THE EFFECT OF LONG-WAVELENGTH THERMAL MODULATIONS ON THE ONSET OF CONVECTION IN AN INFINITE POROUS LAYER HEATED FROM BELOW

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SUMMARY

The onset of finite-amplitude convection in a horizontal porous layer of infinite extent is considered. Attention is focused on the case of spatially periodic heating and cooling on the lower and upper boundaries, respectively. In particular, we analyse the effects of small-amplitude, symmetric, thermal modulations with a wavelength which is large compared with the layer depth.

Weakly nonlinear theory is used to derive Landau–Ginzburg equations for the amplitude of convection in the form of transverse and longitudinal rolls. It is found that these patterns do not necessarily have the same spatial periodicity as the thermal forcing and may even be spatially quasiperiodic. The most unstable transverse roll, however, always has the same wavelength as the thermal modulations. We show also that for certain ranges of values of the modulation wavenumber the first mode to appear as the Rayleigh number is increased is, somewhat surprisingly, a rectangular cell of large-aspect-ratio planform. This mode is a linear superposition of two rolls equally aligned at a small angle away from the direction of the longitudinal roll.

1. Introduction

In this paper we present a study of the onset of convection in an infinite porous layer heated non-uniformly from below. We assume that the temperatures of the horizontal planar boundaries are subject to steady, symmetric, small-amplitude, sinusoidal modulations about their mean values. The modulations are taken to occur over a length scale which is large compared with the layer depth. This analysis forms part of a systematic examination of the effects of boundary non-uniformities on the onset and stability of finite-amplitude convection in porous layers. In (**1** to **4**) we have presented analyses of configurations where the non-uniformity has an associated wavenumber k_w either equal to, near to, or far from the critical wavenumber k_c for convection in the unmodulated or Lapwood problem (after Lapwood (**5**) who studied in detail the criterion for instability in plane, uniformly heated, porous layers). Weakly nonlinear theory was used to derive evolution equations for the amplitudes of certain roll solutions. It was shown that there exists a variety of linearly stable cellular patterns other than rolls; the particular ones which arise depend on the modulation wavenumber and symmetry of the imperfection. These patterns include square cells ($k_w = k_c$), rolls with a spatially-varying amplitude and local phase or a spatially-varying amplitude and local orientation ($k_w \sim k_c$), and rectangular cells ($0 < k_w < 2k_c$) whose aspect ratio depends on k_w. In this study we extend the results of the above papers by examining the onset of convection when $k_w \sim 0$; that is, the thermal modulations at the plane boundaries occur over a long length scale. The range of applicability of this analysis is greater than is perhaps immediately apparent. It may be shown that, after rescaling, the amplitude equations derived here also apply to the following configurations: (i) a horizontal porous layer with antisymmetric heating, (ii) a horizontal porous layer heated isothermally with boundaries exhibiting small-amplitude, symmetric undulations, (iii) an undulating horizontal porous layer of constant thickness heated isothermally, and (iv) similar fluid-layer configurations. Thus the present work is of importance in the study of convection in porous rock strata, for example, for these often undulate and are subject to non-uniform heating.

There are several papers which deal directly with the Rayleigh–Bénard analogue of the present problem, namely the onset of convection in fluid layers heated non-uniformly from below, or a uniformly heated layer with a varying depth. Eagles (**6**) considered weakly nonlinear convection in a uniformly heated layer with the lower boundary perturbed by an $O(\varepsilon^2)$ amount over a length scale which is $O(\varepsilon^{-1})$, where $\varepsilon \ll 1$. For perturbations with a $\tanh^2$ profile there is a finite set of Rayleigh numbers at which corresponding eigenmodes become unstable. For that problem there is also a continuous spectrum of values of the Rayleigh number for which further eigenmodes become linearly unstable. However, the first mode to appear as the Rayleigh number is increased bifurcates from the trivial solution at the lowest value of the discrete set of Rayleigh numbers. It also appears that the wavenumber of this mode is unique and equal to the critical wavenumber of the unperturbed problem. In a comprehensive study of the onset of convective Walton (**7**) sought to include boundary variations with $O(1)$ amplitudes. Eagles's wavenumber selection and discrete critical Rayleigh number phenomena were explained (for sinusoidal perturbations, at least) by appealing to known properties of the solutions to Mathieu's equation. In both these works, however, only two-dimensional convection was studied. Later, Walton (**8**, **9**) considered finite-amplitude convection in the form of both transverse and longitudinal rolls (note that the present definitions of these rolls are the same as in (**8**) but are different from those in our previous work (**1** to **4**)). Convection in an infinite layer of slowly increasing depth was considered in (**8**); in this case the concept of a specific Rayleigh number at which convection ensues becomes untenable since convection exists within the whole layer and has an amplitude which is space-dependent. In (**9**) it is shown that when the temperature drop across the layer varies monotonically with the maximum at an end wall, the first roll to appear is the longitudinal roll. Although such a result is shown later also to apply here, we find that a three-dimensional disturbance composed of two rolls equally oriented about the longitudinal roll direction often has a still lower critical Rayleigh number.

In §2 we present briefly the governing equations for convection of a Boussinesq fluid in a saturated porous medium and define the precise configuration of the porous layer in question. At subcritical values of the Rayleigh number Ra the flow is unique, two-dimensional and driven directly by the thermal modulations at the boundaries; this is termed the quasiconduction regime and is studied in §3.

Close to the critical Rayleigh number Ra_c for the onset of convection in the unmodulated layer, the quasiconduction flow becomes unstable because of the magnitude of the adverse temperature gradient across the layer. The presence of the thermal modulations serves to modify both the spatial form of the unstable modes and their critical Rayleigh numbers. The onset of transverse rolls (that is, rolls aligned such that the resulting flow pattern remains two-dimensional) is analysed in §4 and it is shown that the critical value of the Rayleigh number is found as an eigenvalue of Mathieu's equation.

In §5 we study the onset of longitudinal rolls; in this case the amplitude and critical Rayleigh number are given as an eigenfunction and eigenvalue, respectively, of a fourth-order analogue of Mathieu's equation. Oblique rolls are analysed in §6. We first present results for rolls with an $O(1)$ orientation relative to the longitudinal roll. For rolls with a small relative orientation the equation governing the onset of convection now has spatial derivatives identical in form to those used in describing the zig-zag instability. Asymptotic and numerical results show that these latter 'oblique' modes constitute the most unstable mode for the present problem but we note that, for certain ranges of values of the thermal modulation wavenumbers, the mode assumes the form of a rectangular cell of large-aspect-ratio planform.

The results are discussed in §7.

2. The equations of motion and formulation of the problem

We consider the onset of cellular convection in an unbounded horizontal porous layer which is heated from below. The non-dimensional equations governing the convection of a Boussinesq fluid in a saturated medium are (see (**1**))

$$\nabla \cdot \mathbf{q} = 0, \qquad \mathbf{q} = -\nabla P + \mathrm{Ra}\, T\hat{\mathbf{z}}, \qquad T_t + \mathbf{q} \cdot \nabla T = \nabla^2 T, \tag{1}$$

where $\hat{\mathbf{z}}$ is the unit upward normal, $\mathbf{q}$ is the Darcy velocity vector, P is the pressure, T is the temperature, t is the time, and Ra is the Darcy–Rayleigh number (or, more briefly, the Rayleigh number) defined in terms of the various fluid and medium properties. On eliminating $\mathbf{q}$ from the above equations we obtain

$$\nabla^2 P - \text{Ra}\, T_z = 0, \tag{2}$$

$$\nabla^2 T = \text{Ra}\, TT_z - \nabla P \,.\, \nabla T + T_t. \tag{3}$$

We impose the thermal boundary conditions $T = \mp(1 + \delta g(\varepsilon x))$ on $z = \pm 1$, and, since the horizontal boundaries are assumed to be impermeable, we require $P_z = \text{Ra}\, T$ there. The modulation function $g(\varepsilon x)$ is arbitrary at this point and we assume that both ε and δ are small.

It proves convenient to work relative to the conduction solution for the Lapwood problem and therefore we set $P = p - \frac{1}{2}\text{Ra}\, z^2$ and $T = \theta - z$ in (2) and (3). Likewise we introduce the slow spatial scale $X = \varepsilon x$ to obtain

$$\nabla^2 p - \text{Ra}\, \theta_z = -2\varepsilon p_{xX} - \varepsilon^2 p_{XX}, \tag{4}$$

$$\begin{aligned}\nabla^2 \theta + \text{Ra}\, \theta - p_z = \text{Ra}\, \theta\theta_z - \nabla p \,.\, \nabla\theta + \theta_t - 2\varepsilon\theta_{xX} - \varepsilon^2\theta_{XX} \\ - \varepsilon(p_x\theta_X + p_X\theta_x) - \varepsilon^2 p_X \theta_X,\end{aligned} \tag{5}$$

which are to be solved subject to

$$p_z = \mp\delta\, \text{Ra}\, g(X), \qquad \theta = \mp\delta g(X) \qquad \text{on } z = \pm 1. \tag{6}$$

In addition it is assumed that there is no net fluid flux along the layer (that is, there is a zero mean horizontal pressure gradient) and that the resulting convection patterns are either spatially periodic or quasiperiodic.

