CONVECTIVE PLUME PATHS FROM A LINE SOURCE

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[Received 19 February 2001. Revise 29 November 2001]

Summary

We consider the flow induced by a line source placed at the intersection of two semi-infinite plane surfaces, and our main aim is to determine the influence of these bounding surfaces on the direction the plume takes. To this end a two-term boundary layer theory is presented from which it is deduced that the plume is, in general, orientated away from the vertical. A careful analysis of the boundary-layer solutions is undertaken to provide, subject to the determination of a constant for each Prandtl number, an analytical expression for the plume angle in terms of the orientations of the bounding surfaces. It is found that the plume is affected very strongly by the presence of the surfaces.

1. Introduction

Variations in the direction of the centreline of both line-source and point-source plumes are well known even from casual observation. In the environment such variations arise most frequently because of the presence of mixed or forced convective effects, such as wind in the atmosphere, or groundwater movements in the dispersion of underground pollutants. A further cause, which exists in the porous medium context, is the presence of inhomogeneities or anisotropy; in these cases even free convective plumes may exhibit a centreline which is not vertical (**1**,**2**). The close proximity of a solid surface also plays a significant role by drawing the plume towards itself—numerous excellent photographs of plumes being entrained towards nearby vertical surfaces and of the conjoining of neighbouring plumes may be found in the review by Gebhart (**3**). A detailed mathematical analysis of the curved path taken by a plume in a cavity with specified, but asymmetrically placed, ports of entry and exit of the fluid was undertaken by Shaw (**4**). As far as we are aware, the only other papers which deal with free convection plumes as boundary-layer flows in the presence of bounding walls are the studies of Afzal (**5**) and of Bastiaans *et al.* (**6**). Afzal (**5**) considers the case where two bounding surfaces are placed symmtrically either side of the plume and intersect at the source. In such a configuration the plume continues to rise vertically. In Baastians *et al.* (**6**) numerical simulations of turbulent flows induced by a line heat source in a cavity are presented.

The aim of the present paper is to determine how two inclined plane surfaces affect the motion of a free convective line-source plume placed at the intersection of those surfaces. This is achieved using almost exactly the same techniques as were employed in (**1**,**2**), namely the method of matched asymptotic expansions. The need for this is as follows. If a two-term boundary-layer theory is required, then it is necessary to find the leading-order circulations in the regions on either side

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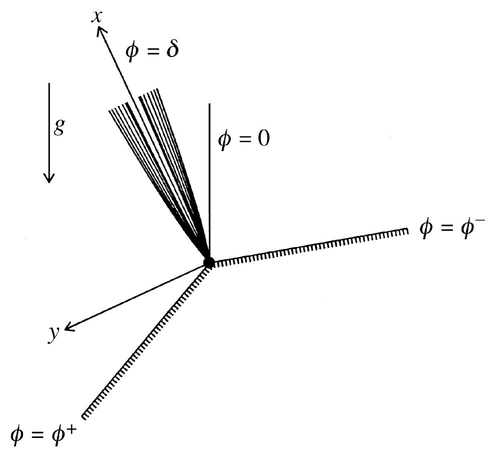

Fig. 1 Schematic diagram of the flow configuration depicting the two bounding surfaces at $\phi = \phi^{+}$ and $\phi = \phi^{-}$, and the plume with centreline along $\phi = \delta$. Also shown are the x- and y-axes ($\phi = \delta$ and $\phi = \delta + 90°$, respectively) and the direction of gravity

of the plume before finding the first-order plume solution (that is, the second boundary layer term). However, if the leading-order plume is assumed to rise vertically, then the equations for the first-order plume cannot be solved in general. The resolution of this mathematical difficulty lies in assuming that the leading-order plume centreline lies at an angle δ to the vertical. Then the imposition of the requirement that a solution must exist for the first-order plume yields a solvability condition from which a unique value of δ may be found in terms of the orientation angles of the bounding surfaces.

The inclination angle of the plume is a function of three variables, the orientations of the bounding surfaces and the Prandtl number. But we show that it is possible to reduce the equations of motion to a form where only one numerical solution is required for each Prandtl number, and this yields a constant which appears in a straightforward analytical expression relating the orientations of the plume and the surfaces. A comprehensive set of results is given for Prandtl numbers equal to 6·7, 0·7 and 0·01.

2. Governing equations

The present work deals with free convective flow induced by a line source of heat which is placed at the intersection of two semi-infinite bounding planes, as shown in Fig. 1. We will be concentrating on the possible behaviour of the plume at relatively large distances from the source by employing the boundary layer approximation. In this study the flow is divided into three regions, a boundary layer (inner) region which comprises the plume and which is thin relative to the flow domain, and two outer regions, either side of the plume. The plume is assumed to have a centreline which is at an angle δ to the vertical, and which is determined from the analysis.

