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A Derivation of the Volume-Averaged Boussinesq Equations for Flow in Porous Media with Viscous Dissipation

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Abstract. The volume-averaged equations are derived for convective flow in porous media. In the thermal energy equation viscous dissipation is taken into account, and a suitable form is obtained which is valid when Brinkman effects are significant.

Key words: viscous dissipation, convective flows, Brinkman terms, volume averaging.

1. Introduction

The last few years have seen a substantial rise in the number of papers which are concerned with the effects of viscous dissipation in porous medium flow. The review article by Magyari *et al.* (2005) cites and discusses many of these papers, which are concerned with free, forced and mixed convection flows. It is well-known that the precise way in which the viscous dissipation term is written down depends on the model used for the momentum equation. For example, if Darcy's law is assumed to be valid and the medium is isotropic, then the viscous dissipation terms are proportional to the sum of the squares of the velocity components, $\underline{u}.\underline{u}$. Although there is no disagreement about this, there is, however, a difference of opinion over how viscous dissipation should be modelled when Brinkman's extension to Darcy's law is valid. Nield (2000) contends that the correct 'Brinkman' dissipation term is proportional to $\underline{u}.\nabla^2\underline{u}$ and has a negative coefficient. This is justified using a drag force argument. Al-Hadhrami *et al.* (2003) use an argument based on the work done by frictional forces to obtain a term proportional to

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$$2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2. \tag{1}$$

This is identical to the term used for viscous dissipation in a clear Newtonian fluid. Al-Hadhrami *et al.* (2003) state correctly that their viscous dissipation term gives a smooth transition between the Darcy-flow limit and the clear-flow limit. In addition they argue that Nield's (2000) form can, in certain circumstances, yield negative values for the viscous dissipation, which is physically unacceptable. On the other hand, Nield (2000) questions the manner in which Al-Hadhrami *et al.* (2003) use the stress tensor in their derivation, even though the final expression appears to give satisfactory behaviour.

The present study, therefore, aims to derive an appropriate parameterization of the dissipation rate by means of a rigorous volume averaging of the Boussinesq equations.

2. Derivation of the Volume-Averaged Boussinesq Equations

In a porous medium, a fluid phase and a solid phase can be distinguished. Some quantities, such as the temperature and the density, are defined in both phases. To distinguish the two phases, the subscripts β and σ are used to refer to the fluid and the solid phase, respectively. Other quantities, such as the velocity and the pressure, are defined only in the fluid phase. In these cases the subscript β is omitted in order to simplify the notation.

The full Boussinesq equations for the fluid phase read

$$\frac{\partial u_i}{\partial x_i}=0, \tag{2}$$

$$\rho_\beta\left(\frac{\partial u_i}{\partial t}+\frac{\partial u_i u_j}{\partial x_j}\right) = -\rho_\beta\beta(\theta_\beta-\theta_{\mathrm{r}})\,g_i-\frac{\partial p}{\partial x_i}+2\mu_\beta\frac{\partial e_{ij}}{\partial x_j}, \tag{3}$$

$$(\rho c_{\mathrm{p}})_\beta\frac{\partial\theta_\beta}{\partial t}+(\rho c_{\mathrm{p}})_\beta\frac{\partial u_j\theta_\beta}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\lambda_\beta\frac{\partial\theta_\beta}{\partial x_j}\right)+2\mu_\beta e_{ij}^2, \tag{4}$$

where u_i is the velocity component in direction i, p the pressure, θ_β the potential temperature of the fluid phase and θ_{r} the reference temperature, $e_{ij}=(1/2)\left((\partial u_i/\partial x_j)+(\partial u_j/\partial x_i)\right)$ is the rate-of-strain tensor, β is the coefficient of cubical expansion, g_i is the gravitational acceleration, ρ is the mass density, μ is the dynamic viscosity, c_{p} is the specific heat at constant pressure, and λ is the thermal conductivity.