3. Quasiconduction regime

In the classical Lapwood problem (that is, a saturated plane porous layer heated uniformly from below) the basic steady state is motionless since heat is conducted uniformly from the lower to the upper boundary. When thermal modulations are present they drive a weak convective motion via baroclinic effects. This motion is two-dimensional and is most easily described in terms of the non-dimensional streamfunction ψ defined by $\mathbf{q} = \text{curl}\,(\psi\hat{\mathbf{y}})$, where $\hat{\mathbf{y}}$ is the unit vector in the y-direction. Since both $\varepsilon \ll 1$ and $\delta \ll 1$ we seek solutions by introducing a perturbation expansion of the form

$$(\psi, \theta) = \sum_{m=0} \sum_{n=0} \delta^m \varepsilon^n (\psi^{mn}, \theta^{mn}). \tag{7}$$

After some routine algebra we eventually obtain the solutions

$$\begin{aligned}\psi = \tfrac{1}{6}\text{Ra}\,(z^3 - z)g'(X)\delta\varepsilon \\ - \left[\text{Ra}\left(\frac{z^5}{60} - \frac{z^3}{18} + \frac{7z}{180}\right) + \text{Ra}^2\left(\frac{z^7}{5040} - \frac{z^5}{720} + \frac{7z^3}{2160} - \frac{31z}{15120}\right)\right] g'''\, \delta\varepsilon^3 \\ + o(\delta\varepsilon^3),\end{aligned} \tag{8}$$

$$\theta = -zg(X)\delta - \left[\tfrac{1}{6}(z^3 - z) + \mathrm{Ra}\left(\frac{z^5}{120} - \frac{z^3}{36} + \frac{7z}{360}\right)\right]g''\,\delta\varepsilon^2$$
$$+ \mathrm{Ra}\left[\left(\frac{z^5}{120} - \frac{z^3}{36} + \frac{7z}{360}\right)gg'' - \left(\frac{z^5}{40} - \frac{z^3}{36} + \frac{z}{360}\right)g'g'\right]\delta^2\varepsilon^2$$
$$+ o(\delta^2\varepsilon^2), \tag{9}$$

where the dashes denote derivatives with respect to X. The leading-order effect on the flow field is a circulation of magnitude $\delta\varepsilon$ consisting of pairs of vertically stacked counter-rotating cells of large aspect ratio. The lowest-order term in the temperature profile comprises a simple modification which is proportional to the local thermal modulation at the boundaries. In the remainder of this paper we choose to study sinusoidal boundary modulations, and therefore the leading-order correction to the mean heat transfer across the layer arises at $O(\delta^2\varepsilon^2)$, where nonlinear terms in $g(X)$ first appear in the expansion. We define the Nusselt number to be the mean heat transfer across the layer per unit length in the x-direction:

$$\mathrm{Nu} = 1 - \frac{\omega\varepsilon}{2\pi}\int_0^{2\pi/\omega\varepsilon} \theta_z(z = \pm 1)\,dx = 1 - \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \theta_z(z = \pm 1)\,dX, \tag{10}$$

where it should be noted that account has been taken of the thermal conduction profile (which was subtracted out when defining θ in §2) and that we have now assumed that the thermal modulations have the explicit form

$$g(X) = \cos(\omega X). \tag{11}$$

Using (9) we find that

$$\mathrm{Nu} = 1 + \frac{\mathrm{Ra}\,\omega^2\delta^2\varepsilon^2}{90} + o(\delta^2\varepsilon^2) \tag{12}$$

and therefore thermal modulations serve to enhance the transfer of heat across the layer.

In (4) the effects on finite-amplitude convection of small-amplitude thermal modulations with finite, non-zero wavenumber k_w were discussed (the present problem corresponds to $k_w = \varepsilon\omega$ for $\varepsilon \ll 1$ and $\omega = O(1)$). The subcritical Nusselt number for symmetric modulations (where the boundary conditions are $\theta = \pm\delta\cos(k_w x)$ on $z = \mp 1$) was found to be given by

$$\mathrm{Nu} \sim 1 + \tfrac{1}{16}k_w\,\mathrm{Ra}^{\frac{1}{2}}[\coth^2\gamma - \coth^2\chi + \chi^{-1}\coth\chi - \gamma^{-1}\coth\gamma]\delta^2, \tag{13}$$

where

$$\gamma^2 = k_w^2 + k_w\,\mathrm{Ra}^{\frac{1}{2}}, \qquad \chi^2 = k_w^2 - k_w\,\mathrm{Ra}^{\frac{1}{2}}, \tag{14}$$

and χ may take real or imaginary values. As $k_w \to 0$, (13) takes the form

$$\mathrm{Nu} \sim 1 + \frac{\mathrm{Ra}\,k_w^2\delta^2}{90} \tag{15}$$

which clearly matches with (12) as $\delta \to 0$.

4. The onset of transverse rolls

The onset and stability of convection in uniformly heated porous layers of infinite extent is now well known. In terms of the present non-dimensionalization, convection ensues when $\text{Ra} \geq \text{Ra}_c = \pi^2$, with a corresponding critical wavenumber $k_c = \frac{1}{2}\pi$. Palm, Weber and Kvernvold (**10**) using the method of Schlüter, Lortz and Busse (**11**) determined that, for a Boussinesq fluid, rolls constitute the only stable planform for finite-amplitude convection (that is, when the Rayleigh number is sufficiently close to Ra_c that convection may be described using weakly nonlinear theory using $\text{Ra} - \text{Ra}_c$ as the small parameter). Furthermore not all possible wavenumbers are stable, for there exists a band of wavenumbers for which the finite-amplitude convection is linearly stable (see (**3**) for quantitative details). In a numerical study, Straus (**12**) extended this stability analysis well into the strongly nonlinear regime; he found that rolls remain stable up to $\text{Ra} \simeq 9\text{Ra}_c$. It is natural therefore that we study rolls at the outset.

Initially we restrict our attention to transverse rolls since the overall flow pattern, which comprises the rolls and the weak base flow (which is discussed in §2), remains two-dimensional. The strategy we use to determine the onset of convection is to develop a weakly nonlinear theory thereby deriving nonlinear equations governing the amplitudes of the convective modes. Although this is not strictly necessary for the present problem, we hope later to extend the present work by using these equations to analyse the stability of finite-amplitude convection.

We begin by expanding p, θ and Ra in a power series in ε, as follows:

$$(p, \theta, \text{Ra}) = \sum_{n=0} \varepsilon^n (p_n, \theta_n, R_n), \tag{16}$$

where $p_0 = \theta_0 = 0$, $R_0 = \text{Ra}_c = \pi^2$, and the summation is over integer values of n. It is necessary at this point to determine a suitable scaling for ε in terms of δ, which is to be regarded as the reference small quantity in accord with our previous work. On defining a local Rayleigh number $\mathcal{R}a$ based on the local temperature drop across the layer, it may be shown that $\mathcal{R}a = \text{Ra}\,(1 + \delta g(X))$. This variation in the local Rayleigh number is consistent with a wavenumber variation of $O(\delta^{\frac{1}{2}})$ so that the associated length scale for variations in the x-direction is $O(\delta^{-\frac{1}{2}})$. It is quite natural to equate this with the modulation length scale $(O(\varepsilon^{-1}))$ and to assess the ramifications a posteriori. Thus it is easily seen that $\varepsilon = O(\delta^{\frac{1}{2}})$ is the appropriate scaling, which is identical with that considered in (**6**).

On substituting $\varepsilon = \delta^{\frac{1}{2}}$ and (16) into (4) to (6) we obtain a set of linear equations for the various unknowns. At $O(\varepsilon)$ we obtain the following homogeneous equations for the unknowns (p_1, θ_1):

$$\nabla^2 p_1 - R_0 \theta_{1z} = 0, \tag{17}$$

$$\nabla^2 \theta_1 + R_0 \theta_1 - p_{1z} = 0; \tag{18}$$

these are to be solved subject to the boundary conditions $p_{1z} = \theta_1 = 0$ on $z = \pm 1$. Although (17) and (18) possess an infinity of eigensolutions we consider transverse modes of the form

$$\begin{pmatrix} p_1 \\ \theta_1 \end{pmatrix} = (Ae^{ik_c x} + \bar{A}e^{-ik_c x})\begin{pmatrix} k_c \sin(k_c z) \\ \frac{1}{2}\cos(k_c z) \end{pmatrix}, \tag{19}$$

where $A = A(X, \tau)$ is a complex amplitude to be determined later, and $\tau = \frac{1}{2}\varepsilon^2 t$ is a slow time scale. We omit further details of the expansion which is rather straightforward, but note that the $O(\varepsilon^2)$ solution is given by

$$\begin{pmatrix} p_2 \\ \theta_2 \end{pmatrix} = -\begin{pmatrix} \frac{1}{2}\pi^2 z^2 \\ z \end{pmatrix} g(X) + \begin{pmatrix} -\frac{1}{2}k_c^2 \cos(2k_c z) \\ \frac{1}{4}k_c \sin(2k_c z) \end{pmatrix} A\bar{A}. \tag{20}$$

At third order in ε the equations do not possess a solution unless a value for R_2 is chosen such that the inhomogeneous terms are orthogonal to the $O(\varepsilon)$ eigensolution. The application of such an orthogonality or solvability condition, which is a standard procedure in problems of this type, yields the following evolution equation for the transverse roll amplitude:

$$A_\tau = [R_2 + 4k_c^2 g(X)]A + 4A_{XX} - k_c^4 A^2 \bar{A}. \tag{21}$$

In this paper we confine our attention to the onset problem and therefore we linearize the steady form of (21). We consider sinusoidal thermal modulations only, and so we let $g(X) = \cos \varepsilon_0 X$, noting that the wave-number of the modulations is, therefore, $k_w = \varepsilon_0 \delta^{\frac{1}{2}}$. We obtain

$$[R_2 + 4k_c^2 \cos \varepsilon_0 X]A + 4A_{XX} = 0, \tag{22}$$

which is the Mathieu equation, and its canonical form is recovered using the substitutions $\xi = \varepsilon_0 X/2$, $\gamma = R_2/\varepsilon_0^2$ and $\chi = \pi^2/2\varepsilon_0^2$, giving

$$A_{\xi\xi} + (\gamma + 2\chi \cos 2\xi)A = 0. \tag{23}$$

Although much of the relevant analysis of (23) for this type of problem is contained in (8) it is worthwhile to summarize some of the results, as they also apply here.

Solutions of (22) and (23) exist with Floquet exponent ν such that

$$A = e^{i\nu\varepsilon_0 X/2} \sum_{n=-\infty}^{\infty} A_n e^{in\varepsilon_0 X} = e^{i\nu\xi} \sum_{n=-\infty}^{\infty} A_n e^{2in\xi}, \tag{24}$$

but ν is real only in certain regions of (R_2, ε_0)- or (γ, χ)-space. These regions are shown in Figs 1 and 2 respectively. Elsewhere ν is complex and the corresponding solutions grow exponentially in space and are thus unphysical. When ν is not an integer, equation (22) has two linearly independent solutions of the form $A_{+\nu}(X)e^{i\nu\varepsilon_0 X/2}$ and $A_{-\nu}(X)e^{-i\nu\varepsilon_0 X/2}$, where both $A_{+\nu}$ and $A_{-\nu}$ have period $2\pi/\varepsilon_0$. When ν takes integer values there are, once more, two linearly independent solutions; these have period $2\pi/\varepsilon_0$ when ν is even and period $4\pi/\varepsilon_0$ otherwise.