A Cartesian frame of reference is chosen where the x-axis is aligned along the centreline of the plume, the z-axis is in the direction of the line source and is horizontal, while the y-axis is perpendicular to both these. The dimensional basic equations for steady two-dimensional flow are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\rho_\infty\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\nabla^2 u+\rho_\infty g\beta(T-T_\infty)\cos\delta, \tag{2}$$

$$\rho_\infty\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\nabla^2 v-\rho_\infty g\beta(T-T_\infty)\sin\delta, \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\kappa\nabla^2 T, \tag{4}$$

where the Boussinesq approximation has been used. The equation for the global conservation of heat takes the form

$$\rho_\infty C_p\int_{-\infty}^{\infty}\left[u(T-T_\infty)-\kappa\frac{\partial T}{\partial x}\right]dy=q', \tag{5}$$

where q' is the rate of heat flux per unit length of the line source. In the above equations u and v are the fluid velocities in the x- and y-directions, respectively, p is the pressure, and T is the temperature of the fluid. The diffusivity, viscosity and coefficient of cubical expansion are given by κ, μ and β, respectively, while g is acceleration due to gravity, ρ is density and C_p is specific heat at constant pressure. Ambient conditions are denoted by the subscript ∞.

Equations (1) to (5) are made dimensionless by using the transformations

$$(u,v)=\frac{\mu}{\rho_\infty d}(u^*,v^*),\qquad (x,y)=d(x^*,y^*),$$
$$p=\frac{\mu^2}{\rho_\infty d^2}p^*,\qquad T-T_\infty=\frac{q'}{\rho_\infty C_p}\theta^*, \tag{6}$$

where the starred variables denote dimensionless quantities, and d is defined as a natural lengthscale by

$$d=\mu\left(\frac{C_p}{g\beta q'\rho_\infty^2}\right)^{1/3}. \tag{7}$$

In non-dimensional terms the governing equations are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{8}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\nabla^2 u+\theta\cos\delta, \tag{9}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\nabla^2 v-\theta\sin\delta, \tag{10}$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\sigma^{-1}\nabla^2\theta, \tag{11}$$

$$\int_{-\infty}^{\infty}\left[u\theta-\frac{1}{\sigma}\frac{\partial\theta}{\partial x}\right]dy=1, \tag{12}$$

where the asterisks have been omitted for clarity of presentation, and where the non-dimensional parameter, σ, is the Prandtl number which is given by

$$\sigma = \frac{\mu}{\rho_\infty \kappa}. \tag{13}$$

We note that there is no Grashof number present in these equations. The non-dimensionalization used here has, in effect, set the appropriate Grashof number equal to unity and it is this which defines the lengthscale, d, given in (7). For the sake of completeness the scalings in (6) may also be written in the form

$$(u, v) = \rho_\infty^{-1/3}\left(\frac{g\beta q'}{C_p}\right)^{1/3}(u^*, v^*), \qquad (x, y) = \frac{\mu}{\rho_\infty^{2/3}}\left(\frac{C_p}{g\beta q'}\right)^{1/3}(x^*, y^*),$$
$$p = \rho_\infty^{1/3}\left(\frac{g\beta q'}{C_p}\right)^{2/3} p^*. \tag{14}$$

The analysis of this paper corresponds to the situation which prevails at large-dimensional distances from the heat source relative to the size of d, and which corresponds to large values of x compared with unity. As the flow is two-dimensional the governing equations reduce to

$$\nabla^4\psi + \frac{\partial\theta}{\partial y}\cos\delta + \frac{\partial\theta}{\partial x}\sin\delta = \frac{\partial\psi}{\partial y}\nabla^2\left[\frac{\partial\psi}{\partial x}\right] - \frac{\partial\psi}{\partial x}\nabla^2\left[\frac{\partial\psi}{\partial y}\right], \tag{15}$$

$$\sigma^{-1}\nabla^2\theta = \frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}, \tag{16}$$

where the pressure, p, has been eliminated, and a streamfunction, ψ, is defined according to

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}. \tag{17}$$

Far from the plume centreline the velocity and temperature approach their respective ambient values,

$$\frac{\partial\psi}{\partial y} \to 0, \quad \theta \to 0 \qquad \text{as} \quad y \to \pm\infty. \tag{18}$$

This condition will need to be made more precise below when we consider the detailed effect of the isothermal flow induced in the regions either side of the plume.