The thermal energy equation for the solid phase reads

$$(\rho c_{\mathrm{p}})_\sigma \frac{\partial \theta_\sigma}{\partial t} \;=\; \frac{\partial}{\partial x_j}\left(\lambda_\sigma \frac{\partial \theta_\sigma}{\partial x_j}\right). \tag{5}$$

The first step in the derivation of the volume-averaged equations is the introduction of the *superficial* volume average, which is denoted by $\langle \ldots \rangle$. The superficial volume-average of an arbitrary quantity Ψ is defined by Quintard and Whitaker (1994) as

$$\langle \Psi_\alpha \rangle(\underline{x}) \;=\; \int_V \gamma_\alpha(\underline{x}+\underline{y}) m(\underline{y}) \Psi_\alpha(\underline{x}+\underline{y}) \mathrm{d}V, \tag{6}$$

where the subscript α indicates the phase ($\alpha = \beta$ or σ), $\underline{y}$ is position vector relative to the centroid $\underline{x}$ of the averaging volume V, m is a weighting function and γ_α is the phase-indicator function equal to 1 in phase α and 0 outside phase α. A superficial volume-average is related to an *intrinsic* volume-average, which is denoted by $\langle \ldots \rangle^\alpha$, according

$$\langle \Psi_\alpha \rangle \;=\; \epsilon_\alpha \langle \Psi_\alpha \rangle^\alpha, \tag{7}$$

where ϵ_α is the porosity or volume fraction of phase α inside the averaging volume

$$\epsilon_\alpha \;=\; \int_V \gamma_\alpha m \, \mathrm{d}V \tag{8}$$

and where we have assumed that all pores are connected. Unfiltered quantities can be decomposed into a contribution from the intrinsic volume -averaged quantity and a deviation thereof (Gray, 1975)

$$\Psi_\alpha \;= \langle \Psi_\alpha \rangle^\alpha + \tilde{\Psi}_\alpha. \tag{9}$$

To proceed further with the analysis the following four assumptions are made:

(1) The porosities are constant in space and time.
(2) The volume-averaged quantities are *well-behaved* (Gray, 1975): $\langle\langle u_i \rangle\rangle \approx \varepsilon_\beta \langle u_i \rangle$, which is equivalent to $\langle \tilde{u}_i \rangle \approx 0$. This assumption implies constraints for the weighting function m in (6), as discussed by Quintard and Whitaker (1994).
(3) Local thermal equilibrium, Whitaker (1999): $\langle \theta_\beta \rangle^\beta = \langle \theta_\sigma \rangle^\sigma = \langle \theta \rangle$.
(4) The deviation flow $(\tilde{u}_i)$ is in local energy equilibrium: production and dissipation of $E = \langle (1/2) \tilde{u}_i \tilde{u}_i \rangle^\beta$ are in local balance.

The derivation of the volume-averaged form of Equations (2) and (3) is similar to the derivation given by Whitaker (1996), and is therefore not repeated here. The Volume-Averaged Navier–Stokes (VANS) equations read

$$\frac{\partial \langle u_i \rangle^\beta}{\partial x_i} = 0, \tag{10}$$

$$\rho_\beta\left(\frac{\partial\langle u_i\rangle^\beta}{\partial t}+\frac{\partial\langle u_i\rangle^\beta\langle u_j\rangle^\beta}{\partial x_j}\right)$$
$$=-\rho_\beta\beta(\langle\theta\rangle-\theta_{\rm r})g_i-\frac{\partial\langle p\rangle^\beta}{\partial x_i}+2\mu_\beta\frac{\partial\langle e_{ij}\rangle^\beta}{\partial x_j}-\rho_\beta\frac{\partial\langle\tilde u_i\tilde u_j\rangle^\beta}{\partial x_j}+f_i, \tag{11}$$

where $\langle e_{ij}\rangle=\frac{1}{2}\left((\partial\langle u_i\rangle/\partial x_j)+(\partial\langle u_j\rangle/\partial x_i)\right)$ is the rate-of-strain tensor of the volume-averaged flow. The penultimate term of (11) represents dispersion by sub-filter motions and f_i is the drag force given by

$$f_i=\frac{1}{\epsilon_\beta}\int_V m[-\tilde p n_i+2\mu_\beta\tilde e_{ij}n_j]\mathrm{d}A, \tag{12}$$

where $\tilde e_{ij}=\frac{1}{2}\left((\partial\tilde u_i/\partial x_j)+(\partial\tilde u_j/\partial x_i)\right)$ is the rate-of-strain tensor of the deviation flow, A is the contact area between the solid and the fluid phase inside the averaging volume V, and n_i and n_j are components of the normal unit vector to A pointing from the fluid into the solid phase.