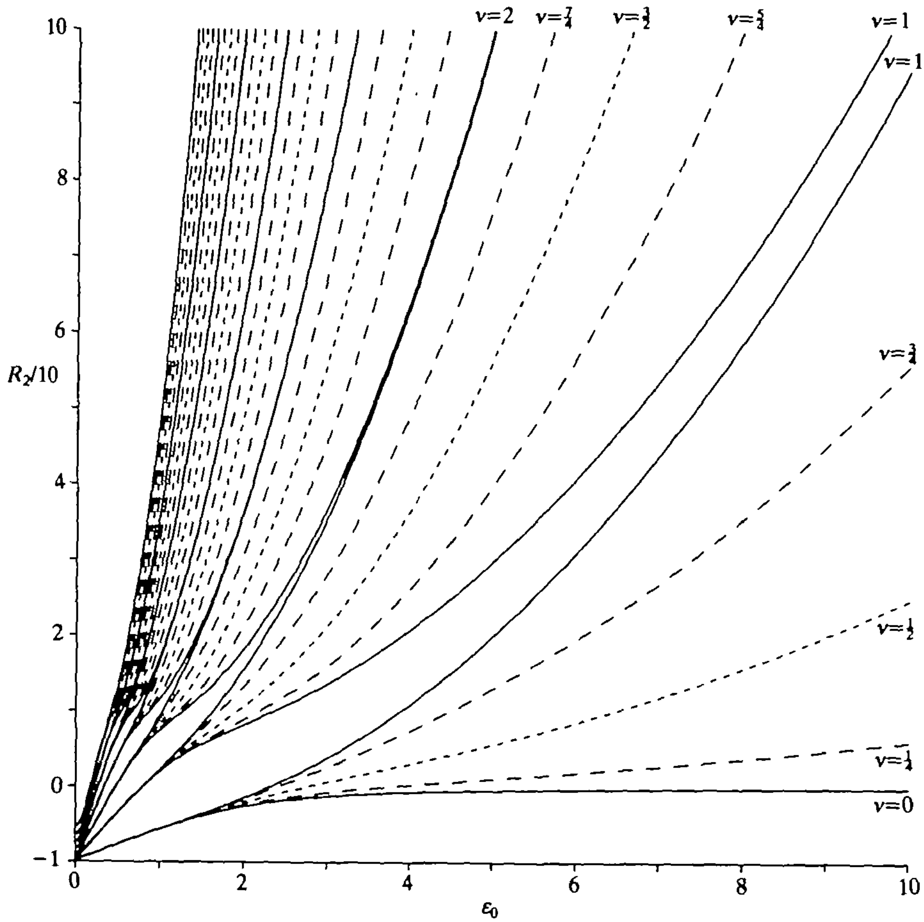

FIG. 1. Critical values of R_2 satisfying equation (22) as a function of the wavenumber of the thermal forcing ε_0, for various values of the Floquet exponent ν

For large values of ε_0 the most unstable disturbance, which is easily shown to correspond to $\nu = 0$, has the form

$$A \sim 1 + (k_c^2/\varepsilon_0^2)\cos\varepsilon_0 X + (k_0^4/8\varepsilon_0^4)\cos 2\varepsilon_0 X, \tag{25}$$

and the second-order (that is, $O(\delta)$) correction to the critical Rayleigh number is given by

$$R_{2c} \sim -2k_c^4/\varepsilon_0^2. \tag{26a}$$

Hence the critical Rayleigh number correct to $O(\delta)$ is

$$R_c \sim \pi^2 - \frac{2k_c^2}{\varepsilon_0^2}\,\delta. \tag{26b}$$

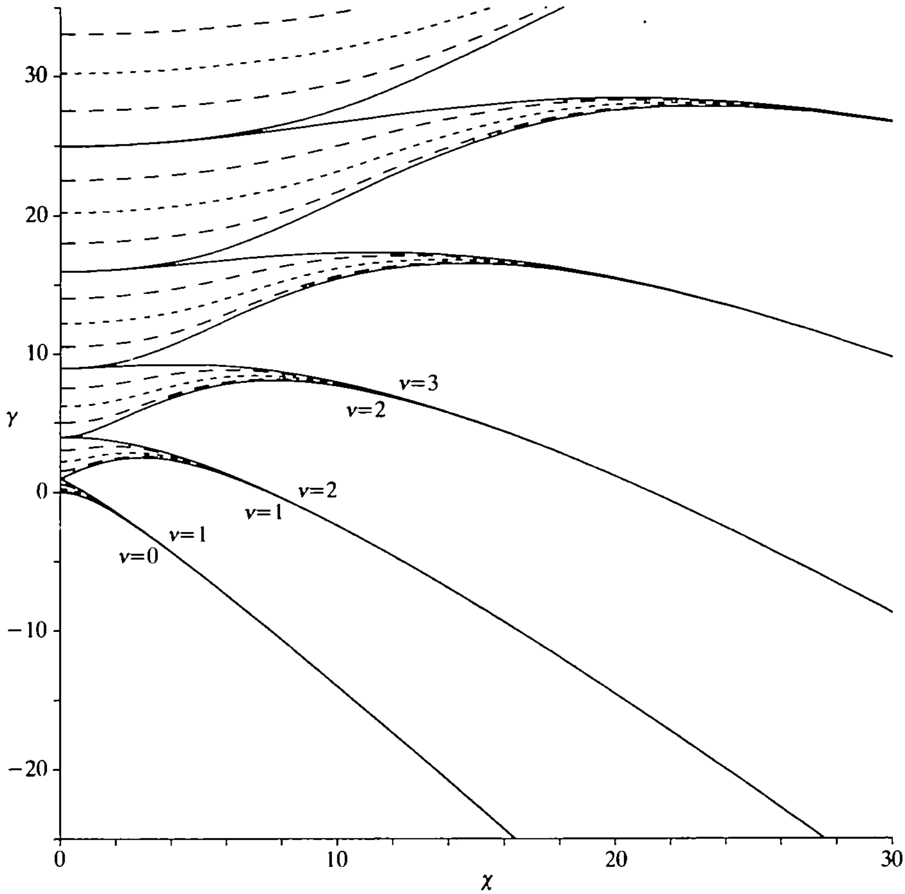

FIG. 2. The stability diagram of the Mathieu equation (23) showing γ as a function of χ for various values of ν

For small values of ε_0, the disturbance becomes concentrated near the regions of maximum local Rayleigh number; that is, near $X = 2m\pi/\varepsilon_0$ for integer m. From Abramowitz and Stegun (13) it may be shown that $R_{2c} \sim -4k_c^2 = -\pi^2$, and the critical curves corresponding to those values of ν between n and $n+1$ become exponentially close. On setting $\zeta = \varepsilon_0^{\frac{1}{2}}X$ and $R_2 = -\pi^2 + R^*\varepsilon_0$ in (22) and expanding for small ε_0, we obtain the following equation, at leading order:

$$(R^* - 2k_c^2\zeta^2)A + 4A_{\zeta\zeta} = 0, \tag{27}$$

which is the parabolic-cylinder equation. The smallest value of R^* for which a solution exists is $2^{\frac{3}{2}}k_c$. We obtain

$$A \sim \exp\left(-\sqrt{2}k_c\varepsilon_0 X^2/4\right), \tag{28}$$

$$R_{2c} \sim -\pi^2 + 2\sqrt{2}k_c\varepsilon_0, \tag{29a}$$

and therefore

$$R_c \sim \pi^2 + (-\pi^2 + 2\sqrt{2}k_c\varepsilon_0)\delta \tag{29b}$$

for small ε_0. It is worth noting that the disturbance is concentrated in the region where ζ is $O(1)$; that is, where $X = O(\varepsilon_0^{-\frac{1}{2}})$, which is small compared with the modulation length scale $O(\varepsilon_0^{-1})$.

For intermediate values of ε_0, the most unstable mode is given by $\nu = 0$ (see Fig. 1); this is a well-known result of the Floquet theory of second-order ordinary differential equations with periodic coefficients (see (14)).

5. The onset of longitudinal rolls

In order to study the onset of convection in the form of longitudinal rolls it is necessary to abandon the $\varepsilon = \delta^{\frac{1}{2}}$ scaling required for transverse rolls. This is because the spatial derivative term drops out of equation (21) for longitudinal rolls for it may be shown that the amplitude of rolls of orientation β, relative to the transverse roll, satisfies (21) with A_{XX} replaced by $A_{XX}\cos^2\beta$. In order to obtain a balance with the next highest derivative available we take $\varepsilon = O(\delta^{\frac{1}{4}})$ and assume that the roll eigensolution appears at $O(\varepsilon^2)$ (but note that the amplitude of the roll remains $O(\delta^{\frac{1}{2}})$). We set $\varepsilon = \delta^{\frac{1}{4}}$ in what follows.

On proceeding with the ε-expansion, we assume the longitudinal roll eigensolution at $O(\varepsilon^2)$,

$$\begin{pmatrix} p_2 \\ \theta_2 \end{pmatrix} = (Ae^{ik_cy} + \bar{A}e^{-ik_cy})\begin{pmatrix} k_c \sin(k_cz) \\ \frac{1}{2}\cos(k_cz) \end{pmatrix}, \tag{30}$$

where A is now a function of X and σ, and $\sigma = \frac{1}{2}\varepsilon^4 t$ is the new slow time scale. Omitting the details, the satisfaction of an orthogonality condition at $O(\varepsilon^6)$ yields

$$A_\sigma = [R_4 + 4k_c^2 g(X)]A - A_{XXXX}/k_c^2 - k_c^4 A^2\bar{A} \tag{31}$$

for the convective amplitude. The onset problem is now

$$[R_4 + 4k_c^2\cos\varepsilon_1 X]A - A_{XXXX}/k_c^2 = 0, \tag{32}$$

where we have set $g(X) = \cos\varepsilon_1 X$. It should be noted that the wavenumber of the modulations is now $k_w = \varepsilon_1\delta^{\frac{1}{4}}$, where $\varepsilon_1 = O(1)$; this is asymptotically larger than the wavenumber considered in §4. At onset, therefore, A satisfies a fourth-order form of Mathieu's equation. In (9) a related equation arises where the functional form $g(X) = -X$ describes the effects of a slowly varying fluid depth in a Bénard layer. Although (8) predicts the form of (32) there is, to our knowledge, no literature on the solutions to such an equation and therefore we present a brief analysis similar in scope to that given above for equation (21).

