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The Development of the Asymptotic Viscous Dissipation Profile in a Vertical Free Convective Boundary Layer Flow in a Porous Medium

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Abstract. The effect of viscous dissipation on the development of the boundary layer flow from a cold vertical surface embedded in a Darcian porous medium is investigated. It is found that the flow evolves gradually from the classical Cheng–Minkowycz form to the recently discovered asymptotic dissipation profile which is a parallel flow.

Key words: free convection, boundary layer, viscous dissipation, asymptotic dissipation profile.

1. Introduction

The viscous dissipation effect, that is, a local production of thermal energy through the mechanism of viscous stresses, serves to modify, sometimes greatly, free, forced and mixed convection flows in both clear viscous fluids and in fluid-saturated porous media. From a mathematical point of view, such modifications in the behaviour of the flows arise because of an additional term in the energy equation which expresses the rate of volumetric heat generation, q''', by viscous dissipation, and which (for a plane boundary layer flow) is given in these two cases by

$$q'''_{\text{clear}} \equiv \mu \left(\frac{\partial u}{\partial y}\right)^2 \quad \text{and} \quad q'''_{\text{Darcy}} \equiv \frac{\mu}{K}u^2 \tag{i}$$

respectively. It would appear that the Expression (i) of q''' for a Darcy flow was deduced for the first time independently by Ene and Sanchez-Palencia (1982) and Bejan (1995). Under the earlier applications of this 'u^2-model' of viscous dissipation in porous media should also be mentioned the papers of Nakayama and Pop (1989) and Ingham *et al.* (1990).

From a physical point of view, the difference between the two cases originates in the fact that u denotes the actual fluid velocity for a clear fluid flow, but denotes a fluid flux velocity (i.e. a macroscopic, averaged quantity) for a porous medium flow. Therefore, at microscopic levels within a porous medium, the convecting fluid is necessarily extruded through the pores of the solid skeleton, and local flows are typically three-dimensional even when the overall macroscopic flow is uniform and unidirectional. This microscopic process is what enhances the rate of heat generation by viscous dissipation considerably. Thus, as it is seen immediately, in a uniform forced convection flow ($u = \text{const.} \equiv u_\infty$), no heat is released by viscous dissipation in clear fluids, but in porous media the heat generation rate increases with u quadratically. As has been shown recently (Magyari *et al.*, 2002) this fact has important consequences for the far-field thermal boundary conditions which have to be imposed for both forced and mixed convection in extended porous media. For free convection flows Expression (i) is compatible with the uniform asymptotic condition $T\,(x, y \to \infty) = \text{const.} = T_\infty$ which is usually imposed on the temperature field since $u \to 0$ as $y \to \infty$. But in forced and mixed convection flows in extended porous media this asymptotic thermal condition contradicts the corresponding energy equation in which the term $q'''_{\text{Darcy}} = (\mu/K)u_\infty^2$ is non-vanishing as $y \to \infty$. Accordingly, some recent results on mixed convection flows in extended porous media (Tashtoush, 2000; Murthy, 2001) should be reconsidered (see Magyari *et al.*, 2003) by taking into account suitably modified (Magyari *et al.*, 2002) boundary conditions on T in the far field.

Although the *quantitative* effect of viscous dissipation is often negligible (for details and exceptions see Gebhart, 1962; Gebhart and Mollendorf, 1969; Nield, 2000), its *qualitative* effect may become significant. Indeed Magyari and Keller (2003a) showed that, for flow over a hot vertical plate embedded in a saturated porous medium in which a uniform horizontal flow ($v = \text{const.} \equiv v_0$) is present, viscous dissipation can give rise to an ascending boundary layer of constant thickness. Such a phenomenon does not exist in the absence of viscous dissipation. Very surprisingly, in such a system (i.e. with $v = \text{const.} \equiv v_0$) the buoyancy forces due to the heat released in the bulk of the fluid by viscous dissipation may cause an ascending wall jet-like (i.e. $u_{\text{wall}} = u_\infty = 0$) free convection flow, when the wall temperature coincides with the ambient temperature of the fluid, $T_\text{w} \equiv T_\infty$ (Magyari and Keller, 2003b). A further interesting effect of the presence of viscous dissipation is that it breaks both the physical and mathematical symmetry which exists otherwise between a free convection boundary layer flow ascending from a hot plate ($T_\text{w} > T_\infty$) and its counterpart, descending from a cold plate ($T_\text{w} < T_\infty$). For this latter case, Magyari and Keller (2003c) have shown that the (small) effect of viscous dissipation is able to cancel the (small) transverse component of the velocity field, thus giving rise to a strictly parallel boundary layer flow of a constant thickness. However, in the case of a free convection flow ascending from a hot plate, this 'self-parallelisation' effect due to viscous dissipation does not exist.

