 3 can only be used with ease when the basic flow is self–similar.

An alternative and quite reasonable view of the method 1 stability analysis is that $Ra_x$ should not be treated formally as a large parameter, but rather that the large $Ra_x$ basic flow is only being used as an approximation to the flow at a finite value of $Ra_x$. This means that there is no inconsistency in computing a finite value of $Ra_x$ in such a stability analysis. The accuracy of the resulting stability criterion then depends on (a) how good an approximation the boundary–layer solution is to the exact solution, and (b) the consequences of neglecting streamwise diffusion in the disturbance equations. With this in mind, the next step is to review the method 1 stability analyses which are contained in the literature.

## LINEAR ANALYSES USING THE LOCAL RAYLEIGH NUMBER

The pioneering work in this field is that of Hsu et al. [1], who studied the vortex instability of convection induced by horizontal heated surfaces with a power–law wall temperature distribution. The basic flow is self–similar (see Cheng and Chang [26]) and the authors follow the method 1 analysis. Their detailed analysis will not be repeated here, but Hsu et al. make mention of the "bottling effect", a term coined by Haaland and Sparrow [34], which is used to allow some terms in the disturbance equations to be neglected. The bottling effect causes the disturbances to be confined to the boundary–layer because there is a net inflow to the boundary–layer caused by fluid entrainment. The streamwise derivative in the disturbance continuity equation is neglected, which allows a disturbance streamfunction to be defined; this is termed the assumption of local similarity. An important feature of the analysis is that the disturbance functions are assumed to display the same functional dependence on $x$ as their basic flow counterparts. Thus both the basic and disturbance temperature fields are taken to be functions of the similarity variable multiplied by $X^m$. Solutions for various exponents, $m$, are presented in terms of neutral curves of $Ra_x$ against a scaled wavenumber, $k$, where $k = aX\,Ra_x^{-1/3}$, and where all disturbances are taken to be proportional to $\exp(iaZ)$, where $a$ is the (constant) wavenumber. These curves are shown in Figures 1 for $0 \le m \le 2$. Hsu et al. [1] also show that the minimum value for $Ra_x$ and the corresponding value of $k$ both vary almost linearly with $m$; this is reproduced in Figure 2.

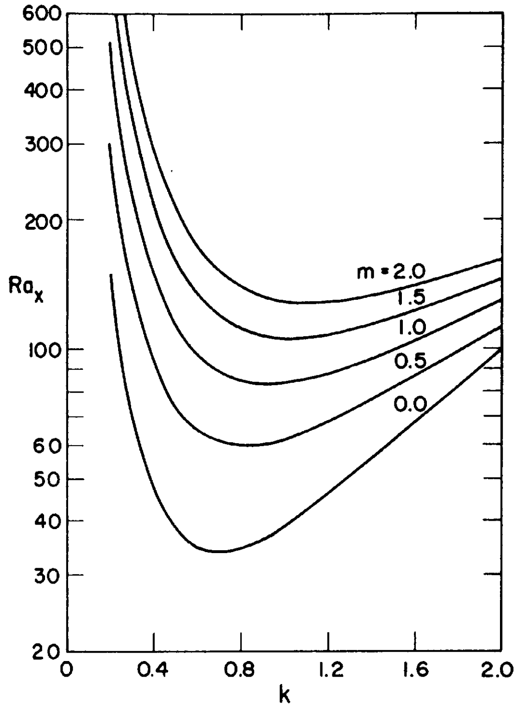



**Figure 1** Rayleigh numbers as a function of spanwise wavenumber at neutral stability. This is figure 2 from Hsu et al. [1]

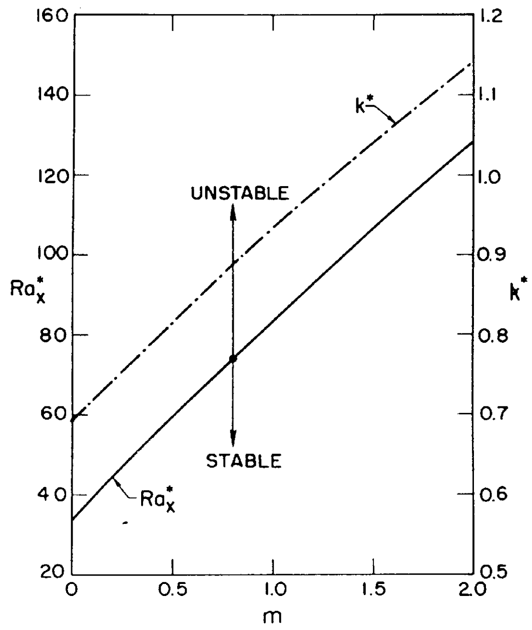



**Figure 2** Critical Rayleigh numbers and associated wavenumbers as a function of $m$. This is figure 3 from Hsu et al. [1]

Typical disturbance streamlines and isotherms corresponding to a cross–section of the vortex system are shown in Figure 3 and confirm that the disturbance is confined to the boundary–layer.

The analysis of Hsu et al. [1] was then applied to inclined surfaces by Hsu and Cheng [2] who followed a method 1 version of the analysis contained between equations (13) and (18). They found that the two parameters, $Ra_x$ and $\phi$ may be combined into a single nondimensional group, an important result, and therefore stability criteria were presented in the form of curves of $Ra_x \cot^2 \phi$ against $k = ax\, Ra_x^{-1/2}$, where $a$ is again the vortex wavenumber. These curves are shown in Figure 4 and the nearly linear dependence of the minimising values of $Ra_x \cot^2 \phi$ and $k$ with $m$ are shown in Figure 5. Note that Hsu and Cheng use $\alpha_0$ as the inclination angle from the vertical and hence $\phi = \pi - \alpha_0$ and $Ra_x \cot^2 \phi = Ra_x \tan^2 \alpha_0$. Clearly these results imply that the vertical thermal boundary–layer is stable, for the critical value of $Ra_x$ becomes asymptotically large as $\phi \to \pi/2$.

Hsu and Cheng [3] then extended the work of Hsu et al. [1] to mixed convection. They found that a free stream velocity proportional to $X^{(2m-1)/3}$ when the surface temperature is proportional to $X^m$ yields a self–similar basic flow. Any other free–stream velocity gives nonsimilarity. The basic flow can be described in terms of $m$ and a mixed convection parameter, $M$, given by $M = Ra_x/Pe_x^{3/2}$, where

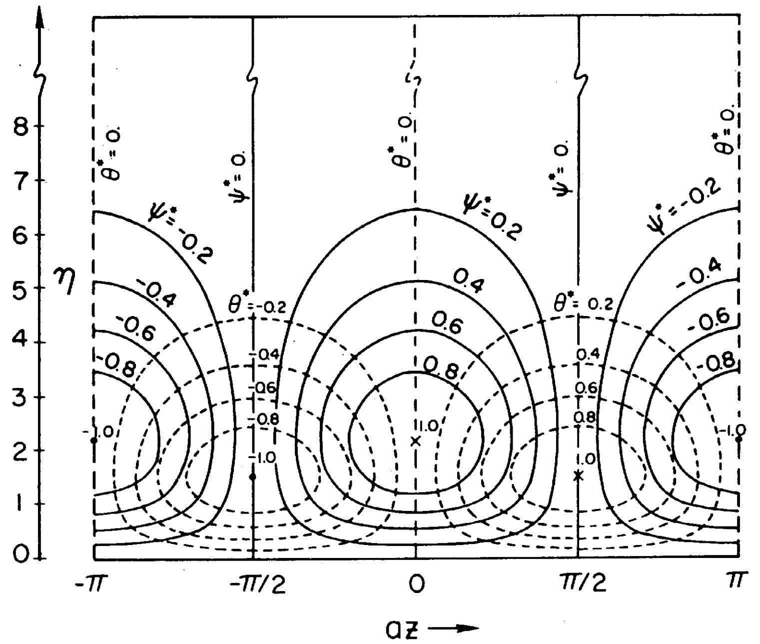
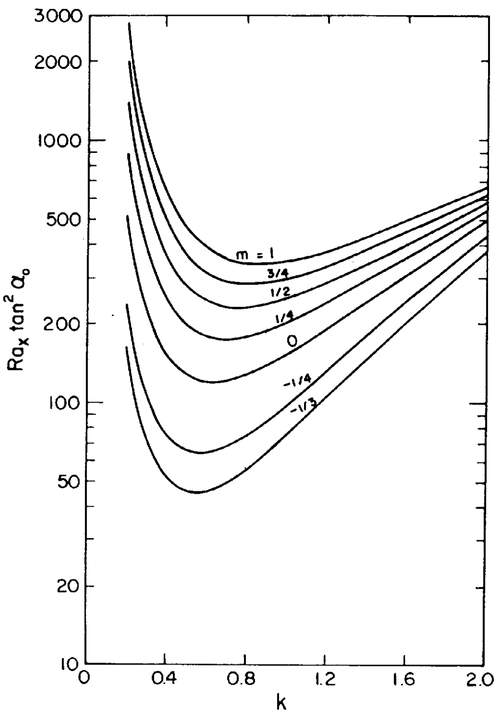