It would seem that a suitable canonical form for (32) is

$$A_{\xi\xi\xi\xi} - (a - 2q\cos 2\xi)A = 0, \tag{33}$$

which may be obtained with the appropriate substitutions. There exist Floquet solutions for (32) and (33) in the form

$$A = e^{i\nu\varepsilon_1 X/2} \sum_{n=-\infty}^{\infty} A_n e^{in\varepsilon_1 X} = e^{i\nu\xi} \sum_{n=-\infty}^{\infty} A_n e^{2in\xi}, \tag{34}$$

where real values of ν correspond to bounded solutions of the equations. Using (34), equations (32) and (33) were solved using a NAG matrix-eigenvalue problem solver to obtain curves of $R_4(\lambda)$ and $a(q)$ for various values of ν. These are shown in Figs 3 and 4, respectively, with successive

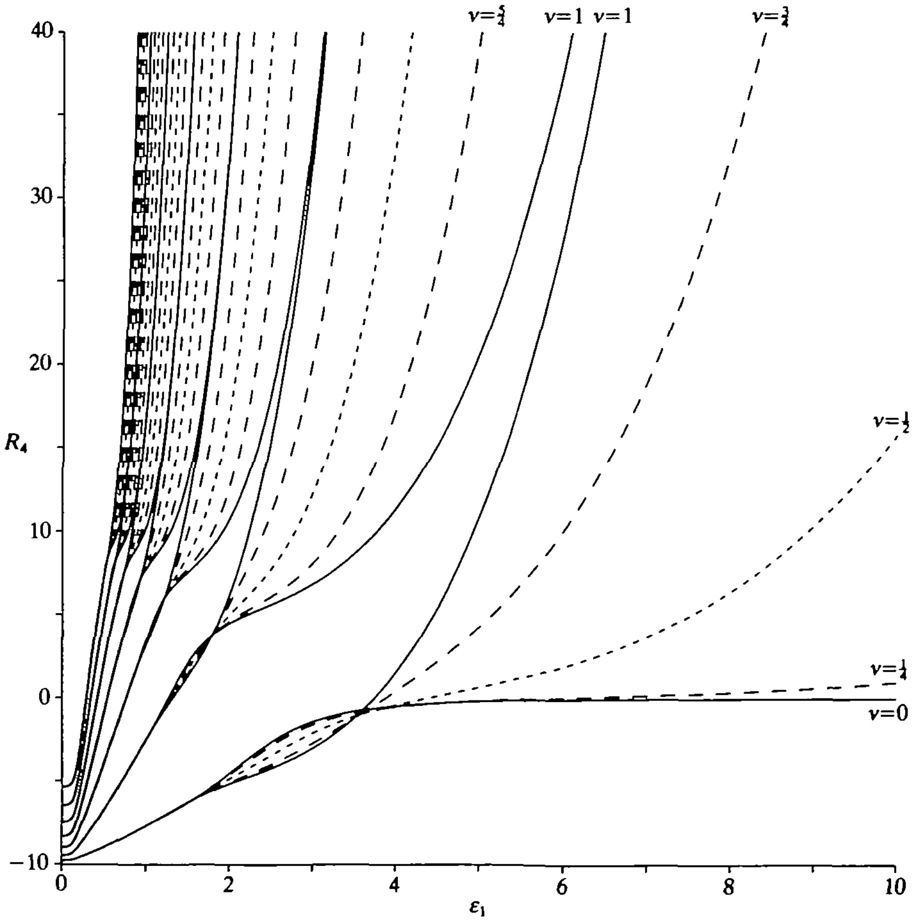

FIG. 3. Critical values of R_4 satisfying equation (32) as a function of the wavenumber of the thermal forcing ε_1, for various values of ν

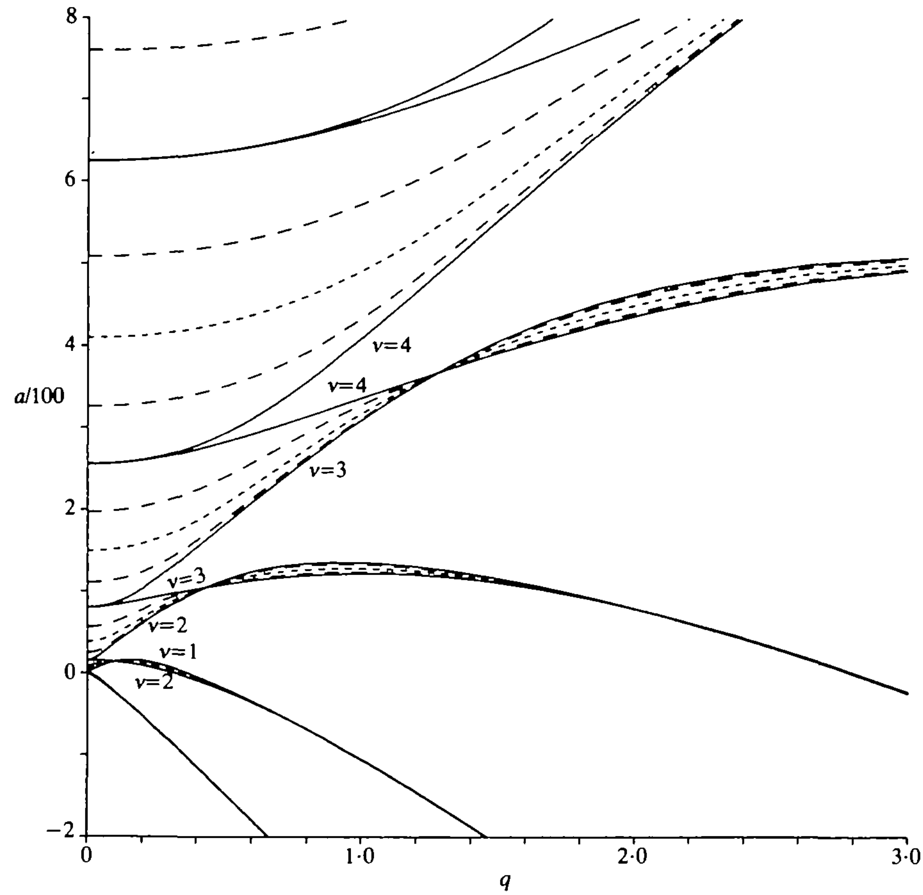

FIG. 4. The stability diagram for the fourth-order analogue of the Mathieu equation (33) showing a as a function of q for various values of v

close-up views of Fig. 3 shown in Figs 5 and 6. For Mathieu's equation there exist two values of the characteristic exponent v ($\pm v_1$, say) for every point in (R_2, ε_0)-space and the general solution is, therefore, a linear superposition of the two corresponding Floquet solutions. For equation (32), however, there are four values of v ($\pm v_2$, $\pm v_3$, say) for each point in (R_4, ε_1)-space, reflecting the higher order of the governing equation, and thus (32) has four linearly independent solutions. This increased multiplicity of v is readily seen in Figs 3 to 6 as the curves for real values of v cross and intertwine, unlike the case for the Mathieu equation. Thus (R_4, ε_1)-space is divided into three regions: (i) where both v_2 and v_3 are complex, (ii) where one or the other is real, and (iii) where both are real. In terms of the classical concept of stability of solutions to equations with periodic

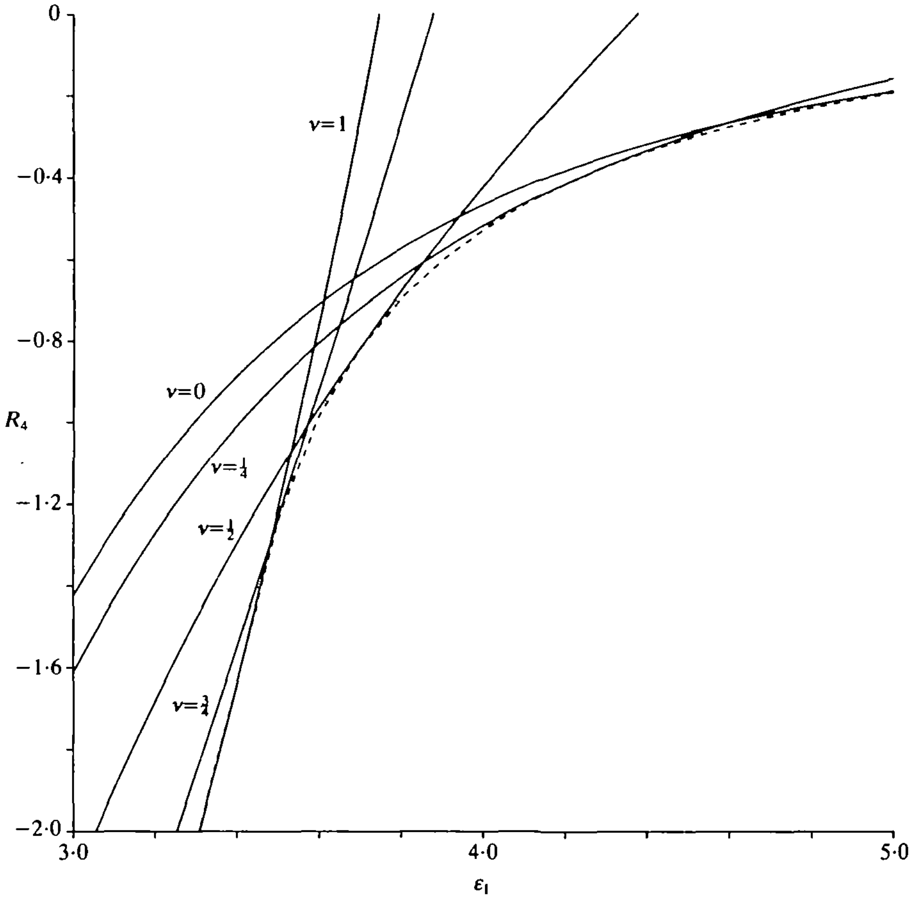

FIG. 5. A close-up of Fig. 3 showing in detail the first interchange process, as ε_1 decreases, whereby the mode with $\nu = 1$ takes over as the mode with the smallest critical value of R_4. The dashed line represents $\min_\nu R_4$ as a function of ε_1

coefficients such as Mathieu's equation, regions (i) and (ii) are unstable, whilst region (iii) is stable. In terms of the present problem, physical solutions exist in regions (ii) and (iii).