Based on the assumption of local thermal equilibrium and a similar analysis as given by Whitaker (1999), the volume-averaged thermal energy equation reads

$$\langle\rho\rangle C_{\rm p}\frac{\partial\langle\theta\rangle}{\partial t}+(\rho c_{\rm p})_\beta\frac{\partial\epsilon_\beta\langle u_j\rangle^\beta\langle\theta\rangle}{\partial x_j}$$
$$=\frac{\partial}{\partial x_j}\left(\Lambda_{jk}\frac{\partial\langle\theta\rangle}{\partial x_k}\right)-(\rho c_{\rm p})_\beta\frac{\partial\epsilon_\beta\langle\tilde u_j\tilde\theta_\beta\rangle^\beta}{\partial x_i}+2\mu_\beta\epsilon_\beta\langle e_{ij}\rangle^\beta\langle e_{ij}\rangle^\beta$$
$$+2\mu_\beta\epsilon_\beta\langle\tilde e_{ij}\tilde e_{ij}\rangle^\beta, \tag{13}$$

where $\langle\rho\rangle=\epsilon_\beta\rho_\beta+\epsilon_\sigma\rho_\sigma$ is the volume-weighted mass density, $C_{\rm p}=[\rho_\beta(\rho c_{\rm p})_\beta+\rho_\sigma(\rho c_{\rm p})_\sigma]/\langle\rho\rangle$ is the mass-weighted specific heat, and Λ_{jk} is the effective thermal conductivity tensor (Whitaker, 1999, p. 78). The last two terms on the right-hand side of (13) represent the production of thermal energy by viscous dissipation of kinetic energy of the volume-averaged flow and of the deviation flow, respectively.

A closure for the last term in Equation (13) can be found from the equation for the kinetic energy $E=\langle(1/2)\tilde u_i\tilde u_i\rangle^\beta$ of the deviation flow, which is derived in Appendix A

$$\rho_\beta\left(\frac{\partial E}{\partial t}+\frac{\partial\langle u_i\rangle^\beta E}{\partial x_j}\right)$$
$$=-\rho_\beta\langle\tilde u_i\tilde u_j\rangle^\beta\frac{\partial\langle u_i\rangle^\beta}{\partial x_j}-\rho_\beta\beta\langle\tilde\theta_\beta\tilde u_i\rangle^\beta g_i+$$
$$+\frac{\partial}{\partial x_j}\left[-\langle\tilde p\tilde u_i\rangle^\beta\delta_{ij}-\rho_\beta\langle\tfrac{1}{2}\tilde u_i\tilde u_i\tilde u_j\rangle^\beta+2\mu_\beta\langle\tilde e_{ij}\tilde u_i\rangle^\beta\right]-$$
$$-2\mu_\beta\langle\tilde e_{ij}\tilde e_{ij}\rangle^\beta-f_i\langle u_i\rangle^\beta, \tag{14}$$

where δ_{ij} is the Kronecker delta. The first term on the right-hand side represents the production of kinetic energy by the work of the volume-averaged flow against the dispersion stress. The second term on the right-hand side represents buoyancy production. The terms between the square brackets [...] represent the spatial *redistribution* of kinetic energy by a spatially varying pressure field $\tilde{p}$, a spatially varying velocity field $\tilde{u}_j$ and viscous diffusion, respectively. The penultimate term of Equation (14) represents viscous dissipation and the last term represents the production by the work of the volume-averaged flow against the drag force. We note that the first and the last term on the right-hand side of (14) appear as loss terms in the equation for the kinetic energy $(1/2)\langle u_i\rangle^\beta\langle u_i\rangle^\beta$ of the volume-averaged flow, which can be easily derived from multiplying Equation (11) by $\langle u_i\rangle^\beta$. These two terms are therefore responsible for the transfer of kinetic energy of the volume-averaged flow to kinetic energy of the deviation flow. The second term on the right-hand side of (15) does *not* appear in the equation for the kinetic energy of the volume-averaged flow. This term originates from the *direct* production of kinetic energy of the deviation flow by buoyancy. From Equation (14) and the assumption that the deviation flow is in local energy equilibrium, the following relation for the dissipation rate is derived:

$$2\mu_\beta\epsilon_\beta\langle\tilde{e}_{ij}\tilde{e}_{ij}\rangle^\beta = -\rho_\beta\epsilon_\beta\langle\tilde{u}_i\tilde{u}_j\rangle^\beta\frac{\partial\langle u_i\rangle^\beta}{\partial x_j} - \rho_\beta\epsilon_\beta\beta\langle\tilde{\theta}_\beta\tilde{u}_i\rangle^\beta g_i - \epsilon_\beta f_i\langle u_i\rangle^\beta. \quad (15)$$