**Figure 3** Secondary flow streamlines (solid lines) and isotherms (dashed lines) at the onset of vortex instability. This is figure 2 from Hsu et al. [1]

$Pe_x = u_\infty X/\alpha$ is a local Peclet number, and $u_\infty(X)$ is the free stream velocity. The stability characteristics are given in terms of the variation of $Pe_x$ with wavenumber for chosen values of $M$ and $m$. It is found that decreasing values of $M$, which correspond to a more strongly forced convection, cause the critical value of $Pe_x$ to increase and the critical wavenumber to decrease. This can be explained by noting that the thinning effect on the basic flow of increasing the free–stream velocity causes the local Rayleigh number based on the local boundary–layer thickness to decrease.

A similar extension to the inclined analysis of Hsu and Cheng [2] was undertaken by the same authors in Hsu and Cheng [4]. In this case the mixed convection parameter is $M = Ra_x/Pe_x$, and the free stream velocity is proportional to $X^m$ in order to retain self–similarity. The qualitative behaviour of the neutral curves is the same as in Hsu and Cheng [4] when the forced flow is in the same direction as that of free convection. But when the forced convection opposes buoyancy forces, the critical wavenumber increases as $M$ decreases. Once more (see Hsu and Cheng [2]) the parameters $Pe_x$ and $\phi$ collapse into a single dimensionless group for presentation of the stability analysis.

The analysis of Hsu and Cheng [2] was re–examined by Jang and Chang [5], who claimed that the former results were not valid as the normal component of the buoyancy force had been neglected. Jang and Chang's formulation of the basic flow is almost identical to that contained in the subsection describing method 1, and this was shown in the first paragraph of the following discussion subsection to be in error. Their basic flow corresponds to a near–horizontal heated surface, rather than a generally inclined one, and therefore their stability results are based on an incorrect basic flow.



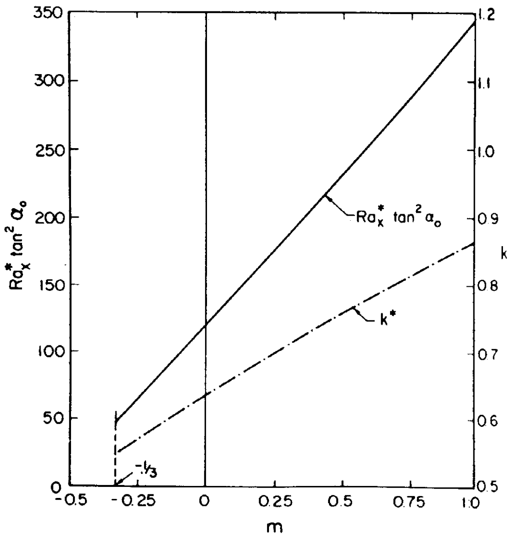




**Figure 4** Values of $Ra_x \tan^2 \alpha_0$ as a function of dimensionless spanwise wavenumber, $k$. This is figure 2 from Hsu and Cheng [2] and is reproduced by permission of A.S.M.E.

**Figure 5** The critical values of $Ra_x \tan^2 \alpha_0$ and wavenumber as functions of $m$. This is figure 3 from Hsu and Cheng [2] and is reproduced by permission of A.S.M.E.

After these first forays into the field of thermal boundary–layer instabilities in porous media, there appeared a long series of papers by Jang, Chang and their co–workers ([6]–[18]) which have detailed the effects of various extensions to the problems considered in [1] to [4]. Each one will be reviewed briefly below, concentrating first on those for which the basic flow is self–similar.

Jang and Chang [6] considered how mass–transfer effects modify the conclusions of Hsu et al. [1]. Jang and Chang, as in all their other papers, assume the principle of local similarity and neglect streamwise derivatives in the perturbation equations. Thus they follow closely the methodology used in Hsu et al. [1]. Attention is restricted to the case of an isothermal and isosolutal surface, since the presence of mass transfer effects already give rise to two extra parameters, namely, the Lewis number, $Le$, and a buoyancy ratio, $N$. The case of $N = 0$ corresponds to having no mass transfer effects, and positive values of $N$ to when mass transfer acts in the same sense as buoyancy effects. Critical values of $Ra_x$ are found to vary markedly with changes in both $Le$ and $N$. It is found that $Ra_x$ decreases when either $Le$ or $N$ increase. $Ra_x$ is independent of $Le$ when $N = 0$, a result which is intuitive. It is worth noting that critical values of $Ra_x$ become very low (less than 10) for large values of $N$ and $Le$. Detailed variations of $Ra_x$ are shown in Figure 6.

Maximum density effects on the stability of inclined free convection flow were studied in Jang and Chang [7]. They considered the stability of water–saturated media near the maximum density point at $T_m = 4°$C. The critical value of

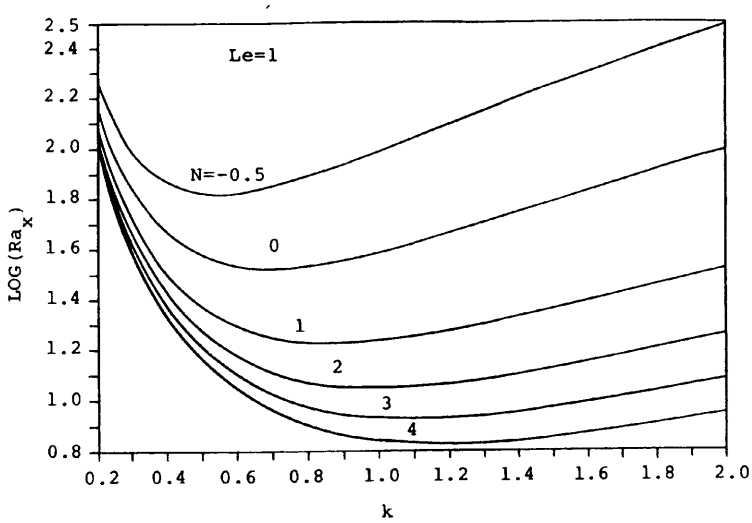



**Figure 6** Neutral stability curves for selected values of $N$ with $Le = 1$. This is figure 7a from Jang and Chang [6]

$Ra_x \cot^2 \phi$ (in our notation) decreases as the absolute value of $(T_m - T_\infty)/(T_w - T_\infty)$ increases. Again it is possible for these values to become less than 10 under some circumstances. Another paper dealing with this problem is by Jang and Chang [8] and it aims to correct the analysis of Jang and Chang [7] in the same way that Jang and Chang [5] attempted to correct that of Hsu and Cheng [2]. As the analysis of Jang and Chang [8] follows very closely that of Jang and Chang [5], save that maximum density effects are included, it is clear that the basic flow presented in Jang and Chang [8] is also invalid, and therefore the stability results are unreliable.

Jang and Leu [9] and Leu and Jang [10] considered how variations with temperature of the viscosity, $\mu$, affect the stability of the horizontal free convective boundary–layer. The former paper is concerned with a uniformly heated surface, whereas the latter, unusually in this review, is concerned with a uniform heat flux surface. The authors assume that the basic flow is self–similar and deduce that an isothermal surface is the only case for which the assumption is true. On assuming an exponential form for the viscosity/temperature relation, the stability analysis shows that the critical value of $Ra_x$ increases as $\mu_w/\mu_\infty$ increases. However, that Nield [35] questions the general conclusion of Leu and Jang [10] because the fact that the Rayleigh number definition is not unique necessarily implies that the occurrence of destabilisation or stabilisation depends on the precise definition of $Ra_x$.