It is evident from Figs 3, 5 and 6 that the first mode to become unstable as R_4 increases is not always that which corresponds to $\nu = 0$, as is the case for the transverse mode whose amplitude satisfies Mathieu's equation. Although it is possible to rewrite (32) as a pair of second-order ordinary differential equations, the general theory of Hill's equation (that is, a second-order ordinary differential equation with periodic coefficients), such as is expounded by Magnus and Winkler (**14**), obviously does not apply here.

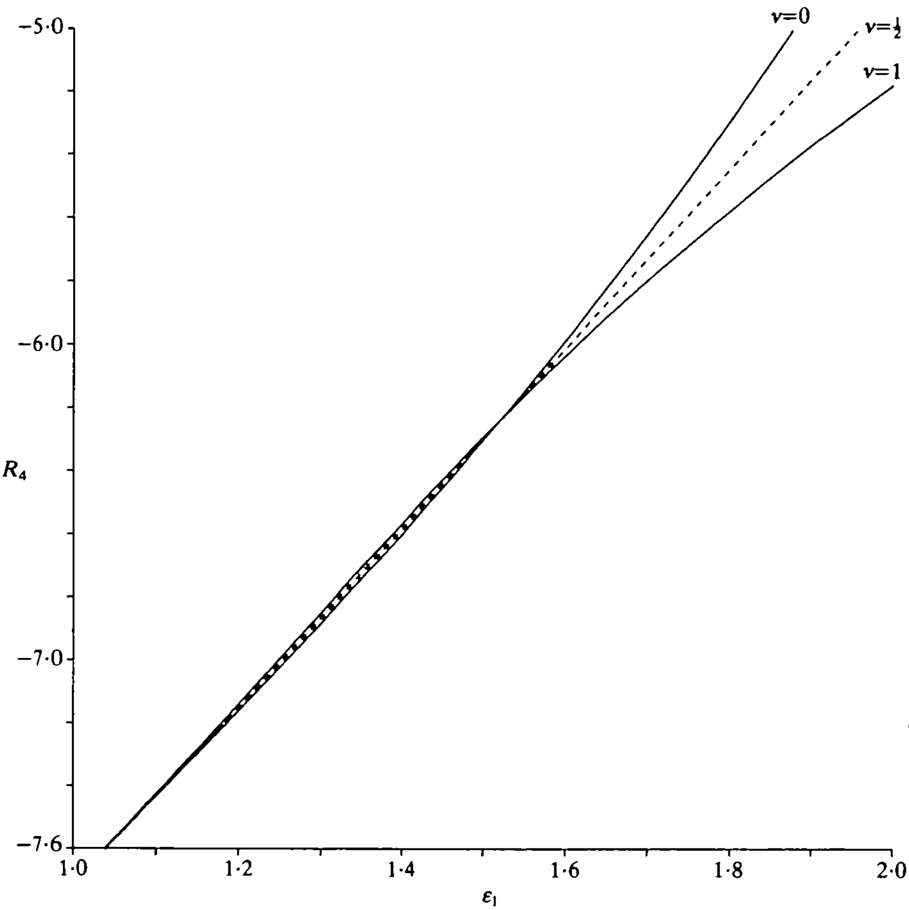

FIG. 6. A close-up of Fig. 3 showing the second interchange process whereby the mode with $v = 0$ again becomes the mode with the smallest critical value of R_4

The intertwining of the critical curves has considerable importance in the determination of the periodicity of the most unstable mode. When this mode corresponds to a rational value of v (for example, m/n) then the period is $4\pi n/\varepsilon_1$ when m is odd, and $2\pi n/\varepsilon_1$ otherwise. For irrational values of v the mode is spatially quasiperiodic. In Fig. 5 we also show $\min_v R_4$ as a function of ε_1 for $3 < \varepsilon_1 < 5$. Although this curve varies smoothly, the spatial period of the solutions it represents does not. The corresponding minimizing values, v_m, of v are shown in Fig. 7, where it may be seen that v_m varies between 0 and 1. It may also be seen that the intervals in ε_1 over which v_m is constant decrease as ε_1 decreases. Our numerical data indicate

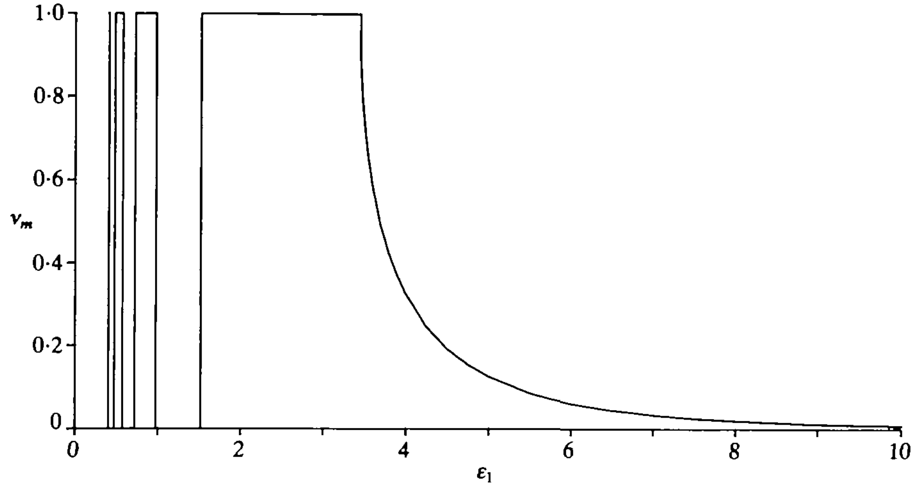

FIG. 7. The value of v_m which minimizes R_4, as a function of ε_1

that when v_m is neither 0 nor 1 the slope of the curve is not infinite but is very large whenever $\varepsilon_1 < 2$, so that v_m varies continuously.

For large values of ε_1 it may be shown that $v_m = O(\lambda^{-2})$ and the most unstable disturbance has the form

$$A \sim 1 + (4k_c^4/\varepsilon_1^4)\cos\varepsilon_1 X + (k_c^8/2\varepsilon_1^8)\cos 2\varepsilon_1 X, \tag{35}$$

whilst the fourth-order (in ε; that is, the $O(\delta)$) correction to the critical Rayleigh number is given by

$$R_{4c} \sim -8k_c^6/\varepsilon_1^4. \tag{36}$$

The disturbance again becomes concentrated about $X = 2m\pi/\varepsilon_1$ for integral m as ε_1 becomes small. On setting $R_4 = -\pi^2 + R^*\varepsilon_1^{\frac{4}{3}}$ and $\mu = \varepsilon_1^{\frac{1}{3}}X$ in (32) and expanding for small ε_1 we obtain

$$(R^* - 2k_c^2\mu^2)A - A_{\mu\mu\mu\mu}/k_c^2 = 0, \tag{37}$$

which is a fourth-order analogue of the parabolic-cylinder equation. The smallest value of R^* for which a solution exists has been found numerically to have the value 2·274508, and therefore the critical value of R_4 has the asymptotic form

$$R_4 \sim -\pi^2 + 2{\cdot}274508\varepsilon_1^{\frac{4}{3}} \tag{38}$$

for small ε_1. Although we have not been able to find an analytic expression for the disturbance, which is shown in Fig. 8, its asymptotic form for large μ is given by

$$A \sim b_0\mu^{-\frac{3}{4}}\exp(-a_0\mu^{\frac{3}{2}})\cos(a_0\mu^{\frac{3}{2}} + c_0), \tag{39}$$

where $a_0 = 2^{-\frac{1}{4}}\pi/3$, b_0 is an arbitrary constant and c_0 is unknown due to the

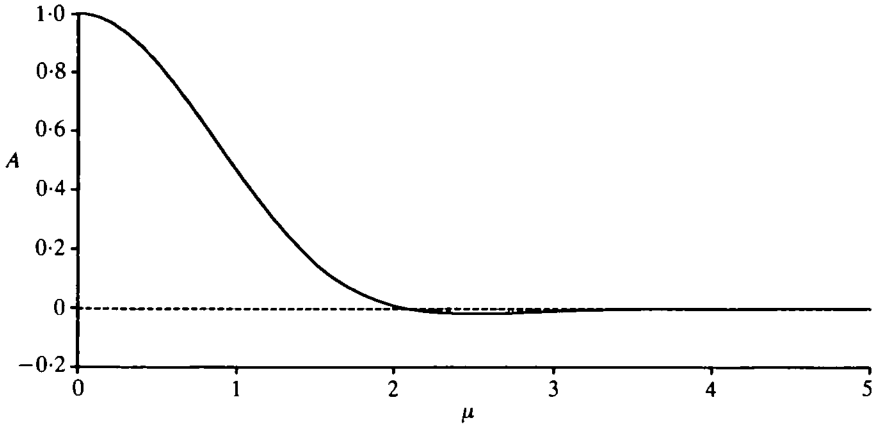

FIG. 8. The disturbance profile for small ε_1 as a function of μ

asymptotic nature of the analysis. We note that the convection planform corresponding to (39) actually takes the form of rectangular cells since $A = 0$ whenever $\cos(a_0\mu^{\frac{3}{2}} + c_0) = 0$, but these should not be confused with those described in the next section which are composed of a pair of rolls.

6. The onset of oblique rolls

At the start of §5 it was stated that, when we consider a roll of orientation β relative to the transverse roll, the A_{XX} term in (21) is replaced by $A_{XX}\cos^2\beta$. However, as the corresponding roll-eigensolution has terms proportional to $\exp(\pm ik_c(x\cos\beta - y\sin\beta))$, it is natural to reflect the additional y-dependence by introducing the slow spatial scale $Y = \varepsilon y$. Thus, for oblique rolls, equation (21) is superseded by

$$A_\tau = [R_2 + 4k_c^2 g(X)]A + 4[A_{XX}\cos^2\beta - 2A_{XY}\cos\beta\sin\beta + A_{YY}\sin^2\beta] - k_c^4A^2\bar{A} \qquad (40)$$

and therefore the A_{YY} term, which is absent in (21), has a non-zero coefficient when $\beta = \frac{1}{2}\pi$, that is, for longitudinal rolls. It is necessary therefore to extend the analysis of §5 to include such y-variations. Before doing so, however, it is instructive to consider the onset of convection in the form of oblique rolls when β is not close to $\frac{1}{2}\pi$.