The main aim of the present paper is to show that this parallel flow, which is obtained as a possible elementary analytical solution of the governing equations (Magyari and Keller, 2003c), does exist and is nothing more than the asymptotic configuration achieved by the natural evolution of the flow away from the classical Cheng–Minkowycz boundary layer profile. This 'asymptotic viscous dissipation profile' may then be said to exist beyond a certain distance x_* from the leading edge of the plate.

2. Basic Equations

A steady free convection boundary layer from a vertical impermeable flat plate embedded in a fluid saturated porous medium of ambient temperature T_∞ is considered. The plate is maintained at the constant temperature $T_\mathrm{w} < T_\infty$, Under these conditions a boundary layer flowing down the plate will be formed. The basic boundary-layer equations, subject to the Boussinesq approximation, are (see Murthy and Singh, 1997; Nield and Bejan, 1999):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial y} = -\frac{g\beta K}{\upsilon}\frac{\partial T}{\partial y} \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\upsilon}{Kc_p}u^2 \tag{3}$$

These equations are to be solved subject to the boundary conditions

$$y = 0:\quad v = 0,\ T = T_\mathrm{w}; \qquad y \to \infty:\quad u \to 0,\ T \to T_\infty > T_\mathrm{w} \tag{4}$$

Here $x \geqslant 0$ and $y \geqslant 0$ are the Cartesian coordinates along and normal to the plate, respectively, u and v are the velocity components along x and y axes, T is the fluid temperature, K is the permeability of the porous medium, g is the acceleration due to gravity, c_p is the specific heat at constant pressure, and α, β and $\upsilon = \mu/\rho$ are the effective thermal diffusivity, thermal expansion coefficient and kinematic viscosity, respectively. The positive x-axis, with its origin at the leading edge of the plate, points vertically downwards in the direction of $\mathbf{g}$.

From (2) and (4) it is straightforward to show that

$$u = \frac{g\beta K}{\upsilon}(T_\infty - T) \tag{5}$$

Bearing in mind that $T_\mathrm{w} < T_\infty$, it is convenient to write T in the form

$$T = T_\infty - (T_\infty - T_\mathrm{w})\,\theta \tag{6}$$

where θ is a non-negative function of x and y, which, as a consequence of Equation (4), satisfies the boundary conditions $\theta = 1$ on $y = 0$ and $\theta \to 0$ as $y \to \infty$.

The introduction of the stream function, ψ, defined according to $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, transform the equations to

$$\frac{\partial\psi}{\partial y} = \frac{g\beta K\,(T_\infty - T_\mathrm{w})}{\upsilon}\theta,$$
$$\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} = \alpha\frac{\partial^2\theta}{\partial y^2} - \frac{\upsilon}{K\,c_p\,(T_\infty - T_\mathrm{w})}\left(\frac{\partial\psi}{\partial y}\right)^2 \tag{7}$$

Now, Equation (7) may be non-dimensionalised by the transformation $(x, y, \psi) \to (\xi, Y, \Psi)$ defined according to

$$x = L\xi, \qquad y = LRa^{-1/2}Y, \qquad \psi = \alpha Ra^{+1/2}\Psi,$$
$$Ra = \frac{g\beta K(T_\infty - T_\mathrm{w})L}{\upsilon\alpha} \tag{8}$$