Substituting Equation (15) into Equation (13) yields

$$\begin{aligned}
&\langle\rho\rangle C_\mathrm{p}\frac{\partial\langle\theta\rangle}{\partial t} + (\rho c_\mathrm{p})_\beta\frac{\partial\epsilon_\beta\langle u_j\rangle^\beta\langle\theta\rangle}{\partial x_j}\\
&\quad = \frac{\partial}{\partial x_j}\left(\Lambda_{jk}\frac{\partial\langle\theta\rangle}{\partial x_k}\right) - (\rho c_p)_\beta\frac{\partial\epsilon_\beta\langle\tilde{u}_j\tilde{\theta}_\beta\rangle^\beta}{\partial x_j}\\
&\qquad + 2\mu_\beta\epsilon_\beta\langle e_{ij}\rangle^\beta\langle e_{ij}\rangle^\beta - \rho_\beta\epsilon_\beta\langle\tilde{u}_i\tilde{u}_j\rangle^\beta\frac{\partial\langle u_i\rangle^\beta}{\partial x_j}\\
&\qquad - \rho_\beta\epsilon_\beta\beta\langle\tilde{\theta}_\beta\tilde{u}_i\rangle^\beta g_i - \epsilon_\beta f_i\langle u_i\rangle^\beta. \qquad (16)
\end{aligned}$$

Further progress can be made by means of four additional assumptions

(5) $-\rho_\beta\langle\tilde{u}_i\tilde{u}_j\rangle^\beta = 2\mu_\mathrm{d}\langle e_{ij}\rangle^\beta$ where μ_d is the sub-filter viscosity due to dispersion (mechanical and/or turbulent). The sub-filter viscosity depends on the structural properties of the porous medium and on the flow: $\mu_\mathrm{d}/\rho_\beta = \mathrm{O}(|\langle\underline{u}\rangle^\beta|d_p) + \mathrm{O}(|\langle\underline{u}\rangle^\beta|d_f)$, where d_p and d_f are, respectively, characteristic length scales of the solid obstacles and the pores.

(6) $-(\rho c_\mathrm{p})_\beta\epsilon_\beta\langle\tilde{u}_j\tilde{\theta}_\beta\rangle^\beta = \lambda_\mathrm{d}(\partial\langle\theta\rangle/\partial x_j)$ where λ_d is the sub-filter thermal conductivity due to dispersion. The ratio $\lambda_\mathrm{d}/(\rho c_\mathrm{p})_\beta$ has the same order of magnitudes as μ_d/ρ_β.

(7) $f_i = -(\mu_\beta/K)(1+F)\epsilon_\beta\langle u_i\rangle^\beta$ where K and F are, respectively the permeability and the Forchheimer parameter (both of which are assumed to be isotropic). Many semi–empirical relations are available for K and F such as the Ergun equation for packed beds (MacDonald *et al.*, 1979).
(8) Isotropic effective thermal conductivity tensor: $\Lambda_{ij} = \Lambda I_{ij}$, where I_{ij} is the unit tensor.

Given the above assumption Equations (10), (11) and (16) read

$$\frac{\partial\langle u_i\rangle^\beta}{\partial x_i} = 0, \tag{17}$$

$$\begin{aligned}
&\rho_\beta\left(\frac{\partial\langle u_i\rangle^\beta}{\partial t} + \frac{\partial\langle u_i\rangle^\beta\langle u_j\rangle^\beta}{\partial x_j}\right)\\
&\quad = -\rho_\beta\beta(\langle\theta\rangle - \theta_r)g_i - \frac{\partial\langle p\rangle^\beta}{\partial x_i} +\\
&\qquad +2\frac{\partial[\mu_\beta+\mu_d]\langle e_{ij}\rangle^\beta}{\partial x_j} - \frac{\mu_\beta}{K}(1+F)\epsilon_\beta\langle u_i\rangle^\beta,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\langle\rho\rangle C_{\mathrm{p}}\frac{\partial\langle\theta\rangle}{\partial t} + (\rho c_{\mathrm{p}})_\beta\frac{\partial\epsilon_\beta\langle u_j\rangle^\beta\langle\theta\rangle}{\partial x_j}\\
&\quad = \frac{\partial}{\partial x_j}\left[(\Lambda+\lambda_{\mathrm{d}})\frac{\partial\langle\theta\rangle}{\partial x_j}\right] + 2\epsilon_\beta[\mu_\beta+\mu_d]\langle e_{ij}\rangle^\beta\langle e_{ij}\rangle^\beta +\\
&\qquad + \frac{\lambda_{\mathrm{d}}}{(\rho c_{\mathrm{p}})_\beta}\rho_\beta\beta g_i\frac{\partial\langle\theta\rangle}{\partial x_i} + \frac{\mu_\beta}{K}(1+F)(\epsilon_\beta\langle u_i\rangle^\beta)^2.
\end{aligned} \tag{19}$$