The onset problem for generally oblique rolls is given by the steady, linearized form of (40). Here we return temporarily to the transverse-roll scaling, $\varepsilon = \delta^{\frac{1}{4}}$, setting $g(X) = \cos(\varepsilon_0 X)$. By defining slow spatial scales perpendicular and parallel to the oblique roll axis, $X^* = X\cos\beta - Y\sin\beta$ and $Y^* = Y\cos\beta + X\sin\beta$, respectively, the steady linearized form of (40) reduces to

$$(R_2 + 4k_c^2\cos\varepsilon_0^*\xi^*)A + 4A_{\xi^*\xi^*} = 0, \qquad (41)$$

where $\xi^* = X^* + Y^* \tan\beta = X \sec\beta$ and $\varepsilon_0^* = \varepsilon_0 \cos\beta$. Hence if the critical value of R_2 for the transverse roll and its associated eigenmode are respectively denoted by $R_{2T}(\varepsilon_0)$ and $A_T(\varepsilon_0, X)$ (cf. (22)) then the corresponding expressions for the oblique mode are

$$R_2 = R_{2T}(\varepsilon_0 \cos\beta) \quad \text{and} \quad A = A_T(\varepsilon_0 \cos\beta, X \sec\beta). \tag{42}$$

Thus the critical Rayleigh number for oblique modes decreases as β increases from 0 (transverse rolls) to $\frac{1}{2}\pi$ (longitudinal rolls), and so transverse rolls do not comprise the most unstable mode. It would seem from this simple analysis that the most unstable mode is the longitudinal roll; we shall now examine this conjecture by studying rolls with β close to $\frac{1}{2}\pi$.

For such values of β the coefficients of the X-derivative terms in (40) are small, and we shall proceed by defining the slow spatial scales $X = \varepsilon x$ (in order to recover the A_{XXXX} term considered in §5) and $Y = \varepsilon^2 y$, where we have taken the longitudinal-roll scaling $\varepsilon = \delta^{\frac{1}{4}}$, set $g(X) = \cos \varepsilon_1 X$ (and therefore the modulation wavenumber is $k_w = \varepsilon_1 \delta^{\frac{1}{4}}$), and where the basic roll eigensolution is the longitudinal roll. The Y-scaling may be seen to be consistent with the $O(\varepsilon^4)$ variation in the Rayleigh number allowed for in §5. The X-scaling is also that appropriate for the study of the zig-zag instability (see (**3**, **15**)); in the absence of thermal non-uniformities any X dependence in A serves to redefine the orientation of the roll from $\frac{1}{2}\pi$ to $\frac{1}{2}\pi - O(\varepsilon)$.

After proceeding with the perturbation expansion the imposition of orthogonality conditions at $O(\varepsilon^6)$ yields the amplitude equation

$$A_\sigma = [R_4 + 4k_c^2 g(X)]A + \left(2\frac{\partial}{\partial Y} - \frac{i}{k_c}\frac{\partial^2}{\partial X^2}\right)^2 A - k_c^4 A^2 \bar{A}, \tag{43}$$

and therefore the required equation for the onset problem is

$$(R_4 + 4k_c^2 \cos \varepsilon_1 X)A + \left(2\frac{\partial}{\partial Y} - \frac{i}{k_c}\frac{\partial^2}{\partial X^2}\right)^2 A = 0. \tag{44}$$

This equation possesses solutions with the form

$$A = e^{i(KY + \nu\varepsilon_1 X/2)} \sum_{n=-\infty}^{\infty} A_n e^{in\varepsilon_1 X}, \tag{45}$$

and it is straightforward to compute critical values of R_4 as a function of K, ν and ε_1. For general values of K the critical curves behave in a manner similar to those corresponding to the longitudinal roll, shown in Fig. 3, and are therefore not presented. We shall now consider the spatial planform of the most unstable mode and its critical value of R_4 for large, small and intermediate values of ε_1.

6a. *Onset for large values of* ε_1

For large values of ε_1, the asymptotic form for A is

$$A \sim e^{iKY}\left[e^{i\nu\varepsilon_1 X/2} + \frac{64k_c^4}{\varepsilon_1^4}\left(\frac{e^{i\varepsilon_1(1+\frac{1}{2}\nu)X}}{(\nu+2)^4-\nu^4} + \frac{e^{i\varepsilon_1(-1+\frac{1}{2}\nu)X}}{(\nu-2)^4-\nu^4}\right)\right], \tag{46}$$

where

$$R_{4c} = \left(\frac{\nu^2\varepsilon_1^2}{4k_c} + 2K\right)^2 + O(\varepsilon_1^{-4}) \tag{47}$$

and $\nu \neq \pm 1$. For rolls with a wavenumber greater than k_c (that is, $K > 0$, since the wavenumber is $k_c + \varepsilon^2 K$), R_{4c} is easily seen to be minimized by setting $\nu = 0$ in (47), in which case we obtain

$$A \sim e^{iKY}\left[1 + \left(\frac{4k_c^4}{\varepsilon_1^4} - \frac{16Kk_c^5}{\varepsilon_1^6} + \frac{64K^2k_c^6}{\varepsilon_1^8}\right)\cos\varepsilon_1 X + \frac{k_c^8}{2\varepsilon_1^8}\cos 2\varepsilon_1 X\right] \tag{48a}$$

and

$$R_{4c} \sim 4K^2 - (8k_c^6/\varepsilon_1^4). \tag{48b}$$

However, for rolls with a wavenumber less than k_c (that is, $K < 0$) it is better to choose $|K| = O(\varepsilon_1^2)$ in order to minimize the critical Rayleigh number. It may be shown that R_4 is minimized by choosing

$$K = K_m = -\varepsilon_1^2/8k_c \quad \text{and} \quad \nu = 1, \tag{49a}$$

whereupon we find that

$$A \sim e^{iK_mY}\left[\cos\frac{\varepsilon_1 X}{2} + \frac{1}{2}\frac{k_c^4}{\varepsilon_1^4}\cos\frac{3\varepsilon_1 X}{2}\right] \tag{49b}$$

and

$$R_{4c} \sim -2k_c^2 - k_c^6/\varepsilon_1^4. \tag{49c}$$

The leading-order form of (49b) may be written as $\frac{1}{2}e^{iK_mY}(e^{i\varepsilon_1 X/2} + e^{-i\varepsilon_1 X/2})$ and therefore this mode is the superposition of two rolls with wavevectors $(\frac{1}{2}\varepsilon_1\delta^{\frac{1}{4}}, k_c + \delta^{\frac{1}{2}}K_m)$ and $(-\frac{1}{2}\varepsilon_1\delta^{\frac{1}{4}}, k_c + \delta^{\frac{1}{2}}K_m)$. From (30) and (49b) the horizontal spatial form of the $O(\varepsilon^2)$ eigensolution is, to leading order, $\cos(k_c + \delta^{\frac{1}{2}}K_m)y \cos\varepsilon_1 X/2$, and is, therefore, a rectangular cell with a large-aspect-ratio planform. We note that this solution, together with the above value of R_{4c}, matches asymptotically, as $\delta \to 0$, the rectangular cellular planform described in (4) and its corresponding critical Rayleigh number. In common with the analysis of (4) (at least for symmetric thermal modulations) this mode is the most unstable mode.

6b. *Onset for small values of* ε_1

We turn now to the onset problem for small values of ε_1. Once more the disturbance becomes concentrated near where the temperature drop across the layer achieves its maxima. We also find that the spatial form of the most unstable mode depends on the sign of K and we proceed by setting

$A = B(X)e^{iKY}$ in (44) to obtain

$$(R_4 - 4K^2 - 4k_c^2 \cos \varepsilon_1 X)B + \frac{4K}{k_c} B_{XX} - \frac{B_{XXXX}}{k_c^2} = 0. \tag{50}$$

Now, for positive values of K both the second- and fourth-derivative terms in (50) are positively diffusive. As ε_1 is small, $\cos(\varepsilon_1 X)$ varies over an X-scale which is large, therefore the fourth derivative term is small compared with the second derivative and we can neglect it. Thus we use the transverse-roll scaling for small ε_1 to obtain the spatial form of this mode; we set $X = \varepsilon_1^{-\frac{1}{2}}\xi$ and $R_4 = -\pi^2 + S\varepsilon_1$, and expand for small ε_1. We obtain

$$(S - 2k_c^2\xi^2)B + \frac{4K}{k_c} B_{\xi\xi} = 0 \tag{51}$$

at leading order, and the solution of this equation with the lowest critical value of S is given by

$$B \sim \exp\left(-\frac{k_c}{2}\left(\frac{k_c}{K}\right)^{\frac{1}{2}}\xi^2\right) = \exp\left(-\frac{k_c}{2}\left(\frac{k_c}{K}\right)^{\frac{1}{2}}\varepsilon_1 X^2\right), \tag{52a}$$

where $S_c = 4(k_c K)^{\frac{1}{2}}$ and so the $O(\delta)$ correction to the critical Rayleigh number is

$$R_{4c} \sim -\pi^2 + 4(k_c K)^{\frac{1}{2}}\varepsilon_1. \tag{52b}$$

Obviously this solution is invalid for negative K, but it may be shown that the asymptotic expansion breaks down when $K = O(\varepsilon_1^{\frac{2}{3}})$. This is because the length scale over which X varies in (52a) is now $\varepsilon_1^{-\frac{1}{3}}$ for $O(\varepsilon_1^{\frac{2}{3}})$ values of K and therefore the fourth-derivative term in (50) is formally of the same order of magnitude as the second-derivative term, and has to be included in the analysis. We analyse the $K = O(\varepsilon_1^{\frac{2}{3}})$ regime below.

For negative values of K in (50) the second-derivative term is negatively diffusive and must be balanced with the positively diffusive fourth derivative. Once more we proceed by setting $X = \varepsilon_1^{\frac{1}{2}}\xi$, but now we use a multiple-scales analysis and expand:

$$R_{4c} = S_0 + \lambda^{\frac{1}{2}}S_1 + \ldots \quad \text{and} \quad B = B_0 + \lambda^{\frac{1}{2}}B_1 + \ldots,$$

where $B_0 = B_0(X, \xi)$. Details of this analysis may be found in the Appendix. We find that the disturbance has the form

$$A \sim \exp\left[iKY + i(-2k_c K)^{\frac{1}{2}}X - \frac{k_c}{4}\left(\frac{-k_c}{K}\right)^{\frac{1}{2}}\varepsilon_1 X^2\right], \tag{53a}$$

where

$$R_{4c} \sim -\pi^2 + 4(-k_c K)^{\frac{1}{2}}\varepsilon_1. \tag{53b}$$

In common with the results presented above for positive K, this solution also breaks down when $K = O(\varepsilon_1^{\frac{2}{3}})$, since the two length scales over which the X-dependent terms vary in (53a) become coincident and equal to $\varepsilon_1^{-\frac{1}{3}}$. It is important, therefore, to consider this small range of values of K since, from (52b), (53b), R_{4c} decreases as the range is approached.