Variable porosity, permeability and thermal diffusivity effects on horizontal free convection were presented by Jang and Chen [11]. Motivated by the fact that porosity changes often occur near a solid boundary (see Vafai [36]), termed the channelling effect, the authors fitted an exponential relation to expressions for these properties. A power–law surface temperature distribution was taken. Self–similarity was ensured by assuming that the variation in permeability, given by $K(Y) = K_\infty(1 + 3e^{-Y/\gamma})$, is made a function of $\eta$; this is done by choosing the "constant", $\gamma$, to be precisely $X/Ra_x^{1/3}$. Although reference is made to other papers citing support for this mathematical device, it is difficult to see how a porous medium could have a region of decreased permeability the thickness of which is an exact proportion of the local boundary–layer thickness. If $K$ is kept as a function of $Y$, a physically reasonable assumption, then the basic flow is nonsimilar and the stability analysis would then need modification.

A study of the combined effects of fluid inertia (Forchheimer form–drag) and thermal dispersion on horizontal free convection was undertaken by Jang and Chen [12]. The basic flow is self–similar only when the power–law exponent of the surface temperature is precisely 0.5. In addition, a uniform external flow allows mixed convection effects also to be considered without loss of self–similarity. Thermal dispersion is found to stabilise the flow, and this becomes increasingly pronounced as the free stream velocity increases. Inertia, on the other hand, destabilises the flow by increasing the thickness of the basic boundary–layer flow.

The remaining papers by Jang and co–workers use nonsimilar flows as the basic flow and therefore the methodology of computing neutral values of $Ra_x$ has to change slightly. Generally the nonsimilar basic flow is computed using the Keller–box method [27], and the value of $Ra_x$ is found by solving the ordinary differential disturbance equations. Basic flow solutions at values of $Ra_x$ lying between the streamwise gridpoints of the Keller–box computation are interpolated to find the neutral curves. It is interesting to note that the Keller–box method itself may be modified to compute eigenvalues such as growth rates (see Lewis et al. [37] and Shu and Wilks [38]), but such an extension of the basic numerical method has not yet been used in stability calculations.

Jang and Lie [13] considered mixed convection flow from horizontal and inclined power–law heated surfaces; in this regard the problem is identical to that of Hsu and Cheng [3]. Again these authors considered that Hsu and Cheng [3] were in error by neglecting the normal component of the buoyancy force. In this case Jang and Lie [13] retain the generally inclined pseudo–similarity variable, $\eta = Ra_x^{1/2}Y/X^{1/2}$. They also assume that both the $X$ and $Y$ temperature derivatives are of the same order of magnitude at $O(1)$ inclination angles. Again, the flow must be elliptic under these conditions and hence the boundary–layer approximation is either invalid, or their flow corresponds only to asymptotically small inclinations from the horizontal.

Inertia (Forchheimer drag) effects on horizontal free convection were considered in Chang and Jang [14] where the surface temperature varies as $X^m$. In general the flow is nonsimilar, but when $m = \frac{1}{2}$ it becomes self–similar. The authors present a thorough study of the basic flow, as this had not previously appeared in the literature. We note that an analysis for the isothermal case has only recently been published, see Rees [39]. Generally it is found that the presence of inertia destabilises the flow. A secondary purpose of his paper was to investigate the effect of the presence of those terms with first degree streamwise derivatives which originate in the advective terms of the full disturbance equations. When present, they tend to stabilise the flow relative to when they are absent. The additional effects of including the advective inertia terms and the Brinkman terms were investigated by Chang and Jang [15]. Much of this paper is taken up with details of the basic flow as that had not been investigated previously. There still remains scope for further analysis in this area. Indeed, even the effect on the basic flow of having the Brinkman term as the only extension to Darcy's law has not yet been studied in detail — unpublished work by the present author indicates a developing two–layer structure within the main boundary–layer at large distances from the leading edge, which is similar to the vertical boundary–layer flow considered by Kim and Vafai [40]. Returning to Chang and Jang [15], the stability analysis depends on a large number of independent parameters and therefore it is difficult to condense the detailed conclusions. The further inclusion of variable porosity, permeability and thermal diffusivity effects, but with no advective inertia terms may be found in Jang and Chen [16].

Horizontal mixed convection flow from a surface with power–law heating, inertia (Forchheimer drag) and boundary (Brinkman) effects are studied in Lie and Jang [17]. Boundary effects are again found to stabilise the flow, as are inertia effects (in contradistinction to the results of Chang and Jang [14]). The addition effect of uniform suction or blowing at the heated surface was considered by Jang et al. [18]. The main conclusion is that suction stabilises the flow, while blowing destabilises it, and, again, this may be related to the local boundary–layer thickness.

## OTHER LINEARISED ANALYSES

All the papers reviewed above consider vortex disturbances. In the analogous field of clear–fluid convection it is well–known that the most dangerous disturbance changes from streamwise vortices to two–dimensional travelling waves as the inclination of the uniformly heated surface becomes more nearly vertical, see Lloyd and Sparrow [41]. It is possible that waves are more dangerous, i.e. have a lower critical $Ra_x$, in porous medium convection. Of all the modifications to Darcy's law considered by Jang, Chang and co–workers, it is the Brinkman term, which adds a 'viscous' term to Darcy's law, which is most likely to affect the qualitative nature of the most dangerous disturbance. It is possible that a remark in Hsu et al. [1], stating that experimental evidence favours vortices, has caused effort to be concentrated almost entirely on vortices, while neglecting waves. However, the paper by Rees and Bassom [19] concludes that waves grow beyond $Ra_x = 28.90$ (in the notation of [1]) compared with $Ra_x = 33.47$ for vortices, see Hsu et al. [1] — this is for an isothermal horizontal surface. Behind this rather surprising and counter–intuitive result lies the question: what is the role played by the assumed $x$–dependence of the disturbances? At present the answer is unknown, but it is extremely likely that these critical values will depend on the chosen form of the $x$–dependence.

A paper by Storesletten and Rees [20] dealing with instability from uniformly heated horizontal or inclined surfaces is concerned with the accuracy of the basic flow. This concern was motivated by the facts that, high–order analyses of the basic flow, such as that of Riley and Rees [31] for horizontal convection, proceeds in powers of $x^{-1/3}$, and the critical value of $x$ for vortices is 33.47, according to Hsu et al. [1]. Thus successive terms in the high–order analysis do not vary greatly in their magnitude at the point of incipient instability, and therefore the leading–order boundary–layer flow is inaccurate there. The aim of Storesletten and Rees [20] was to investigate the effect on stability of including further terms in the representation of the basic flow. As in other high–order analyses, the porous medium is assumed to bounded by two semi–infinite surfaces, the second of which is either insulated or at the ambient temperature, and the included angle between the surfaces is termed the wedge angle. The main conclusion of Storesletten and Rees [20] is that stability results for generally inclined surfaces are very reliable when the heated surface is near the vertical, but that they become increasingly unreliable as the surface tends