3. Dimensionless Equations

To nondimensionalize equations (17)–(19) appropriate velocity, temperature and length scales need to be chosen. It is assumed here that the momentum Equation (18) is dominated by the balance between the buoyancy force and the Darcy drag force. This yields the following characteristic velocity scale:

$$U = \frac{\rho_\beta\beta\Delta T g K}{\mu_\beta} = Ra\left[\frac{\Lambda}{(\rho c_p)_\beta}\frac{1}{L}\right], \tag{20}$$

where Ra is the Darcy–Rayleigh number

$$Ra = \frac{\rho_\beta\beta\Delta T g L K}{\mu_\beta\Lambda/(\rho c_{\mathrm{p}})_\beta}. \tag{21}$$

The characteristic temperature scale is denoted by the imposed temperature difference ΔT. The typical length scale for variations in space is denoted by L. For the case considered here $L = \mathrm{O}(\sqrt{K})$ close to a solid wall, but far away from a solid wall $L \gg \sqrt{K}$. Based on the above scales the dimensionless equations read

$$\frac{\partial \langle u_i \rangle^\beta}{\partial x_i} = 0, \tag{22}$$

$$\begin{aligned} &\frac{\partial \langle u_i \rangle^\beta}{\partial t} + \frac{\partial \langle u_i \rangle^\beta \langle u_j \rangle^\beta}{\partial x_j} \\ &\quad = -\frac{Pr}{RaDa}\frac{g_i}{g}(\langle \theta \rangle - \theta_{\mathrm{r}}) - \frac{\partial \langle p \rangle^\beta}{\partial x_i} \\ &\quad\quad + 2\frac{Pr}{Ra}\frac{\partial \left[1 + \frac{\mu_d}{\mu_\beta}\right] \langle e_{ij} \rangle^\beta}{\partial x_j} - \frac{Pr}{RaDa}(1+F)\epsilon_\beta \langle u_i \rangle^\beta, \end{aligned} \tag{23}$$

$$\begin{aligned} &\left[\frac{\langle \rho \rangle C_{\mathrm{p}}}{(\rho c_p)_\beta}\right]\frac{\partial \langle \theta \rangle}{\partial t} + \frac{\partial \epsilon_\beta \langle u_j \rangle^\beta \langle \theta \rangle}{\partial x_j} \\ &\quad = \frac{1}{Ra}\frac{\partial}{\partial x_j}\left[\left(1 + \frac{\lambda_d}{\Lambda}\right)\frac{\partial \langle \theta \rangle}{\partial x_j}\right] + GeDa2\epsilon_\beta \left[1 + \frac{\mu_d}{\mu_\beta}\right]\langle e_{ij} \rangle^\beta \langle e_{ij} \rangle^\beta + \\ &\quad\quad + \frac{Ge}{Ra}\frac{\lambda_d}{\Lambda}\frac{g_i}{g}\frac{\partial \langle \theta \rangle}{\partial x_i} + Ge(1+F)(\epsilon_\beta \langle u_i \rangle^\beta)^2, \end{aligned} \tag{24}$$

where Pr the Prandtl number, Ge the Gebhart number and Da the Darcy number. These numbers are defined according

$$Pr = \frac{\nu_\beta}{\Lambda/(\rho c_{\mathrm{p}})_\beta}, \tag{25}$$

$$Ge = \frac{\beta g L}{(c_{\mathrm{p}})_\beta}, \tag{26}$$

$$Da = \frac{K}{L^2}. \tag{27}$$

4. Order-of-Magnitude Analysis

From Equation (23) it follows that the velocity scale given by (20) is only appropriate when $RaDa/Pr \ll 1$ and $F \ll 1$. The first condition implies that the left-hand side of (23) can be neglected. The third term on the right-hand side of (23), which is known as the Brinkman correction, is only significant when $Da = \mathrm{O}(1)$. This condition is satisfied close to a solid wall. The ratio $\mu_d = \mu_\beta = \mathrm{O}(Ud_p/\nu)$ is of the same order as F and therefore can also be neglected. Thus, when $RaDa/Pr \ll 1$ and $F \ll 1$, Equations (22)–(24) can be simplified to