The equation for the global conservation of heat now takes the form

$$\int_{-\infty}^{\infty}\left[\frac{\partial\psi}{\partial y}\theta - \frac{1}{\sigma}\frac{\partial\theta}{\partial x}\right]dy = 1, \tag{19}$$

where it is noted that the infinite limits represent the mathematical problem from the point of view of the asymptotically thin plume boundary layer.

3. Analysis

In solving equations (15), (16), (18) and (19) we use the method of matched asymptotic expansions to determine series solutions for ψ and θ. Entrainment into the plume induces a flow in the outer regions which, in turn, causes a small correction to the plume solution. But the equations governing this leading correction have a solution only when the centreline angle, δ, takes the correct value.

3.1 *The inner region*

In the inner, or plume boundary-layer region, the solution takes the following form as $x \to \infty$:

$$\psi = x^{3/5} f_0(\eta) + f_1(\eta) + \cdots, \qquad \theta = x^{-3/5} g_0(\eta) + x^{-6/5} g_1(\eta) + \cdots, \tag{20}$$

where the similarity variable is

$$\eta = y/x^{2/5}. \tag{21}$$

After substitution of (20) and (21) into the governing equations (15) and (16), the boundary-layer approximation is invoked ($x \gg y$) and like powers of x are equated to obtain

$$f_0''' + g_0 \cos\delta + \tfrac{3}{5} f_0 f_0'' - \tfrac{1}{5} f_0' f_0' = 0, \tag{22}$$

$$\sigma^{-1} g_0' + \tfrac{3}{5} f_0 g_0 = 0, \tag{23}$$

$$f_1'''' + \tfrac{3}{5} f_0 f_1''' + \tfrac{4}{5} f_0' f_1'' + \tfrac{1}{5} f_0'' f_1' + g_1' \cos\delta = (\tfrac{3}{5} g_0 + \tfrac{2}{5}\eta g_0') \sin\delta, \tag{24}$$

$$\sigma^{-1} g_1'' + \tfrac{3}{5} g_0 f_1' + \tfrac{6}{5} f_0' g_1 + \tfrac{3}{5} f_0 g_1' = 0, \tag{25}$$

where equations (22) and (23) have already been integrated once with respect to η. The heat flux constraint becomes

$$\int_{-\infty}^{\infty} f_0' g_0 \, d\eta = 1, \qquad \int_{-\infty}^{\infty} \big(f_0' g_1 + f_1' g_0\big)\, d\eta = 0. \tag{26}$$

It is convenient to introduce the functions h_0 and h_1 defined by

$$h_0 = \int_0^{\eta} f_0'(\xi) g_0(\xi)\, d\xi, \qquad h_1 = \int_0^{\eta} [f_0'(\xi) g_1(\xi) + f_1'(\xi) g_0(\xi)]\, d\xi, \tag{27}$$

so that equations (26) may be written in the form

$$h_0' = f_0' g_0, \qquad h_1' = f_0' g_1 + f_1' g_0, \tag{28}$$

subject to the boundary conditions given below. We assume that the leading-order solution displays the appropriate symmetries about $\eta = 0$ (that is, f_0 is odd and g_0 is even), and then we use these symmetries to integrate the equations for positive values of η only. The fifth-order system formed by equations (22), (23) and $(28)_1$ is completed, for numerical purposes, by five boundary conditions,

$$\eta = 0: \quad f_0 = f_0'' = h_0 = 0, \qquad \eta \to \infty: \quad f_0' \to 0, \quad h_0 \to \tfrac{1}{2}. \tag{29}$$

Given that $f_0' \to 0$ as $\eta \to \infty$, it is clear that f_0 asymptotes to a constant and, given the form of equation (23) at large values of η, we conclude that g_0 will decay exponentially, thereby rendering it unnecessary to apply a condition on the temperature field. We will define a_0 as the limiting value of f_0 at large values of η and therefore

$$\lim_{\eta \to \pm\infty} f_0 = \pm a_0, \qquad \text{where} \qquad a_0 = a_0(\sigma, \delta). \tag{30}$$

3.2 *The outer regions*

The aim of this subsection is the determination of the flow in the relatively large regions outside the plume. We have seen that the temperature field decays exponentially, and therefore we may ignore it when considering the outer flow. As shown in Fig. 1 there are impermeable surfaces placed at $\phi = \phi^+$ and $\phi = \phi^-$, and the plume, which is asymptotically thin in the tangential direction, is located at $\phi = \delta$. We have assumed that the line source is at the intersection of the planes and hence the streamfunction is taken to be zero on the planes.