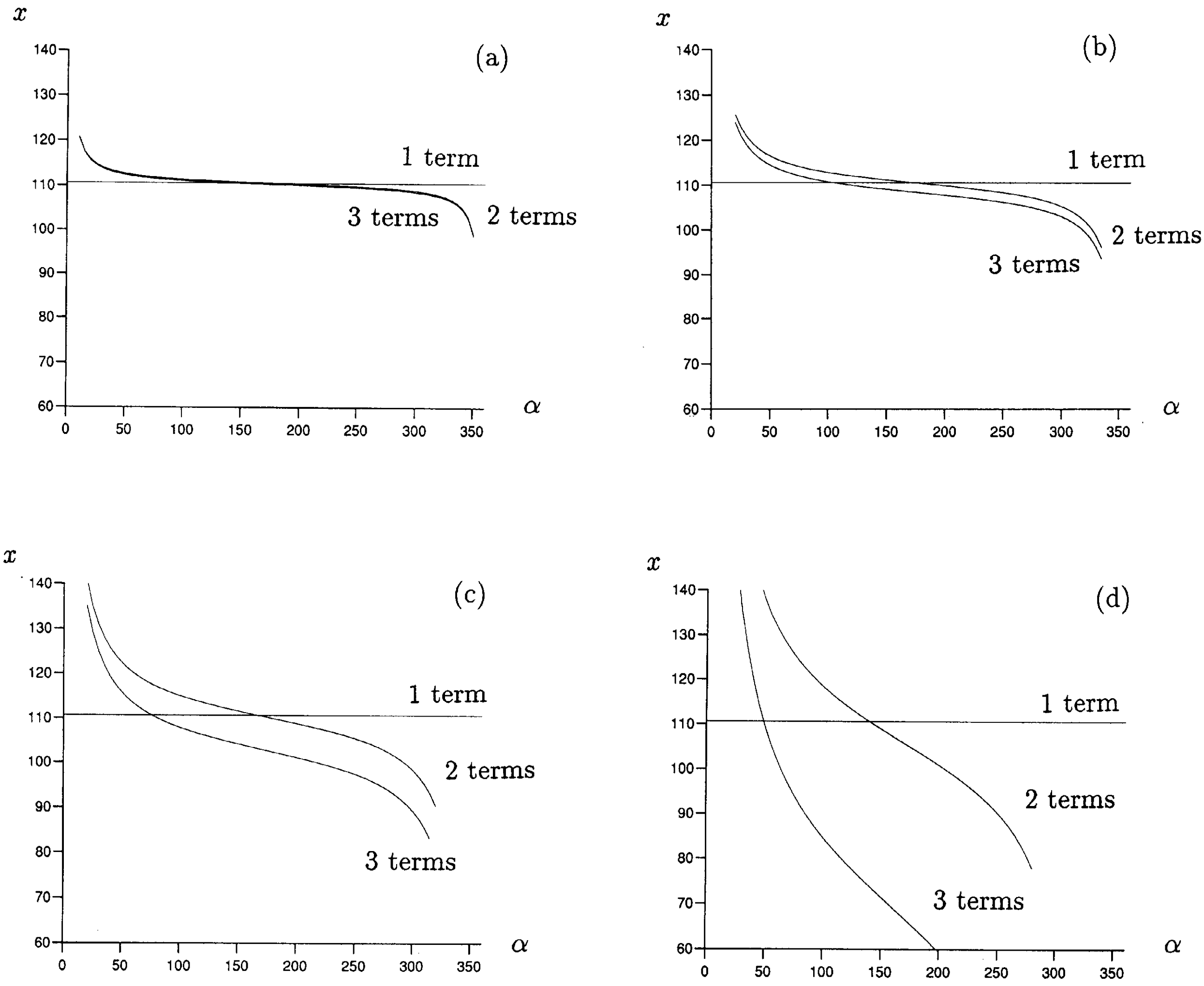



**Figure 7** Variation of the critical distance, $x$, with wedge angle, $\alpha$, and the number of terms in the asymptotic series for the basic flow. (a) $\phi = 85°$; (b) $\phi = 75°$; (c) $\phi = 60°$; (d) $\phi = 30°$. This is figure 2a–d from Storesletten and Rees [20]

towards the horizontal; see Figure 7.

When the surface is horizontal the similarity variable is different, and the analysis has to be reworked. The critical value of $x$ depends very strongly on the wedge angle between the surfaces; see Figure 8. In fact, when the wedge angle is less than about 62° the wavespeed of travelling waves is negative, implying that the instability travels upstream. This unphysical result shows that the basic flow obtained even with a high–order analysis (which goes only as far as the third term because of the appearance of eigensolutions) is still insufficiently accurate for horizontal convection.

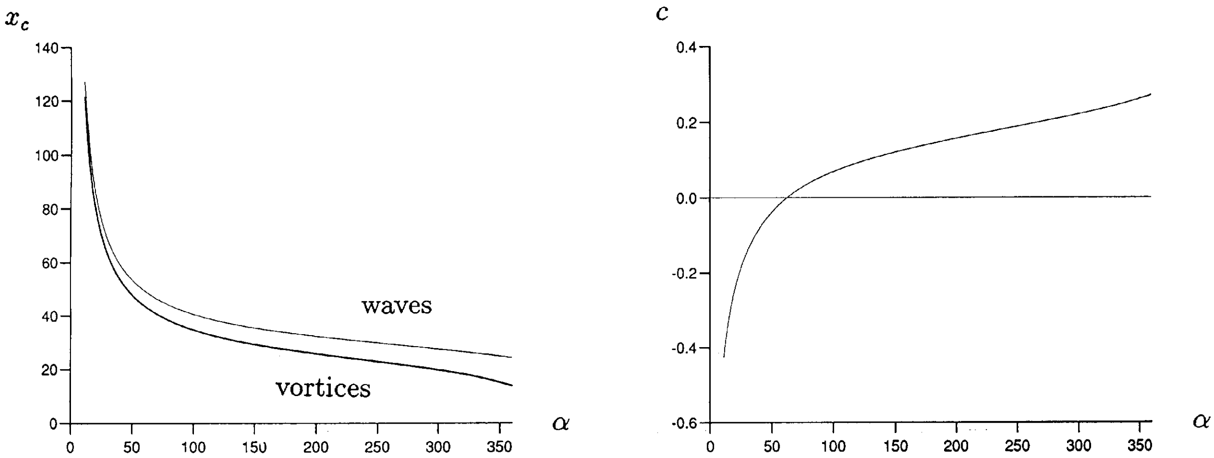



**Figure 8** Variation with wedge angle, $\alpha$, of the critical value of $x$ for waves and vortices, $x_c$, and the wavespeed, $c$. This is figure 4 and figure 5 from Storesletten and Rees [20]

## THE NATURE OF LINEAR STABILITY THEORY

Without exception, papers [1]–[18] use what Jang and Chang [6] term the principle of local similarity to develop their linearised stability theories. In practice streamwise derivatives in the diffusion terms are neglected in the perturbation equations. Then the vortex perturbations are assumed to take the form, $x^m \mathcal{F}(\eta) e^{iaz}$, and the disturbance equations reduce to an ordinary differential eigenvalue problem for the critical distance, $x$ or $Ra_x$, depending on the formulation. This procedure is also known as the parallel–flow approximation. In this section we will explore what it means for linear stability theory to include the effects of non–parallelism of the basic flow. To do this we consider the analogous problem of convection in layers of finite thickness heated from below. In particular, we will consider the Bénard–like problem of Walton [42] which deals with a layer where the gap between the surfaces increases linearly, but very gradually. This nonuniformity is accounted for by using a weakly nonlinear theory and it models the increasing thickness of most boundary–layer flows together with a nonuniform/nonparallel basic flow. A similar analysis for convection in a porous layer results in the same canonical equation. This qualitative analysis is novel, and has not appeared elsewhere in the literature.

The weakly nonlinear analysis of Walton [42] is assumed and not reproduced here. The analysis results in an equation governing the amplitude, $A$, of vortex disturbances to the basic flow:

$$A_\tau = XA - A_{XXXX} - A^3. \tag{19}$$

Here, $X$ is a 'slow' lengthscale (much greater than the vortex wavelength) and its value in the term $XA$ represents the increase in the depth of the layer. The slow timescale is denoted $\tau$. In this discussion the definition of $X$ has nothing to do with that used in the rest of the paper. Equation (19) governs how weakly nonlinear convection evolves in time, and the $A_{XXXX}$ term represents how disturbances diffuse in the streamwise direction, this is a nonparallel term. The Rayleigh number does not appear in this equation but its definition is based on the layer depth at $X = 0$ and its value is precisely $4\pi^2$, the value corresponding to the onset of convection in a uniform layer; see Lapwood [43]. The $A^3$ term in equation (19) arises from nonlinear terms in the governing equations, and it restrains the unbounded growth of linearly unstable convection.

Linear theory will neglect the nonlinear term, but parallel analyses also neglect the $X$-derivative term. Thus the equivalent of papers [1]–[19] in Walton's problem is the equation,

$$A_\tau = XA, \tag{20}$$

which has a solution $A = A_0(X)e^{X\tau}$, where $A_0(X)$ is the disturbance profile at $\tau = 0$. Clearly $X = 0$ corresponds to neutral stability in this context, exponential decay occurs when $X < 0$ and growth when $X > 0$. Thus a well-defined neutral position has been found, qualitatively the same as in [1]–[19].

Consider now the effect of nonparallelism by including the diffusion term. For simplicity we will assume that we are at neutral stability and therefore the time derivative term is absent. Now equation (19) reduces to

$$0 = XA - A_{XXXX}, \tag{21}$$

which is a fourth-order version of Airy's equation. We will assume that solutions are sought in the region $-\infty < X < \infty$, and for physical reasons we require the solution to decay as $X \to -\infty$, but the appropriate solution also decays as $X \to \infty$. A straightforward asymptotic analysis of equation (21) for large negative $X$ shows that $A$ may take any one of the four functional forms,

$$A \sim (-X)^{-5/6} \exp\left[\frac{4}{5\sqrt{2}}(\pm 1 \pm i)(-X)^{5/4}\right], \tag{22a}$$

whereas, for large positive $X$,

$$A \sim X^{-5/6} \exp\left[\pm(4i/5)X^{5/4}\right], \qquad A \sim X^{-5/6} \exp\left[\pm(4/5)X^{5/4}\right], \tag{22b}$$

with these forms being valid up to a multiplicative factor. It is possible to show that there is a unique function which exhibits decay as $X \to \pm\infty$, and we note that this decay is superexponential for negative $X$ but is algebraic (with a decreasing period of oscillation) for positive $X$; in this regard it is similar to the Airy function, $Ai(-X)$. Thus the neutral mode cannot be considered in terms of a location beyond which instability grows, but rather it is a distinct mode whose profile must be computed rather than assumed in advance.