$$\frac{\partial \langle u_i \rangle^\beta}{\partial x_i} = 0, \tag{28}$$

$$0=-\frac{Pr}{RaDa}\frac{g_i}{g}(\langle\theta\rangle-\theta_r)-\frac{\partial\langle p\rangle^\beta}{\partial x_i}+2\frac{Pr}{Ra}\frac{\partial\langle e_{ij}\rangle^\beta}{\partial x_j}-\frac{Pr}{RaDa}\epsilon_\beta\langle u_i\rangle^\beta, \tag{29}$$

$$\begin{aligned}\left[\frac{\langle\rho\rangle C_\mathrm{p}}{(\rho c_\mathrm{p})\beta}\right]\frac{\partial\langle\theta\rangle}{\partial t}+\frac{\partial\epsilon_\beta\langle u_j\rangle^\beta\langle\theta\rangle}{\partial x_j}&=\frac{1}{Ra}\frac{\partial}{\partial x_j}\left[\left(1+\frac{\lambda_\mathrm{d}}{\Lambda}\right)\frac{\partial\langle\theta\rangle}{\partial x_j}\right]\\&+GeDa2\epsilon_\beta\langle e_{ij}\rangle^\beta\langle e_{ij}\rangle^\beta+\frac{Ge}{Ra}\frac{\lambda_\mathrm{d}}{\Lambda}\frac{g_i}{g}\frac{\partial\langle\theta\rangle}{\partial x_i}+Ge(\epsilon_\beta\langle u_i\rangle^\beta)^2.\end{aligned} \tag{30}$$

The ratio of $(\lambda_\mathrm{d}/\Lambda)=\mathrm{O}\big(Ra(d_\mathrm{p}/L)\big)=\mathrm{O}\big((1/Pr)(Ud_\mathrm{p}/\nu_\beta)\big)$ and is not negligible when $Pr \ll F$. Equation (30) can not be simplified further without any information on the values of Ra, Ge, Da and d_p/L.

Close to a solid wall it is expected that $Da=\mathrm{O}(1)$ and also that d_p/L is not negligible. Therefore the last three terms on the right-hand side in (30) are all $\mathrm{O}(Ge)$ close to a solid wall, whereas the first term on the right-hand side is on the order $\mathrm{O}(1/Ra)+\mathrm{O}(1)$. However, because Ge and Ra both get small close to a solid wall, the first term on the right-hand side of (30) may strongly dominate over all other terms in this equation.

Far away from a solid wall $Da \ll 1$ and $d_\mathrm{p}/L \ll 1$. The second and third terms on the right-hand side of (30) are negligible with respect to the fourth term. Also the Brinkman correction in (29) is negligible, which is consistent with neglecting the second term on the right-hand side of (30). As Ra and Ge become both relatively large, especially Ra, the fourth term on the right-hand side of (30) may be dominant over the first term and may be in balance with the terms at the left-hand side of (30).

5. Summary and Conclusion

The most general form of the volume-averaged Boussinesq equations for flow in porous media is given by Equations (10), (11) and (16). These equations are valid both for a very small permeability and for a very large permeability. In the latter case they reduce to the standard Boussinesq equations for a clear fluid. The constraints for the validity of these equations are: a constant porosity, well-behaved volume-averaged quantities, local thermal equilibrium, and local energy equilibrium for the sub-filter motions. To proceed further with the analysis closures have been proposed for the dispersion terms and the drag force in combination with the assumption of an isotropic effective conductivity tensor. The equations thus obtained are made dimensionless on the assumptions that $RaDa/Pr \ll 1$ and $F \ll 1$. These conditions are often met in practical applications. When these conditions hold, the dimensionless Boussinesq equations are given by (28)–(30), which may be further simplified depending on the values of Ra, Ge, Da and d_p/L.

We conclude by introducing a note of caution. Although we have demonstrated, using volume averaging, that the Brinkman component of the viscous dissipation term corresponds with that derived by Al-Hadhrami *et al.* (2003), some recent numerical work by Rees (2004) indicates that even the Darcy component might not be an accurate representation when Darcy's law is valid. This author considered convection in a sidewall-heated square cavity and found that it is possible for the internal temperature within the cavity to exceed the highest temperature which is imposed on the boundary. While one might expect such behaviour in a forced convective flow, such as channel flow under a constant pressure gradient with uniform and identical temperatures on the boundaries, it seems unlikely that a fully buoyancy-driven flow should generate such high temperatures without violating the second law of thermodynamics. Ingham (2003) is also concerned about the possibility of thermal runaway. Therefore it is to be recommended that other effects, such as the work done against pressure forces, should be included in future studies, and these new models tested.