On neglecting the temperature field in equation (15) we obtain

$$\nabla^4 \psi = \psi_y \nabla^2 \psi_x - \psi_x \nabla^2 \psi_y. \tag{31}$$

However, on rewriting this equation in polar coordinates, and noting that $\psi = O(r^{3/5})$ as $r \to \infty$, we see that the left-hand side terms are of $O(r^{-17/5})$ while the right-hand side terms are $O(r^{-9/5})$, which is asymptotically larger. Therefore we must neglect the diffusion terms for the leading-order external flow. The equation formed by setting to zero the right-hand side of (31), is satisfied by solutions of the equation

$$\nabla^2 \psi = 0, \tag{32}$$

which corresponds to the usual potential flow. In polar coordinates (r, ϕ) this is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} = 0, \tag{33}$$

where the upward vertical corresponds to $\phi = 0$, the plume centreline to $\phi = \delta$ and the two bounding surfaces to $\phi = \phi^+$ and $\phi = \phi^-$, respectively.

Equation (33) must be solved separately in the regions on either side of the plume. Given the large-η behaviour of f_0 given in (30), and the definition of f_0 in (20), the boundary conditions corresponding to the plume are obtained by asymptotic matching. Therefore (33) must be solved subject to

$$\lim_{\phi \to \delta^+} \psi = a_0 r^{3/5} \qquad \text{and} \qquad \psi(\phi = \phi^+) = 0, \tag{34}$$

in the region to the left of the plume, while on the right we have

$$\psi(\phi = \phi^-) = 0 \qquad \text{and} \qquad \lim_{\phi \to \delta^-} \psi = -a_0 r^{3/5}. \tag{35}$$

If we set

$$\begin{aligned} \psi &= r^{3/5} \mathcal{F}^+(\phi) \quad (\phi > \delta), \\ \psi &= r^{3/5} \mathcal{F}^-(\phi) \quad (\phi < \delta), \end{aligned} \tag{36}$$

then equation (33) reduces to

$$\mathcal{F}'' + \tfrac{9}{25}\mathcal{F} = 0. \tag{37}$$

The solutions are

$$\mathcal{F}^+ = a_0 \frac{\sin \frac{3}{5}(\phi^+ - \phi)}{\sin \frac{3}{5}(\phi^+ - \delta)} \qquad \text{and} \qquad \mathcal{F}^- = -a_0 \frac{\sin \frac{3}{5}(\phi - \phi^-)}{\sin \frac{3}{5}(\delta - \phi^-)}. \tag{38}$$

These solutions may now be used to obtain matching conditions for the first-order plume equations. For small positive values of $(\phi - \delta)$ the streamfunction may be expanded in the form

$$\psi \sim a_0 r^{3/5}[1 - \tfrac{3}{5}(\cot \tfrac{3}{5}(\phi^+ - \delta))(\phi - \delta)] \quad \text{as } (\phi - \delta) \to 0^+. \tag{39}$$

This may be rewritten in terms of x and η by making use of the asymptotic forms

$$x \sim r \qquad \text{and} \qquad \eta = y/x^{2/5} \sim r^{3/5}(\phi - \delta), \tag{40}$$

and therefore we have

$$\psi \sim a_0[x^{3/5} - (\tfrac{3}{5}\cot(\phi^+ - \delta))\eta] \quad \text{as } \eta \to \infty. \tag{41}$$

Similarly, we find that

$$\psi \sim a_0[-x^{3/5} - (\tfrac{3}{5}\cot(\delta - \phi^-))\eta] \quad \text{as } \eta \to -\infty. \tag{42}$$

The behaviour of the second terms in (41) and (42) may be translated into the following boundary conditions for the first-order plume:

$$f_1' \to -\tfrac{3}{5}a_0 \cot(\phi^+ - \delta) \quad \text{as} \quad \eta \to \infty, \tag{43}$$

$$f_1' \to -\tfrac{3}{5}a_0 \cot(\delta - \phi^-) \quad \text{as} \quad \eta \to -\infty. \tag{44}$$

4. Numerical solution of the plume equations

Our aim here is to solve the leading-order system (22), (23), $(28)_1$ and (29) and the first-order system (24), (25), $(28)_2$, (43) and (44). As mentioned earlier, we cannot solve these equations in general by taking $\delta = 0$, and therefore it is necessary to find δ as a function of the three parameters ϕ^+, ϕ^- and σ. Given that the boundary conditions for the first-order equations do not display a particular symmetry in general, it is not straightforward to consider solely the $\eta \geqslant 0$ part of the domain. Thus it would seem necessary to split the first-order equations (24), (25) and $(28)_2$ into two parts, one corresponding to $\eta \geqslant 0$ and one for $\eta \leqslant 0$ and to solve the larger system subject to additional matching conditions at $\eta = 0$. However, this approach is very computationally intensive and requires the solution of this new enlarged system over many combinations of the three parameters.