When temporal growth is included, equation (19) becomes

$$A_\tau = XA - A_{XXXX}, \tag{23}$$

which has the solution,

$$A = e^{\lambda\tau} B(X - \lambda), \tag{24}$$

where $B(X)$ satisfies equation (21), and $\lambda$ has an arbitrary real value. As this solution is valid for any value of $\lambda$ we have now a continuous set of modes, indistinguishable by profile, but distinguished by their growth rate, or, equivalently, by their location in space. Assuming that it is possible to extend Fourier analysis to cover solutions of equation (21), then the general solution may be written in terms of an initial disturbance profile using a fourth–order–Airy version of a Fourier integral.

If now a sidewall is placed at some finite, negative value of $X$, thereby restricting the layer to be semi–infinite, the above continuous set of modes becomes a discrete set since the sidewall boundary conditions confer a restriction on which of the original set of modes can be chosen. Such a qualitative change in the nature of the modal set does not occur in the semi–infinite *uniform* Bénard problem with a sidewall, see Cross et al. [44], but once the layer ceases to be uniform, then such a qualitative change takes place. Thus the problem is characterised by a set of modes with discrete values of the exponential growth rate.

How do these qualitative results from the model problem carry over to the instability of boundary–layer flows? First, we note that the basic flow is nonparallel and the depth of the layer increases as $X$ increases, thereby increasing the Rayleigh number based on the local layer depth. Second, it is necessary to be convinced that the presence of a sidewall in the model problem is a good analogue of the leading edge in boundary–layer flows. To see this, consider the inclined uniformly heated surface, the equations for which are given in equations (15). This analysis will follow method 3 and therefore we set $Ra = 1$ in equations (15). Additionally, equations (15) will be extended by assuming that the flow may be unsteady. For illustration purposes we assume that the flow is two–dimensional and therefore let $u = \psi_y$, $v = -\psi_x$ and $w = 0$. Equations (15) reduce to

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial \theta}{\partial y} \sin \phi - \frac{\partial \theta}{\partial x} \cos \phi, \qquad \frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}, \quad (25a, b)$$

where $t$ is nondimensional time. Assume that the porous medium is bounded by a flat surface at $y = 0$, where $\theta = 1$ on $x > 0$ and $\theta_y = 0$ on $x < 0$. As $y \to \infty$ then $\theta \to 0$. When this surface is vertical, the introduction of parabolic coordinates (a Schwartz–Christoffel transformation) renders these full equations soluble precisely in terms of the Cheng and Minkowycz [28] leading order boundary–layer solution. If the parabolic coordinates are defined according to

$$x = \frac{\xi^2 - \eta^2}{4}, \qquad y = \frac{\xi \eta}{2}, \tag{26}$$

then equations (25) are transformed to

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = \tfrac{1}{2}\left(\eta \frac{\partial \theta}{\partial \xi} + \xi \frac{\partial \theta}{\partial \eta}\right) \sin \phi - \tfrac{1}{2}\left(\xi \frac{\partial \theta}{\partial \xi} - \eta \frac{\partial \theta}{\partial \eta}\right) \cos \phi, \tag{27a}$$

$$\frac{(\xi^2 + \eta^2)}{4}\frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial \eta}\frac{\partial \theta}{\partial \xi} - \frac{\partial \psi}{\partial \xi}\frac{\partial \theta}{\partial \eta} = \frac{\partial^2 \theta}{\partial \xi^2} + \frac{\partial^2 \theta}{\partial \eta^2}. \tag{27b}$$

The flow domain within this coordinate system is $0 \leq \xi < \infty$ and $0 \leq \eta < \infty$. Therefore $\xi = 0$, which is the insulated surface, $y = 0$, $x < 0$, may be seen to correspond to the sidewall in the model problem.

There is one difference between this boundary–layer problem and the model problem. The former has a boundary–layer thickness in terms of either $y$ or $\eta$ which varies over $O(1)$ distances in terms of $x$ or $\xi$, whereas the thickness of the layer in the model problem varies very slowly. However, this is only a quantitative difference, the advantage of slow variation in the model problem is that much more analytical progress can be made. The quick variation in the boundary–layer thickness of the present problem means that a weakly nonlinear analysis cannot be undertaken, but it is possible to derive from equations (27) a linearised set of equations which are equivalent to equation (23). If the steady basic flow satisfying equations (27) is denoted by $(\psi^B, \theta^B)$, then equations (27), after linearisation about this solution becomes,

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = \tfrac{1}{2}\Big(\eta\frac{\partial\theta}{\partial\xi} + \xi\frac{\partial\theta}{\partial\eta}\Big)\sin\phi - \tfrac{1}{2}\Big(\xi\frac{\partial\theta}{\partial\xi} - \eta\frac{\partial\theta}{\partial\eta}\Big)\cos\phi, \qquad (28a)$$

$$\frac{(\xi^2+\eta^2)}{4}\frac{\partial\theta}{\partial t} + \frac{\partial\psi^B}{\partial\eta}\frac{\partial\theta}{\partial\xi} + \frac{\partial\psi}{\partial\eta}\frac{\partial\theta^B}{\partial\xi} - \frac{\partial\psi^B}{\partial\xi}\frac{\partial\theta}{\partial\eta} - \frac{\partial\psi}{\partial\xi}\frac{\partial\theta^B}{\partial\eta} = \frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^2\theta}{\partial\eta^2}. \qquad (28b)$$

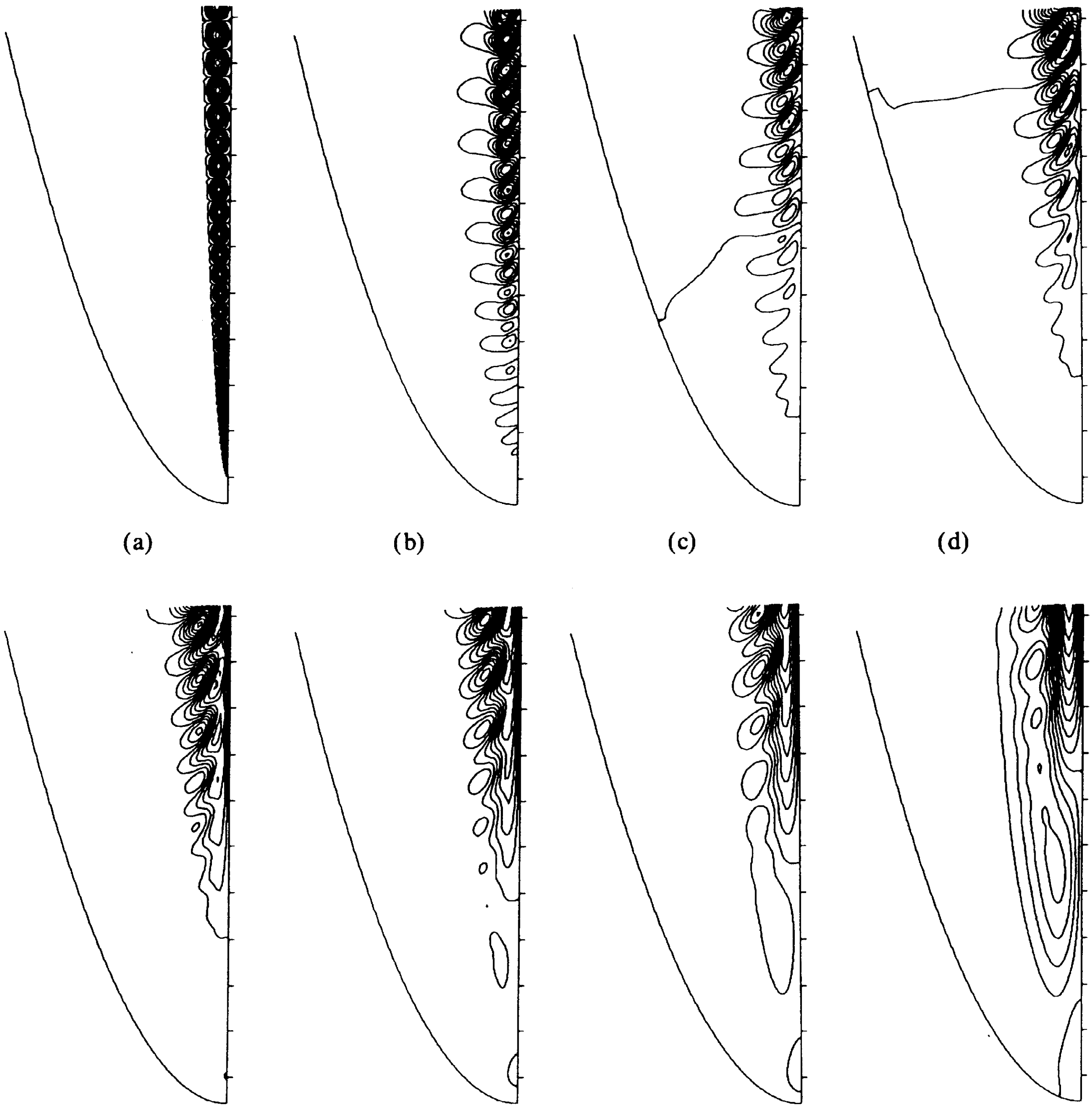
This system is fourth–order in $\xi$, equivalent to $X$ in equation (19), and, when $\psi$ and $\theta$ are set proportional to $e^{\lambda t}$, there results a partial differential eigenvalue problem for the growth rate $\lambda$. Apart from an additional cross–stream dependence on $\eta$, equations (28) are identical in form to equation (23). Although no numerical solutions of equation (28) yet exist, it is clear, given the analogy, that the full linearised problem incorporating nonparallel and streamwise diffusion effects will yield an infinite set of modes each with their associated growth rate. Moreover, this set of growth rates will form a discrete set, rather than a continuous set. Thus the full linearised equations, like the model problem, cannot give a critical distance from the leading edge beyond which disturbances grow. This paradigm shift may be difficult to accept, but it is the inevitable consequence of the presence of streamwise diffusion in the linearised equations.