Appendix A: Transport Equation for the Kinetic Energy of the Deviation Flow

In this appendix a derivation is given of Equation (14) for the kinetic energy $E=\langle(1/2)\tilde{u}_i\tilde{u}_i\rangle^\beta$. A similar derivation of this equation was also given by Wang and Takle (1995), who however did not consider the effect of buoyancy. The assumptions in the analysis below are that

(1) The porosities are constant in space and time.
(2) The volume-averaged quantities are *well-behaved*, Gray (1975): $\langle\langle u_i\rangle\rangle\approx\varepsilon_\beta\langle u_i\rangle$, which is equivalent to $\langle\tilde{u}_i\rangle\approx 0$.

The transport equation for $\tilde{u}_i=u_i-\langle u_i\rangle^\beta$ is obtained by subtracting Equation (11) for $\langle u_i\rangle^\beta$ from Equation (3) for u_i. The result reads

$$\rho_\beta\left(\frac{\partial\tilde{u}_i}{\partial t}+\frac{\partial}{\partial x_j}[\tilde{u}_i\langle u_j\rangle^\beta+\langle u_i\rangle^\beta\tilde{u}_j+\tilde{u}_i\tilde{u}_j]\right)$$
$$=-\rho_\beta\beta\tilde{\theta}_\beta g_i-\frac{\partial\tilde{p}}{\partial x_i}+2\mu_\beta\frac{\partial\tilde{e}_{ij}}{\partial x_j}+\rho_\beta\frac{\partial\langle\tilde{u}_i\tilde{u}_j\rangle}{\partial x_j}-f_i. \tag{A1}$$

The continuity equation for $\tilde{u}_i$ is found from subtracting Equation (10) from Equation (2)

$$\frac{\partial\tilde{u}_i}{\partial x_i}=0. \tag{A2}$$

The equation for $\tilde{u}_i\tilde{u}_i/2$ may be obtained by multiplying Equation (A1) by $\tilde{u}_i$. The resulting equation is then volume-averaged in order to obtain the equation for E. The preliminary result reads

$$\rho_\beta\left(\frac{\partial\langle(1/2)\tilde{u}_i^2\rangle}{\partial t}+\langle u_j\rangle^\beta\left\langle\frac{\partial(1/2)\tilde{u}_i^2}{\partial x_j}\right\rangle+\langle\tilde{u}_i\tilde{u}_j\rangle\frac{\partial\langle u_i\rangle^\beta}{\partial x_j}+\left\langle\frac{\partial(1/2)\tilde{u}_i\tilde{u}_i\tilde{u}_j}{\partial x_j}\right\rangle\right)$$
$$=-\rho_\beta\beta\langle\tilde{u}_i\tilde{\theta}_\beta\rangle g_i-\left\langle\frac{\partial\tilde{p}\tilde{u}_i}{\partial x_i}\right\rangle+2\mu_\beta\left\langle\frac{\partial\tilde{u}_i\tilde{e}_{ij}}{\partial x_j}\right\rangle-2\mu_\beta\left\langle\tilde{e}_{ij}\tilde{e}_{ij}\right\rangle$$
$$+\rho_\beta\left\langle\tilde{u}_i\frac{\partial\langle\tilde{u}_i\tilde{u}_j\rangle}{\partial x_j}\right\rangle-\langle\tilde{u}_i f_i\rangle. \tag{A3}$$

To proceed further with the derivation, we make use of the spatial averaging theorem (Whitaker, 1969) which relates the volume average of a gradient to the gradient of a volume average. For instance the volume average of the pressure gradient is equal to

$$\left\langle\frac{\partial p}{\partial x_i}\right\rangle=\frac{d\langle p\rangle}{\partial x_i}+\int_A mn_i p\ \mathrm{d}A. \tag{A4}$$

From the substitution of $p=1$ in the above equation, the following relation is obtained for the gradient of the porosity ϵ_β:

$$\frac{\partial\epsilon_\beta}{\partial x_i}=-\int_A mn_i\ \mathrm{d}A. \tag{A5}$$