and therefore (7) becomes,

$$\frac{\partial\Psi}{\partial Y}\frac{\partial^2\Psi}{\partial Y\partial\xi} - \frac{\partial\Psi}{\partial\xi}\frac{\partial^2\Psi}{\partial Y^2} = \frac{\partial^3\Psi}{\partial Y^3} - Ge\left(\frac{\partial\Psi}{\partial Y}\right)^2, \qquad \theta = \frac{\partial\Psi}{\partial Y} \tag{9}$$

where L is a reference length, Ra denotes the Rayleigh number and Ge is the Gebhart number $Ge = g\beta L/c_p$ which controls the contribution of the viscous dissipation. Since in the present problem no prescribed length scale exists, the value of Ge may be specified arbitrarily by the choice of L. In this paper it is therefore possible to set $Ge = 1$, which implies that

$$L = \frac{c_p}{g\beta} \tag{10}$$

is a natural length-scale based upon properties of the porous medium and the fluid and upon gravity. The (dimensional) velocity field of the flow is thus given by

$$u = \frac{\alpha}{L}Ra\,\theta, \qquad v = -\frac{\alpha}{L}Ra^{1/2}\frac{\partial\Psi}{\partial\xi} \tag{11}$$

3. The ‘Asymptotic Dissipation Profile’ (ADP)

In general, the velocity field of free convection boundary layer flows has both parallel and transverse components u and v, respectively, where the transverse component is much smaller than the parallel component. Accordingly, such flows are called non-parallel flows. However, as shown recently by Magyari and Keller (2003c), the (small) buoyancy forces due to heat release by viscous dissipation are able to cancel the (small) transversal velocity component v of the free convection field, thus giving rise to a strictly parallel flow. The single non-vanishing component u of the corresponding velocity field depends only on the coordinate y and therefore $u = u(y)$ and this flow has a constant thickness. Such ‘self-parallelisation’ of the velocity field in the presence of viscous dissipation can only happen in a free convection flow which descends over a cold plate, but never in its ascending counterpart over a hot plate (Magyari and Keller, 2003c). The reason is that in the latter case, the buoyancy forces due to heat release by viscous dissipation assist the ‘main’ buoyancy forces sustained by the wall temperature gradient, while in the former case of the cold plate, they oppose them (Magyari and Keller, 2003c).

The parallel-flow solution of Magyari and Keller (2003c) can easily be recovered from Equation (9), by assuming that Ψ and θ depend only on Y (but not on ξ). Indeed, in this case the equation for Ψ given in Equation (9) becomes

$$\frac{\partial^3\Psi}{\partial Y^3} = \left(\frac{\partial\Psi}{\partial Y}\right)^2 \tag{12}$$

which may be solved to obtain:

$$\Psi = -\frac{6}{Y+\sqrt{6}}, \qquad \theta = \frac{6}{(Y+\sqrt{6})^2}, \qquad u = \frac{\alpha}{L} Ra\,\theta, \qquad v = 0 \tag{13}$$

The surface heat flux of this algebraically decaying parallel flow is given by

$$Q_0 = -\left.\frac{\partial\theta}{\partial Y}\right|_{Y=0} = +\sqrt{\frac{2}{3}} \tag{14}$$

and its 1% thickness (i.e. the value Y_δ of Y for which $\theta(Y_\delta) = 0.01$) by $Y_\delta = 9\sqrt{6}$.

The main concern of the present paper is to answer the question of whether the special solution (13) of the boundary value problem (1)–(4) represents a physically realisable state of the descending free convection flows or not. The answer, which is presented in the next section, is that it is realisable. The parallel flow solution given by Equation (13), which will be referred to hereafter as ADP, is attained by a certain distance downstream of the leading edge, x_*, and this value is computed.

4. Flow Development Toward the ADP

The development of the descending free convection flow toward the ADP, as given by Equation (13), will be examined numerically in this section. The starting point of this investigation is the following simple physical reasoning. In the neighbourhood of the leading edge, where the effect of viscous dissipation is negligible, the steady flow has the character of the classical Cheng–Minkowycz boundary layer (see Cheng and Minkowycz, 1977) whose thickness increases with the wall coordinate as $x^{1/2}$. Thus, if the viscous dissipation term in the energy equation is neglected, the boundary layer grows indefinitely according to the Cheng–Minkowycz similarity solution. This holds both for an ascending free convection flow from a hot plate as well as one descending from a cold plate. But the heat released by viscous dissipation, however, warms up the moving fluid. This in turn accelerates the growth of the ascending boundary layer but decelerates that of the descending one. It is therefore expected that far enough from the leading edge, the thickness of the cold boundary layer will be limited by the warming effect of viscous dissipation to a constant asymptotic value. This limiting state of this boundary layer flow, which is approached at some distance x_* from the leading edge, should be precisely the ADP which is described by Equation (13).