## ANALYSES BASED ON THE EXACT SOLUTION

The remaining papers in this review do not use the boundary–layer approximation to the steady basic flow as a basis for stability analyses. Rather, the exact solutions of Rees and Bassom [45] were used. In all cases in this section the Rayleigh number is scaled out of the full governing equations, as per method 3.

### The vertical surface

The simplest possible case of Darcy flow induced by a uniformly heated vertical surface was treated in Rees [21] and Lewis et al. [22]. Both papers use the parabolic coordinate system defined by expressions (26), and equations (28) were solved with $\phi = \pi/2$. In Rees [21], direct numerical simulations using the full equations were presented, and it was concluded that even large perturbations to the basic steady



(a) (b) (c) (d)

(e) (f) (g) (h)

**Figure 9** Perturbation isotherms for the disturbance shown in (a). Isotherms are plotted at intervals of $|\delta\theta|_{\max}/10$ for each frame. Frame (a) has $|\delta\theta|_{\max} = 1$, and frame (h) has $|\delta\theta|_{\max} = 1.147 \times 10^{-3}$. (a) $t = 0$; (b) $t = 100$; (c) $t = 200$; (d) $t = 300$; (e) $t = 400$; (f) $t = 500$; (g) $t = 600$; (h) $t = 850$. Reproduced by permission of Birkhäuser–Verlag. This is figure 5 from Lewis et al. [21]

flow decay eventually; a typical solution in terms of perturbation isotherms is presented in Figure 9.

If an analogy is drawn between this flow and that of a vertical channel with a horizontal temperature drop (the Darcy–Bénard problem rotated through 90°), then such a result is not surprising for Gill [46] demonstrated analytically that the layer problem is linearly stable for all values of the Rayleigh number. However, the result of Rees [21] cannot be considered definitive because a numerical simulation of this sort must, necessarily, involve only a finite computational domain. This observation motivated the asymptotic analysis of Lewis et al. [22]. In this paper the authors first re–examined Gill's work to determine the growth rates of wave disturbances in the limit of large Rayleigh numbers and showed that the exponential growth rate is given by

$$\lambda = i\alpha Ra - 0.56(1 + i\sqrt{3})(\alpha Ra)^{2/3} + \dots , \tag{29}$$

as $Ra \to \infty$, where $\alpha$ is the wavenumber. As the leading order real term is negative, this confirms and quantifies Gill's result. The disturbance is found to split into a two–layer structure: one region is of thickness $O(Ra^{-1/3})$ immediately next to one of the walls, and the other fills the rest of the channel. Equation (29) applies to modes concentrated near the cold wall. For the vertical boundary–layer, the same analysis yields

$$\lambda = -\frac{2i\alpha}{\xi} - 1.46(1 - i\sqrt{3})\Big(\frac{\alpha}{\xi}\Big)^{4/3} + \dots . \tag{30}$$

From this we see that the magnitude of the real part of $\lambda$ decays as $\xi$ increases, but that it remains negative. Thus the boundary–layer becomes less stable further downstream, but it always remains stable. Again, in the large–$\xi$ limit the disturbance develops a two layer structure with the modes concentrated mainly in a thin layer of thickness $\eta = O(\xi^{-1/3})$.

### The horizontal surface

When the heated surface is horizontal, instability is expected *a priori* since the steady basic boundary–layer flow is unstably stratified. According to Hsu et al. [1], the point of incipient instability is very close to the leading edge. We have already seen how the inaccuracy caused by a poor representation of the basic flow affects stability criteria, and it is very likely that the principle of local similarity used by many authors is similarly affected. Thus there are only two possible avenues of study for the horizontal problem. The first is to develop an asymptotic analysis which is valid far downstream of the leading edge, and the other is to solve the full governing equations. The former is undertaken in Bassom and Rees [23], and first steps towards the latter in Rees [24] and Rees and Bassom [25].

In Bassom and Rees [23] the analysis was carried out in the coordinate system defined by

$$x = \xi(\xi^2 - 3\eta^2)/27, \qquad y = \eta(3\xi^2 - \eta^2)/27, \tag{31}$$

which is again a Schwartz–Christoffel transformation; see [45]. The stability results of Hsu et al. [1] were recomputed in this coordinate system in order to determine the neutral curve in terms of $\xi$ against the vortex wavenumber. This curve gives slightly different values from that of Hsu et al. [1] because the exact solution is used as the basic profile. The thrust of the paper is to determine the asymptotic structure of disturbances at large values of $\xi$ (or $x$). This is a long and involved analysis, and the reader is referred to that paper for details, but again disturbances are found to inhabit a thin region (in terms of $\eta$) immediately adjacent to the heated surface. In this case it is shown that $\eta = O(\xi^{-1/3})$ is the thickness of this sublayer.

Numerical simulations of the full, unsteady two–dimensional equations of motion,

$$\psi_{\xi\xi} + \psi_{\eta\eta} = \frac{1}{9}\Big[2\xi\eta\theta_\eta + (\eta^2 - \xi^2)\theta_\xi\Big], \qquad \theta_t = \frac{81}{(\xi^2+\eta^2)^2}\Big[\theta_{\xi\xi} + \theta_{\eta\eta} + \psi_\xi\theta_\eta - \psi_\eta\theta_\xi\Big], \tag{32}$$

are presented in Rees [24] and in much more detail in Rees and Bassom [25]. The central implications of this study are (i) localised disturbances placed well beyond the theoretical point of neutral stability propagate both upstream and downstream, showing that ellipticity in an inherent part of the evolution of disturbances; (ii) developing waves may be ejected from the boundary–layer to form plumes; (iii) the strength of such eruptions can be such that the additional entrainment of fluid they induce causes substantial thinning of the boundary–layer even very close to the leading edge; (iv) cell merging occurs; (v) the flow is essentially chaotic, though the flow may, for short periods of time and at locations sufficiently close to the leading edge, remain roughly periodic in time before events downstream affect the flow. An example of a simulation from Rees [24] is shown in Figure 10, where cell–merging and the creation of recirculating regions may be seen. This figure shows that waves grow in strength while advecting downstream and that these waves seem to maintain a roughly constant wavelength, at least when they are fairly close to the leading edge. As other figures in Rees and Bassom [25] indicate, the further away from the leading edge, the more chaotic the behaviour of the flow seems to be. Given that waves seem not to propagate upstream closer to the leading edge than about $x = 80$ (c.f. the critical value of roughly 28 given in Rees and Bassom [19]) it is clear that these numerical simulations also support the misgivings about the accuracy of the basic flow used in linearised analyses using the parallel–flow approximation. Clearly Rees and Bassom [25] only represents the first faltering steps in what remains to be a large research programme, but it does indicate that there is much exciting work still to be done. Of particular interest are vortex simulations, but since the flow will then be three–dimensional, the numerical effort will be substantially greater.