On setting $K = -\kappa\varepsilon_1^{\frac{2}{3}}$, $R_4 = -\pi^2 + R^*\varepsilon_1^{\frac{4}{3}}$ and $X = \varepsilon^{-\frac{1}{3}}\mu$ in (50) we obtain, at leading order,

$$(R^* - k_c^2\mu^2)B - \frac{4\kappa}{k_c} B_{\mu\mu} - \frac{B_{\mu\mu\mu\mu}}{k_c^2} = 0, \tag{54}$$

which is a generalization of (37). This equation has to be solved numerically to find values of R^* for which non-trivial solutions exist. We find that the smallest value of R^* is 1·940 corresponding to $\kappa = 0{\cdot}3199$. This disturbance, which has a spatial profile similar to the longitudinal roll, has a critical Rayleigh number

$$R_{4c} \sim -\pi^2 + 1{\cdot}940\varepsilon_1^{\frac{4}{3}} \tag{55a}$$

when

$$K \sim -0{\cdot}3199\varepsilon_1^{\frac{2}{3}}. \tag{55b}$$

It is evident that this mode appears at a lower value of the Rayleigh number than any other mode for small values of ε_1 (see (38)).

6c. *Onset for intermediate values of* ε_1

The solution of (44) for intermediate values of ε_1 has to be effected numerically. Since we are interested in the most unstable mode it is necessary to minimize the critical values of R_4 over both v and K for each value of ε_1. The easier way is to minimize first with respect to K and the results of this procedure are shown in Figs 9, 10 and 11. In Fig. 9 we show $\min_K R_4(K, \varepsilon_1, v)$ and we see that these curves intersect one another. However, unlike those shown in Fig. 3, the minimum value of R_4 is taken only by $v = 0$ and $v = 1$. Thus, there is now a simple interchange of critical values of v rather than a continuous one. This is seen clearly in Fig. 10 where a close-up view is shown of the first point of interchange as ε_1 decreases. When $v = 1$ the disturbance consists of rectangular cells similar to those comprising the most unstable mode for $\varepsilon_1 \gg 1$ but with a spatial amplitude modulation. However, when $v = 0$, the disturbance is a roll of orientation $\frac{1}{2}\pi$, a longitudinal roll, the wavenumber of which is $k_c + \delta^{\frac{1}{2}}K$.

In Fig. 9 we see, and it is easily proved, that the minimum values of R_4 tend to zero as $\varepsilon_1 \to \infty$ if $0 \leq v < 1$. When $v = 1$, however, the minimum value tends to $-\frac{1}{2}\pi^2$, in agreement with (49c). In Fig. 11 we show the minimizing values of K as a function of λ and v. Numerically, we find that these values are consistent with the above asymptotic expansions (see (49a), (55b)). It has already been noted that certain modes with negative values of K constitute the most unstable modes, at least for $\varepsilon_1 \ll 1$ and $\varepsilon_1 \gg 1$. A comparison of Figs 3 and 9 now shows that modes with negative K comprise the most unstable modes for all values of ε_1. Whenever $v = 0$ this mode is a

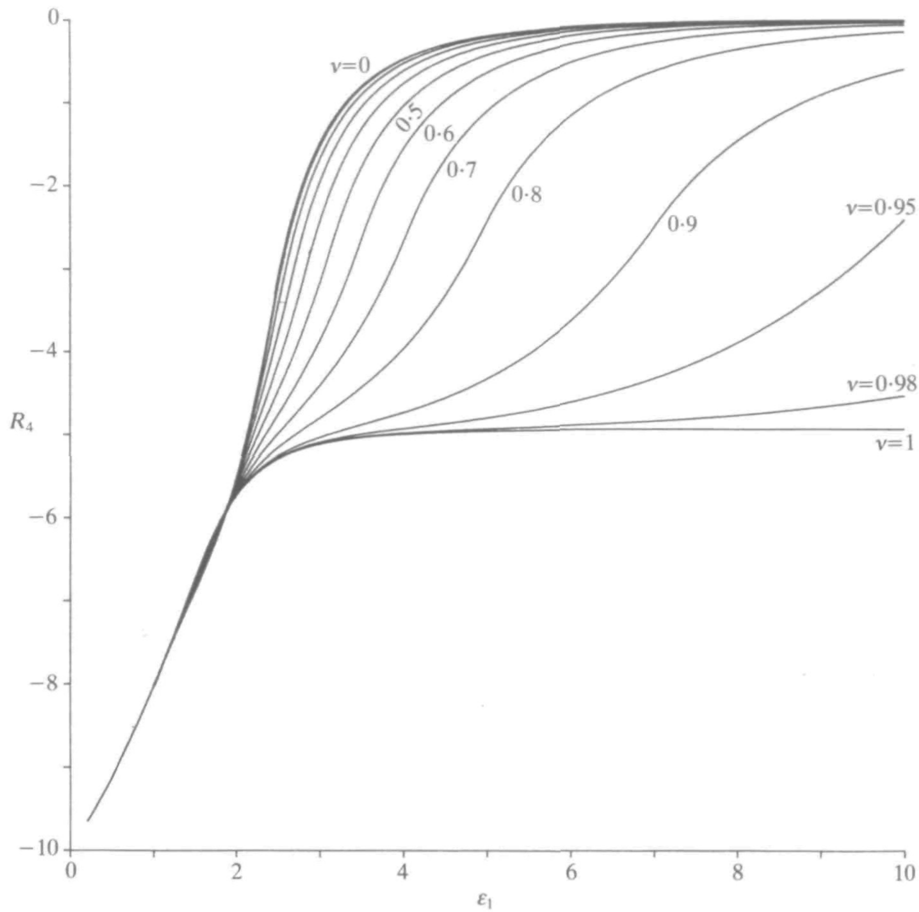

FIG. 9. Minimum values of R_4 with respect to K, as a function of ε_1 for various values of ν

roll but when $\nu = 1$ it is a large-aspect-ratio rectangular cell. Hence the planform of the most unstable mode depends on the precise value of ε_1.

7. Conclusions

We have presented a study of the onset of convection in a porous layer where symmetric non-uniform heating at the horizontal boundaries has an associated length scale which is large compared with the depth of the layer. Thus this paper complements and extends others (cf. (**1** to **4**)) dealing with the effects of boundary non-uniformities with a wavelength comparable with the layer depth. Attention here has been focused on the onset of rolls of various orientations and, in particular, on the onset of transverse and

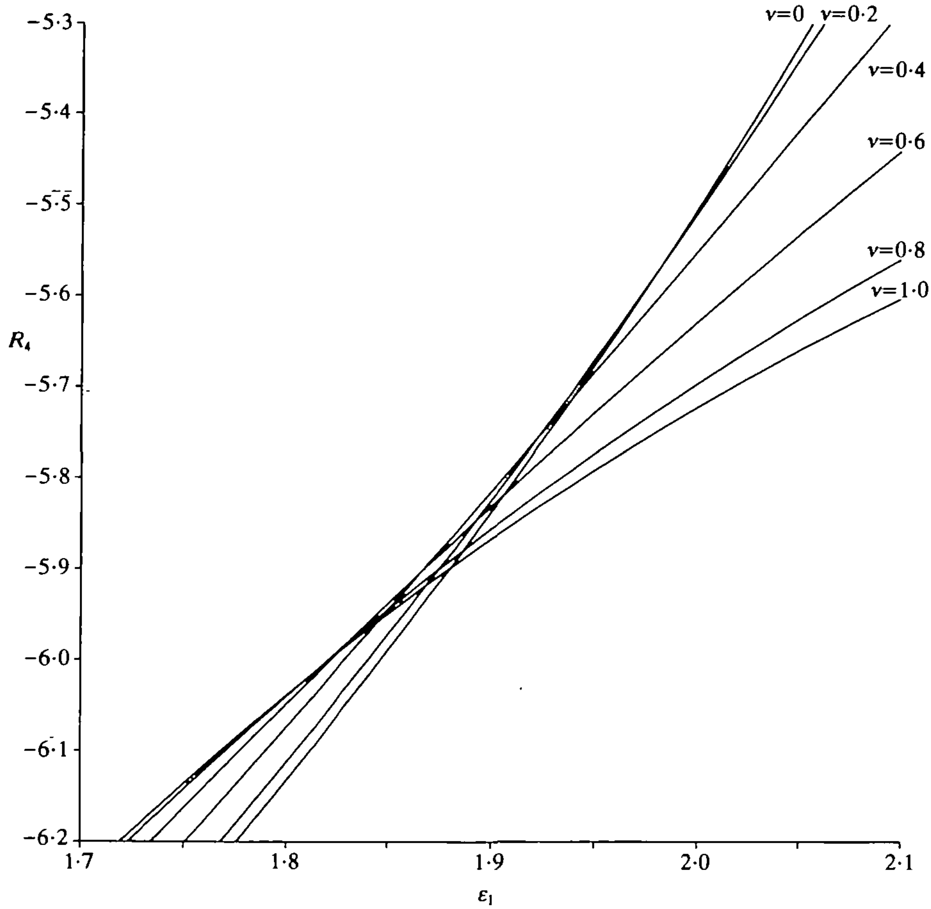

FIG. 10. A close-up of the first exchange process in Fig. 9 for decreasing ε_1, showing the discontinuous transition between $\nu = 0$ and $\nu = 1$ as the minimizing value of ν. The second interchange is similar

longitudinal rolls. Within the thermal-modulation length scales considered, asymptotic results for both large and small modulation wavenumbers have been deduced. These results were confirmed and extended to intermediate wavenumbers using numerical methods.

We have found that the appropriate order of magnitude of the modulation wavenumber depends on the orientation of the roll disturbance. For transverse rolls and oblique rolls (in general) the modulation wavenumber was taken to be $\varepsilon_0\delta^{\frac{1}{2}}$ and, for longitudinal rolls and oblique rolls at an $O(\delta^{\frac{1}{4}})$ orientation relative to the longitudinal roll, it was taken to be $\varepsilon_1\delta^{\frac{1}{4}}$, which is an asymptotically larger value. In all cases, however, the most unstable mode has a critical Rayleigh number lying between $\pi^2(1 - \delta)$ and π^2.