In the present numerical experiment aiming to prove the above conjecture, it is necessary to introduce the usual Cheng–Minkowycz similarity variable for boundary layer flow from a uniform temperature surface in order to describe the beginning stages of the evolution of the flow, while Equation (9) may be used further downstream. Therefore the following transformations

$$\eta = \xi^{-1/2} Y, \qquad \Psi = \xi^{+1/2} f(\eta, \xi), \qquad \theta = \theta(\eta, \xi) \tag{15}$$

transform Equation (9) to

$$f''' + \frac{1}{2} f f'' - Ge\, \xi f'^2 = \xi \left(f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi} \right), \qquad \theta = f' \tag{16}$$

where the primes denote differentiation with respect to η and where $Ge = 1$ is taken, as discussed earlier. In this form of the basic equations it may be seen explicitly that the viscous dissipation term disappears at the origin, where $\xi = 0$.

In the numerical simulation, Equation (16) are solved in the range $0 \leqslant \xi \leqslant 1$, and Equation (9) in the range $\xi \geqslant 1$. This means that the developing boundary layer flow is well approximated near the leading edge, but that the approach to the constant thickness ADP arises naturally within the context of Equation (9). When $\xi \leqslant 1$ Equation (16) are solved subject to the boundary conditions

$$\eta = 0: \quad f = 0, \; f' = 1; \qquad \eta \to \infty: \quad f' \to 0 \tag{17}$$

but when $\xi > 1$ Equation (9) are solved subject to

$$Y = 0: \quad \Psi = 0, \; \frac{\partial \Psi}{\partial Y} = 1; \qquad Y \to \infty: \quad \frac{\partial \Psi}{\partial Y} \to 0 \tag{18}$$

The respective pairs of equations were solved by a straightforward application of the well-known Keller box method. The solution at the leading edge ($\xi = 0$) is readily seen to satisfy a pair of ordinary differential equations, and the solutions there are same as those presented by Riley and Rees (1985). The leading edge profiles were then marched forward in ξ. In the simulation we took constant steps of size 0.1 in the ξ-direction, and 63 unequally spaced intervals between 0 and 100 in the η or Y-direction. The accuracy of our numerical scheme is such that the steady value of Q_0 is 0.816454, which has a relative error of 0.00005 on comparison with Equation (14).

Figure 1 shows the surface rate of heat transfer in two forms as functions of ξ. More specifically the figure depicts

$$Q_1 = -\xi^{-1/2} \left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} \quad \text{for } \xi \leqslant 1, \qquad Q_1 = -\left.\frac{\partial \theta}{\partial Y}\right|_{Y=0} \quad \text{for } \xi \geqslant 1 \tag{19}$$

and

$$Q_2 = -\left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} \quad \text{for } \xi \leqslant 1, \qquad Q_2 = -\xi^{+1/2} \left.\frac{\partial \theta}{\partial Y}\right|_{Y=0} \quad \text{for } \xi \geqslant 1 \tag{20}$$

The value Q_1 shows how the surface rate of heat transfer evolves compared with that of the uniform thickness ADP to which the flow tends as $\xi \to \infty$. Near