## CONCLUSION

In this review an exhaustive survey has been presented of papers dealing with the instability of thermal boundary layers in porous media. In addition, a large amount of space has necessarily been devoted to various theoretical issues which have a very strong bearing on the topic. The methods used to analyse not only the basic flow, but also their instability, have been subject to careful scrutiny.

Many works have appeared which use linearised analyses based on the parallel–flow approximation. Although a large number of extensions to Darcy's law have been studied, the overall method itself has been called to account. The analysis of Storesletten and Rees [20], in particular, shows that the inaccuracy of the leading order boundary–layer solution has a strong effect on the stability criteria. It also shows that when the heated surface is nearly vertical, then such results may be regarded as reliable. Detailed asymptotic analyses valid for large distances from the leading have been undertaken, and these account for nonparallel effects. For the vertical thermal boundary–layer such results are very useful in determining that the flow is stable. For the horizontal boundary–layer it is doubtful whether the results

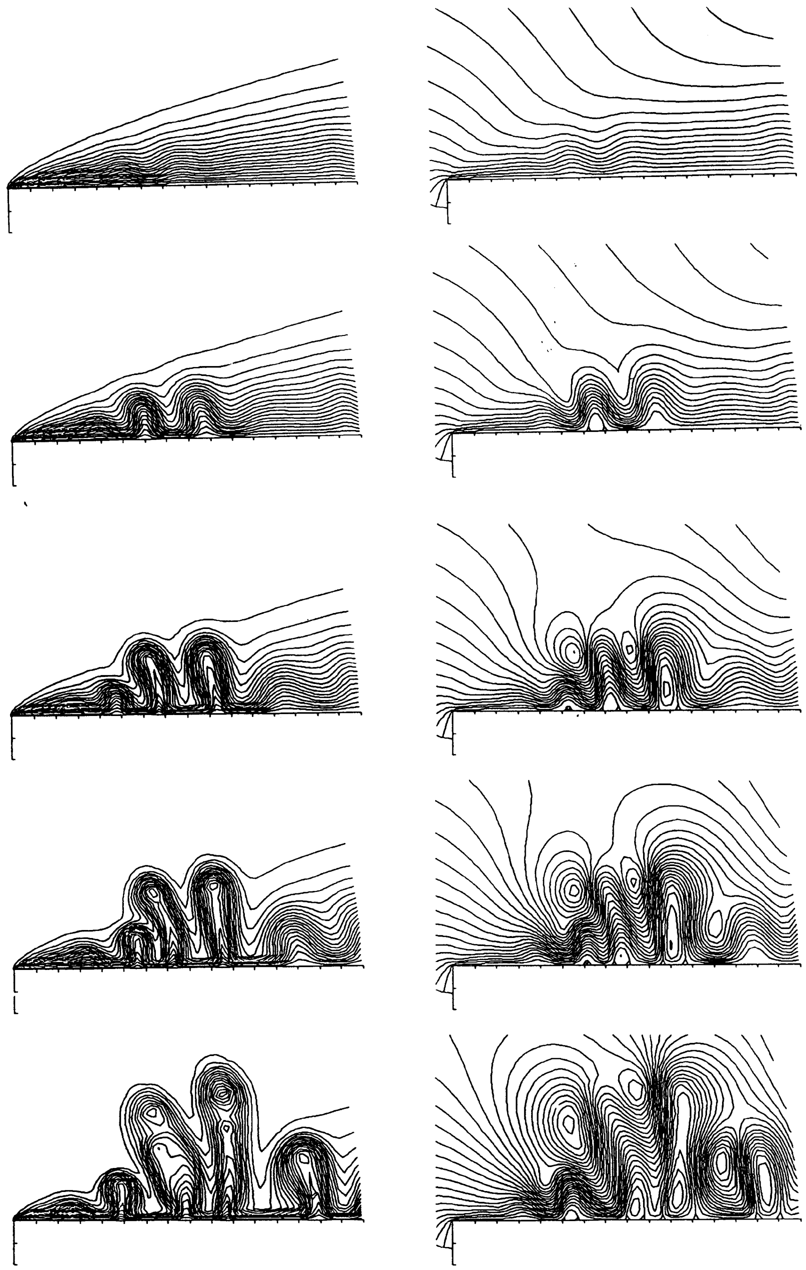


**Figure 10** Instantaneous isotherms (left) and streamlines (right) at successive points in time. Tickmarks on the axes are at intervals of 250. Reproduced by permission of Springer–Verlag. This is figure 1 from Rees [24]

of the analysis are of practical importance, since instabilities enter the strongly nonlinear regime even at stations fairly close to the leading edge. Solutions of the full linearised equations are not yet available; these would enable us to determine modal growth rates and hence the detailed effect of ellipticity (or streamwise diffusion, or nonparallelism). No weakly nonlinear analyses have appeared in the literature, although it may be possible to apply the techniques of Chen et al. [47] to vortex disturbances in near–vertical boundary layers.

Only a few numerical simulations of the full governing equations have been undertaken, and those have concentrated only on two–dimensional flows. At present nothing is known with any assurance about the form of the preferred mode of instability when the heated surface is close to the horizontal, even for a relatively simple flow such as that given by Darcy's law. Therefore is very likely that the results even of Rees and Bassom [25] may not be definitive; this conjecture can only be tested by solving the full three–dimensional equations. Another numerical approach which may prove informative but with a much reduced computational time is the Parabolised Stability Equations method. This method has been used to great effect in the computation of the nonlinear development of instabilities in the Blasius boundary–layer, see Bertolotti et al. [48], and many other boundary–layer flows. A very recent review of this methodology is given in Herbert [49].

Clearly the work which remains in order to give a good understanding of instability mechanisms and their subsequent temporal evolution is substantial. When other effects, such as those considered by Jang, Chang and co–workers, are included in the analysis, it could very well be the case that qualitative results, such as the form of the most dangerous disturbance, may change. Indeed this is quite likely to be the case for Darcy–Brinkman flow, as the resulting equations are much closer to the familiar Navier–Stokes equations than are those for Darcy flow, and as the vertical thermal boundary–layer flow of a clear fluid is destabilised by waves, rather than by vortex disturbances.

**Acknowledgements**

The author would like to express his indebtedness to his co–workers, Prof. D. S. Riley, Prof. L. Storesletten, Dr. A. P. Bassom and Miss Samantha Lewis for their part in the work described here. The author is also grateful to Prof. D. A. Nield for bringing references Leu and Jang [10] and Nield [35] to his attention.

## REFERENCES

1. C.T. Hsu, P. Cheng, and G.M. Homsy, Instability of free convection flow over a horizontal impermeable surface in a porous medium, *Int. J. Heat Mass Transf.* **21**, 1221–1228 (1978).
2. C.T. Hsu and P. Cheng, Vortex instability in buoyancy induced flow over inclined heated surfaces in porous media, *Trans. A.S.M.E. J. Heat Transf.* **101**, 660–665 (1979).
3. C.T. Hsu and P. Cheng, Vortex instability of mixed convection flow in a semi–infinite porous medium bounded by a horizontal surface, *Int. J. Heat Mass Transf.* **23**, 789–798 (1980).
4. C.T. Hsu and P. Cheng, The onset of longitudinal vortices in mixed convective flow over an inclined surface in a porous medium, *Trans. A.S.M.E. J. Heat Transf.* **102**, 544–549 (1980).
5. J.Y. Jang and W.J. Chang, Vortex instability of buoyancy–induced inclined boundary–layer flow in a saturated porous–medium, *Int. J. Heat Mass Transf.* **31**, 759–767 (1988).
6. J.Y. Jang and W.J. Chang, The flow and vortex instability of horizontal natural–convection in a porous–medium resulting from combined heat and mass buoyancy effects, *Int. J. Heat Mass Transf.* **31**, 769–777 (1988).