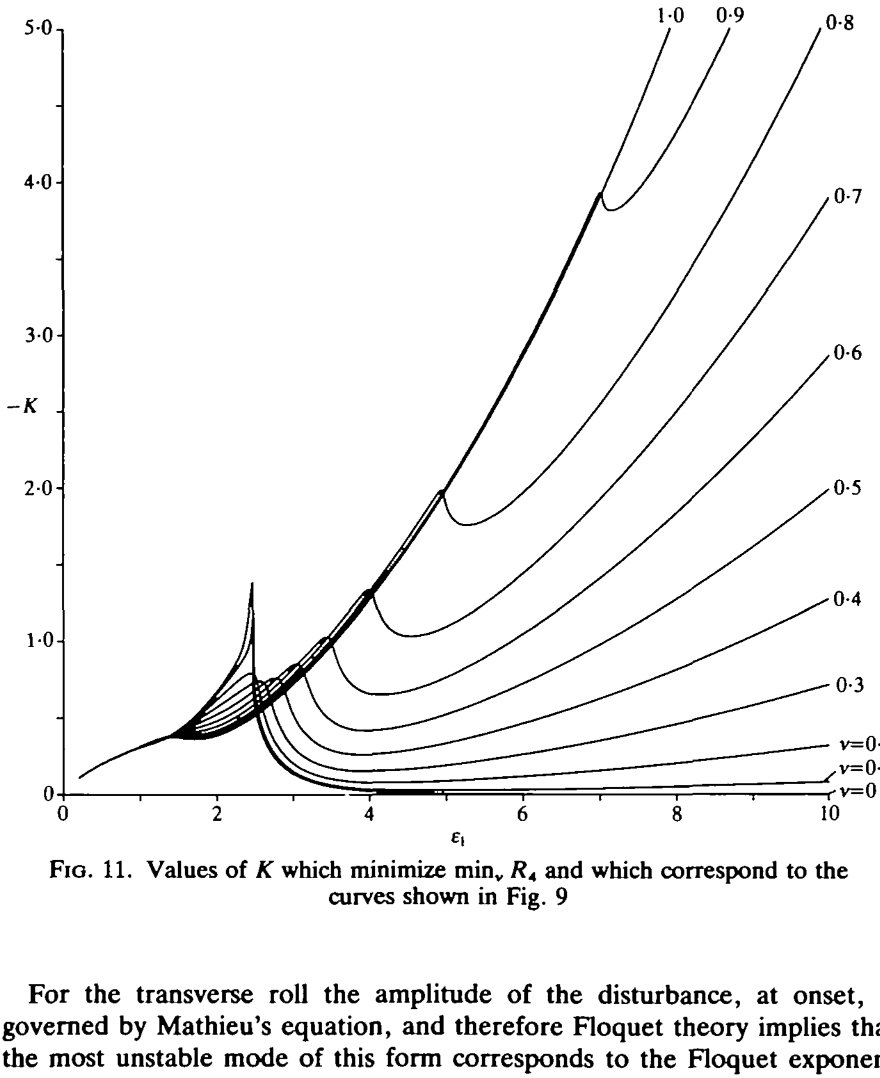

FIG. 11. Values of K which minimize $\min_\nu R_4$ and which correspond to the curves shown in Fig. 9

For the transverse roll the amplitude of the disturbance, at onset, is governed by Mathieu's equation, and therefore Floquet theory implies that the most unstable mode of this form corresponds to the Floquet exponent $\nu = 0$ for all values of ε_0. For large values of ε_0 the amplitude of the disturbance is constant to leading order. As ε_0 decreases, however, the amplitude develops a spatial structure which is most pronounced when ε_0 is small, in which case the disturbance is concentrated near the regions of maximum temperature drop across the layer. Furthermore, the critical value of Ra decreases monotonically as ε_0 decreases.

We have shown that the onset of modes in the form of oblique rolls may be deduced directly from the above results for the transverse roll. It is found that, for a given value of ε_0, all oblique modes have a lower critical Rayleigh number than the transverse roll. Moreover, the critical value decreases as the orientation relative to the transverse roll increases towards $\frac{1}{2}\pi$, the orientation of the longitudinal roll.

The onset of convection in the form of longitudinal rolls was found to be governed by a fourth-order form of Mathieu's equation. In this case $\nu = 0$ was no longer necessarily the value of the Floquet exponent which minimized the critical Rayleigh number. Instead, the minimizing value of ν was found to vary smoothly between 0 and 1 as the modulation wavenumber ε_1 varied, with finite intervals in ε_1 where ν took one or other of its extreme values. Since the wavelength of disturbances depends on ν, the wavelength of the most unstable mode varies discontinuously as ε_1 varies, and the mode is quasiperiodic whenever ν is irrational. Again, the critical Rayleigh number decreases monotonically as ε_1 decreases.

The analysis of the onset of rolls orientated at a small angle to the longitudinal roll was facilitated by considering the effects of the zig-zag instability on the longitudinal roll. In effect we relaxed the assumption that the longitudinal roll has a wavenumber precisely equal to k_c, and the departure from this value is measured by the value of K. Minimization of the critical Rayleigh number over all values of ν and K yields the rather surprising result, in view of the results for the longitudinal roll ($K = 0$), that the minimizing value of ν is either 0 or 1, depending on the precise value of ε_1. Another surprising result is that the most unstable mode for large values of ε_1 corresponds to $\nu = 1$ and therefore the mode has a large-aspect-ratio rectangular planform. Numerical and asymptotic results show that modes with negative values of K constitute the most unstable modes for this problem.

All the results we have presented concerning the onset of convection for large modulation wavenumbers may be shown to match asymptotically, as $\delta \to 0$, with the results of (4) for symmetric thermal modulations with small, but $O(1)$, wavenumbers. In particular, this includes the result that rectangular cells are the most unstable mode for large ε_1.

Here we have concentrated on layers of doubly-infinite horizontal extent and it is natural to question the effects of finite dimensions in either or both horizontal directions. We shall restrict our comments to layers with insulating end walls. For a layer which is of finite extent in the x-direction some of the above analysis would need to be modified. For example, only certain Floquet exponents would be realizable for the transverse-roll analysis but the spatial form of the remaining modes and the identity of the most unstable transverse roll remains unchanged. The spatial form of the longitudinal roll would be modified in order to satisfy the boundary conditions $A_X = A_{XXX} = 0$ at both end walls. For a layer which is narrow in the spanwise direction the longitudinal analysis applies if the width of the layer is an exact multiple of k_c; otherwise transverse rolls may constitute the most unstable mode. It should be noted that unlike the corresponding situation for a fluid layer the spatial forms of transverse rolls of finite length are identical to those of infinite length due to the presence of slip conditions at boundaries.

As nonlinear amplitude equations have been derived it still remains to calculate the effects of spatial modulations on the stability of finite-amplitude convection. Preliminary results for two-dimensional flow indicate that the transverse roll corresponding to $v = 0$ is stable (to two-dimensional disturbances) and transports more heat than other transverse modes. All other transverse modes are unstable at onset, although some regain stability after one or more pitchfork bifurcations.

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APPENDIX

In this Appendix we present a brief multiple-scales analysis of equation (50):

$$(R_4 - 4K^2 - 4k_c^2 \cos \varepsilon_1 X)B + \frac{4K}{k_c} B_{XX} - \frac{B_{XXXX}}{k_c^2} = 0 \tag{A1}$$

for negative values of K and small values of ε_1. We set $X = \varepsilon_1^{\frac{1}{2}}\xi$ and expand both R_4 and B as a power series in $\varepsilon_1^{\frac{1}{2}}$:

$$R_4 = S_0 + \varepsilon_1^{\frac{1}{2}} S_1 + \varepsilon_1 S_2 + \ldots, \tag{A2a}$$

$$B = B_0 + \varepsilon_1^{\frac{1}{2}} B_1 + \varepsilon_1 B_2 + \ldots. \tag{A2b}$$

At leading order (A1) reduces to

$$(S_0 - 4K^2 + 4k_c^2)B_0 + \frac{4K}{k_c} B_{0XX} - \frac{B_{0XXXX}}{k_c^2} = 0 \tag{A3}$$

and hence $B_0 = C(\xi)e^{iMX}$ if M satisfies

$$M^2 = -2Kk_c + k_c[S_0 + 4k_c^2]^{\frac{1}{2}}. \tag{A4}$$

At $O(\varepsilon_1^{\frac{1}{2}})$ we obtain the equation

$$(S_0 - 4K^2 + 4k_c^2)B_1 + \frac{4K}{k_c}B_{1XX} - \frac{B_{1XXXX}}{k_c^2} = -e^{iMX}\left[S_1 C + \frac{4iM}{k_c^2}(2Kk_c + M^2)C_\xi\right]. \tag{A5}$$

All the terms on the right-hand side are multiples of the eigensolution of the left-hand side, and therefore a solution does not exist unless $S_1 = 0$ and

$$M^2 = -2Kk_c. \tag{A6}$$

In conjunction with (A4) this implies further that $S_0 = -4k_c^2$, and that $B_1 = 0$ is a solution of (A5).

At $O(\varepsilon_1)$ we find that

$$(S_0 - 4K^2 + 4k_c^2)B_2 + \frac{4K}{k_c}B_{2XX} - \frac{B_{2XXXX}}{k_c^2} = -e^{iMX}\left[(S_2 - 2k_c^2\xi^2)C - \frac{8K}{k_c}C_{\xi\xi}\right], \tag{A7}$$

and so the existence of a solution requires that C satisfies

$$(S_2 - 2k_c^2\xi^2)C - \frac{8K}{k_c}C_{\xi\xi} = 0. \tag{A8}$$

We recall that K is negative so that the second-derivative term is positively diffusive. Hence

$$C = \exp\left[-\frac{k_c}{4}\left(\frac{-k_c}{K}\right)^{\frac{1}{2}}\xi^2\right] \quad \text{and} \quad S_2 = 4(-k_c K)^{\frac{1}{2}} \tag{A9}$$

and therefore the disturbance has the leading-order form

$$A \sim \exp\left[iKY + i(-2k_c K)^{\frac{1}{2}}X - \frac{k_c}{4}\left(\frac{-k_c}{K}\right)^{\frac{1}{2}}\varepsilon_1 X^2\right]. \tag{A10}$$