With the help of the spatial averaging theorem the second term on the left-hand side of (A3) is equal to

$$\langle u_j\rangle^\beta\left\langle\frac{\partial(1/2)\tilde{u}_i^2}{\partial x_j}\right\rangle=\langle u_j\rangle^\beta\frac{\langle\partial\frac{1}{2}\tilde{u}_i^2\rangle}{\partial x_j}+\langle u_j\rangle^\beta\int_A mn_j\frac{1}{2}\tilde{u}_i^2\ \mathrm{d}A$$
$$=\langle u_j\rangle^\beta\frac{\partial\langle\frac{1}{2}\tilde{u}_i^2\rangle}{\partial x_j}+\langle u_j\rangle^\beta\left[\int_A mn_j\ \mathrm{d}A\right]\frac{1}{2}\langle u_i\rangle^\beta\langle u_i\rangle^\beta, \tag{A6}$$

where we used the fact that, according to Equation (9), $\tilde{u}_i=-\langle u_i\rangle^\beta$ at the surface A of the solid phase, and that $\langle\langle u_i\rangle\rangle\approx\varepsilon_\beta\langle u_i\rangle$; hence $\langle u_i\rangle^\beta$ may be taken out of the surface integral. By virtue of Equation (A5) and the assumption of a constant porosity ϵ_β, the last term on the right-hand side of the above equation is equal to zero. In a similar manner a formula for the last term at the left-hand side of Equation (A3) may be derived

$$\left\langle\frac{\partial(1/2)\tilde{u}_i\tilde{u}_i\tilde{u}_j}{\partial x_j}\right\rangle=\frac{\partial\langle(1/2)\tilde{u}_i\tilde{u}_i\tilde{u}_j\rangle}{\partial x_j}-\left[\int_A mn_j\ \mathrm{d}A\right]\frac{1}{2}\langle u_i\rangle^\beta\langle u_i\rangle^\beta\langle u_j\rangle^\beta$$
$$=\frac{\partial\langle(1/2)\tilde{u}_i\tilde{u}_i\tilde{u}_j\rangle}{\partial x_j}. \tag{A7}$$

The second and third terms on the right-hand side of Equation (A3) may be written as

$$-\left\langle\frac{\partial\tilde{p}\tilde{u}_i}{\partial x_i}\right\rangle=-\frac{\partial\langle\tilde{p}\tilde{u}_i\rangle}{\partial x_i}+\left[\int_A mn_i\tilde{p}\ \mathrm{d}A\right]\langle u_i\rangle^\beta, \tag{A8}$$

$$2\mu_\beta\left\langle\frac{\partial\tilde{u}_i\tilde{e}_{ij}}{\partial x_j}\right\rangle = 2\mu_\beta\frac{\partial\langle\tilde{u}_i\tilde{e}_{ij}\rangle}{\partial x_j} - 2\mu_\beta\left[\int_A mn_j\frac{\partial\tilde{u}_i}{\partial x_j}\,\mathrm{d}A\right]\langle u_i\rangle^\beta. \tag{A9}$$

The last two terms on the right-hand side of (A3) are equal to zero

$$\rho_\beta\left\langle\tilde{u}_i\frac{\partial\langle\tilde{u}_i\tilde{u}_j\rangle}{\partial x_j}\right\rangle = \rho_\beta\langle\tilde{u}_i\rangle\frac{\partial\langle\tilde{u}_i\tilde{u}_j\rangle}{\partial x_j} = 0, \tag{A10}$$

$$\langle\tilde{u}_i f_i\rangle = \langle\tilde{u}_i\rangle f_i = 0. \tag{A11}$$

The final form of the transport equation for E is obtained from the substitution of (A6)–(A11) into Equation (A3) and a division by ϵ_β. This yields

$$\begin{aligned}
&\rho_\beta\left(\frac{\partial E}{\partial t} + \frac{\partial\langle u_i\rangle^\beta E}{\partial x_j}\right) \\
&\quad = -\rho_\beta\beta\langle\tilde{u}_i\tilde{u}_j\rangle^\beta\frac{\partial\langle u_i\rangle^\beta}{\partial x_i} - \rho_\beta\beta\langle\tilde{\theta}_\beta\tilde{u}_i\rangle^\beta g_i \\
&\qquad + \frac{\partial}{\partial x_j}\left[-\langle\tilde{p}\tilde{u}_i\rangle^\beta\delta_{ij} - \rho_\beta\langle\frac{1}{2}\tilde{u}_i\tilde{u}_i\tilde{u}_j\rangle^\beta + 2\mu_\beta\langle\tilde{e}_{ij}\tilde{u}_i\rangle^\beta\right] \\
&\qquad - 2\mu_\beta\langle\tilde{e}_{ij}\tilde{e}_{ij}\rangle^\beta - f_i\langle u_i\rangle^\beta,
\end{aligned} \tag{A12}$$

where we made use of Equation (A12) for the drag force f_i.

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