7. J.Y. Jang and W.J. Chang, Vortex instability of inclined buoyant layer in porous media saturated with cold water, *Int. Comm. Heat Mass Transf.* **14**, 405–416 (1987).
8. J.Y. Jang and W.J. Chang, Maximum density effects on vortex instability of horizontal and inclined buoyancy–induced flows in porous media, *Trans. A.S.M.E. J. Heat Transf.* **111**, 572–574 (1989).
9. J.Y. Jang and J.S. Leu, Variable viscosity effects on the vortex instability of free–convection boundary–layer flow over a horizontal surface in a porous–medium, *Int. J. Heat Mass Transf.* **36**, 1287–1294 (1993).
10. J.S. Leu and J.Y. Jang, Variable viscosity effects on the vortex instability of the convective boundary layer flow over a horizontal uniform heat flux surface in a saturated porous medium, *Proc. 6th Int. Symp. on Transport Phenomena and Thermal Engineering, Seoul, Korea,* **1**, 203–208 (1993).
11. J.Y. Jang and J.L. Chen, Variable porosity effect on vortex instability of a horizontal mixed convection flow in a saturated porous–medium, *Int. J. Heat Mass Transf.* **36**, 1573–1582 (1993).
12. J.Y. Jang and J.L. Chen, Thermal dispersion and inertia effects on vortex instability of a horizontal mixed convection flow in a saturated porous–medium, *Int. J. Heat Mass Transf.* **36**, 383–389 (1993).
13. J.Y. Jang and K.N. Lie, Vortex instability of mixed convection flow over horizontal and inclined surfaces in a porous–medium, *Int. J. Heat Mass Transf.* **35**, 2077–2085 (1992).
14. W.J. Chang and J.Y. Jang, Inertia effects on vortex instability of a horizontal natural–convection flow in a saturated porous–medium, *Int. J. Heat Mass Transf.* **32**, 541–550 (1989).
15. W.J. Chang and J.Y. Jang, Non–Darcian effects on vortex instability of a horizontal natural–convection flow in a porous–medium, *Int. J. Heat Mass Transf.* **32**, 529–539 (1989).
16. J.Y. Jang and J.L. Chen, Variable porosity and thermal dispersion effects on vortex instability of a horizontal natural–convection flow in a saturated porous–medium, *Wärme–Stoffübertrag.* **29**, 153–160 (1994).
17. K.N. Lie and J.Y. Jang, Boundary and inertia effects on vortex instability of a horizontal mixed convection flow in a porous–medium, *Num. Heat Transf. Part A – Applications* **23**, 361–378 (1993).
18. Jang, J. Y., K.N. Lie and J.L. Chen, The influence of surface mass flux on vortex instability of a horizontal mixed convection flow in a saturated porous–medium, *Int. J. Heat Mass Transf.* **38**, 3305–3311 (1995).
19. D.A.S. Rees and A.P. Bassom, The linear wave instability of boundary–layer flow–induced by a horizontal heated surface in porous media, *Int. Comm. Heat Mass Transf.* **21**, 143–150 (1994).
20. L. Storesletten and D.A.S. Rees, The influence of higher–order effects on the linear instability of thermal boundary layer flow in porous media, Submitted to *Int. J. Heat Mass Transf.*
21. S. Lewis, A.P. Bassom and D.A.S. Rees, The stability of vertical thermal boundary layer flow in a porous medium, *Eur. J. Mech. B: Fluids* **14**, 395–408 (1995).
22. D.A.S. Rees, Nonlinear wave stability of vertical thermal boundary layer flow in a porous medium, *Journal of Applied Mathematics and Physics (Z.A.M.P.)* **44**, 306–313 (1993).

23. A.P. Bassom and D.A.S. Rees, The linear vortex instability of flow–induced by a horizontal heated surface in a porous–medium, *Quart. J. Mech. Appl. Math.* **48**, 1–19 (1995).
24. D.A.S. Rees, Numerical simulation of thermal boundary layer instabilities in porous media, *Proc. I.U.T.A.M. Symposium on "Nonlinear instabilities of nonparallel flows", Clarkson University, New York* 242–247 (1993).
25. D.A.S. Rees and A.P. Bassom, The nonlinear nonparallel wave instability of boundary–layer flow–induced by a horizontal heated surface in porous media, *J. Fluid Mech.* **253**, 267–295 (1993).
26. P. Cheng and I.–D. Chang, On buoyancy-induced flows in a saturated porous medium adjacent to horizontal heated surfaces, *Int. J. Heat Mass Transf.* **19**, 1267–1272 (1976).
27. H.B. Keller and T. Cebeci, Accurate numerical methods for boundary layer flows 1. Two dimensional flows, *Proc. Int. Conf. Numerical Methods in Fluid Dynamics. Lecture Notes in Physics* Springer–New York (1971).
28. P. Cheng and W.J. Minkowycz, Free convection about a vertical flat plate imbedded in a porous medium with application to heat transfer from a dike, *J. Geophys. Res.* **82**, 2040–2044 (1977).
29. D.A.S. Rees and D. S. Riley, Free convection above a near–horizontal semi–infinite heated surface embedded in a saturated porous medium, *Int. J. Heat Mass Transf.* **28**, 183–190 (1985).
30. D.B. Ingham, J.H. Merkin and I. Pop, Natural convection from a semi–infinite flat plate inclined at a small angle to the horizontal in a saturated porous medium, *Acta Mechanica* **57**, 183–202 (1985).
31. D.S. Riley and D.A.S. Rees, Non–Darcy natural convection from arbitrarily inclined heated surfaces in saturated porous media, *Quart. J. Mech. Appl. Math.* **38**, 277–295 (1985).
32. I.–D. Chang and P. Cheng, Matched asymptotic expansions for free convection about an impermeable horizontal surface in a porous medium, *Int. J. Heat Mass Transf.* **26**, 163–174 (1983).
33. P. Cheng and C.T. Hsu, Higher order approximations for Darcian free convection about a semi–infinite vertical flat plate, *Trans. A.S.M.E. J. Heat Transf.* **106**, 143–151 (1984).
34. S.E. Haaland and E.M. Sparrow, Vortex instability of natural convection flow on inclined surfaces, *Int. J. Heat Mass Transf.* **16**, 2355–2367 (1973).
35. D.A. Nield, Estimation of an effective Rayleigh number for convection in a vertically inhomogeneous porous medium or clear fluid, *Int. J. Heat Fluid Flow* **15**, 337–340 (1994).
36. K. Vafai, Convective flow and heat transfer in variable–porosity media, *J. Fluid Mech.* **147**, 233–259 (1984).
37. S. Lewis, D.A.S. Rees and A.P. Bassom, High wavenumber convection in tall porous containers heated from below, To appear in *Quart. J. Mech. Appl. Math.*
38. J.J. Shu and G. Wilks, Mixed convection laminar–film condensation on a semi–infinite vertical plate, *J. Fluid Mech.* **300**, 207–229 (1995).
39. D.A.S. Rees, The effect of inertia on free convection from a horizontal surface embedded in a porous medium, *Int. J. Heat Mass Transf.* **39**, 3425–3430 (1996).

40. S.J. Kim and K. Vafai, Analysis of natural convection about a vertical surface embedded in a porous medium, *Int. J. Heat Mass Transf.* **32**, 665–677 (1989).
41. J.R. Lloyd and E.M. Sparrow, On the instability of natural convection flow on inclined plates, *J. Fluid Mech.* **42**, 465–470 (1970).
42. I.C. Walton, On the onset of Rayleigh–Bénard convection in a fluid layer of slowly varying depth, *Stud. Appl. Math.* **67**, 199–216 (1982).
43. E.R. Lapwood, Convection of a fluid in a porous medium, *Proc. Camb. Phil. Soc.* **44**, 508–521 (1948).
44. M.C. Cross, P.G. Daniels, P.C. Hohenberg and E.D. Siggia, Phase–winding solutions in a finite container above the convective threshold, *J. Fluid Mech.* **127**, 155–183 (1983).
45. D.A.S. Rees and A.P. Bassom, Some exact solutions for free convective flows over heated semi–infinite surfaces in porous media, *Int. J. Heat Mass Transf.* **34**, 1564–1567 (1991).
46. A.E. Gill, A proof that convection in a porous medium is stable, *J. Fluid Mech.* **35**, 545–547 (1969).
47. C.C. Chen, A. Labhari, H.–C. Chang, and R.E. Kelly, Spanwise pairing of finite–amplitude longitudinal vortex rolls in inclined free–convection boundary layers, *J. Fluid Mech.* **231**, 73–111 (1991).
48. F.P. Bertolotti, Th. Herbert and P.R. Spalart, Linear and nonlinear stability of the Blasius boundary layer, *J. Fluid Mech.* **242**, 441–474 (1992).
49. Th. Herbert, Parabolized stability equations, *Ann. Rev. Fluid Mech.* **29**, 245–283 (1997).