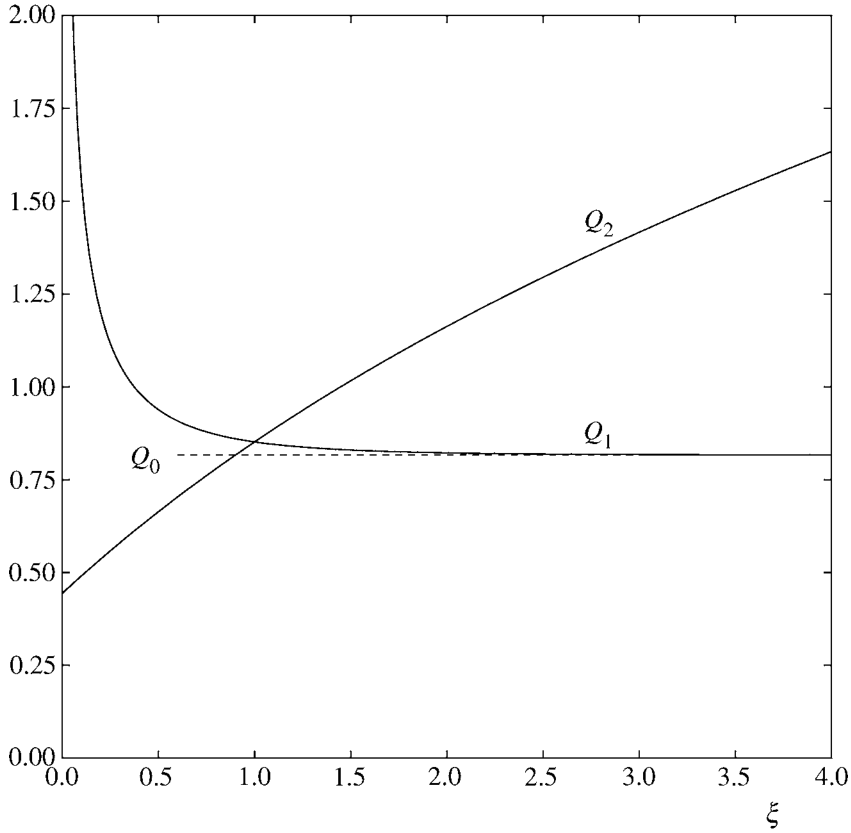

Figure 1. The variation with ξ of the rate of heat transfer as represented by Q_1 and Q_2, as defined in Equations (19) and (20), respectively. The Cheng–Minkowycz value of Q_2 is 0.44376 which corresponds to $\xi = 0$. Also shown as a dashed line is the value of Q_0 corresponding to the ADP.

the leading edge the heat transfer is large simply because the boundary layer is thin relative to the ADP. On the other hand, Q_2 represents a rate of heat transfer which is scaled in the same way as for free convection in the absence of viscous dissipation. In this context the rate of heat transfer increases because the boundary layer becomes relatively thin as ξ increases.

From the data from which Figure 1 was generated, the curve Q_1 is found to be within 1% of the ADP value of $Q_0 = +\sqrt{2/3} = 0.816496$ when $x = 1.79$, and therefore this value may be chosen as being the appropriate value for x_*. In dimensional terms this is equivalent to

$$x \equiv x_* = 1.79L = 1.79\frac{c_p}{g\beta} \tag{21}$$

being the distance from the leading edge beyond which the uniform thickness ADP solution applies. The dependence of this 'self-parallelisation length' of the flow on the parameters β and c_p corresponds to our physical expectation. Indeed, the stronger the buoyancy forces ($\sim g\beta\Delta T$), the stronger the self-parallelisation effect and accordingly the shorter must be the distance x_*. This explains why both β and g appear in the denominator of Equation (21). Furthermore, the smaller the heat capacity c_p, the larger is the temperature increase due to the heat being released by viscous dissipation, which again shortens the distance x_* at which the growth of the cold boundary layer ends. This explains the place of c_p in the numerator of Equation (21).

5. Summary and Conclusion

The objective has been to determine whether or not the ADP of Magyari and Keller (2002c) is achievable in practice, and if so, where it may be found. The conclusion is that such boundary layer flows do exist, but that the boundary layer displays a smooth transition from the well-known Cheng and Minkowycz (1977) similarity solution to the uniform thickness ADP. Equation (21) gives the minimum distance from the leading edge where the ADP may be assumed to exist with a maximum error (at least in terms of the surface rate of heat transfer) of 1%. It should be stressed however, that in usual applications the order of magnitude of x_* amounts to several hundreds of metres so that self-parallelisation of free convection flows due to dissipative effects is likely to occur only in geologically sized applications.

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