In (**1**) we were able to solve the analogous porous medium problem analytically, but a careful scrutiny of the first-order solutions in that paper reveals that the component parts (that is, the complementary functions and particular integral) display either odd or even symmetries. The result of that analysis was the deduction of a straightforward trigonometrical formula for δ in terms of ϕ^+ and ϕ^-. These facts motivate the following analysis which effectively reduces the number of parameters from three to one, although numerical solutions remain to be found.

First we can eliminate $\cos\delta$ from both the leading-order and first-order systems by means of the substitutions

$$f_n(\eta) = (\cos\delta)^{1/5} F_n(\zeta), \quad g_n(\eta) = (\cos\delta)^{-1/5} G_n(\zeta), \quad h_n(\eta) = H_n(\zeta) \qquad \text{for } n = 0, 1, \tag{45}$$

where

$$\eta = (\cos\delta)^{-1/5}\zeta. \tag{46}$$

Hence the leading-order system of equations and its boundary conditions reduce to

$$F_0''' + G_0 + \tfrac{3}{5}F_0F_0'' - \tfrac{1}{5}F_0'F_0' = 0, \tag{47}$$

$$\sigma^{-1}G_0' + \tfrac{3}{5}F_0G_0 = 0, \tag{48}$$

$$H_0' = F_0'G_0, \tag{49}$$

subject to

$$\zeta = 0: \qquad F_0 = 0, \quad F_0'' = 0, \quad H_0 = 0, \tag{50}$$

$$\zeta \to \infty: \qquad F_0' \to 0, \quad H_0 \to \tfrac{1}{2}, \tag{51}$$

where primes indicate derivatives with respect to ζ for the F, G and H variables. The solution of this system depends on only one parameter, σ, and the large-η asymptotic value of f_0 is given by

$$A_0 = \lim_{\zeta\to\infty} F_0(\zeta), \quad \text{where} \quad a_0 = (\cos\delta)^{1/5}A_0. \tag{52}$$

The first-order system of equations and its boundary conditions become

$$F_1'''' + G_1' + \tfrac{3}{5}F_0F_1''' + \tfrac{4}{5}F_0'F_1'' + \tfrac{1}{5}F_0''F_1' = \left[\tfrac{3}{5}G_0 + \tfrac{2}{5}\zeta G_0'\right]\frac{\sin\delta}{(\cos\delta)^{6/5}}, \tag{53}$$

$$\sigma^{-1}G_1'' + \tfrac{3}{5}F_1'G_0 + \tfrac{6}{5}F_0'G_1 + \tfrac{3}{5}F_0G_1' = 0, \tag{54}$$

$$H_1' = F_0'G_1 + G_0F_1', \tag{55}$$

subject to

$$\zeta \to \infty: \qquad F_1' \to -\tfrac{3}{5}A_0\frac{\cot(\phi^+ - \delta)}{(\cos\delta)^{1/5}}, \quad G_1, H_1 \to 0, \tag{56}$$

$$\zeta \to -\infty: \qquad F_1' \to -\tfrac{3}{5}A_0\frac{\cot(\delta - \phi^-)}{(\cos\delta)^{1/5}}, \quad G_1, H_1 \to 0. \tag{57}$$

Strictly speaking (53) to (55) forms a seventh-order system but we have specified only six boundary conditions. However, since δ is unknown we may form an eighth equation, $\delta' = 0$, and apply two extra boundary conditions, $F_1'' \to 0$ as $\zeta \to \pm\infty$. But this approach still retains three independent parameters with its consequently large number of numerical simulations.

A much more efficient approach to solving the system (53) to (57) is to split it into its odd and even components. Therefore we set

$$F_1 = F_{1o} + F_{1e}, \qquad G_1 = G_{1o} + G_{1e} \qquad \text{and} \qquad H_1 = H_{1o} + H_{1e}. \tag{58}$$

Here it is to be understood that F_{1o} is an odd function, but its thermal counterpart, G_{1o}, is in fact even; the subscripts are meant to denote the fact that F_{1o} and G_{1o} are associated with one another. Likewise F_{1e} is even and G_{1e} is odd. If, for convenience, we define the values α^+ and α^- according to

$$\alpha^+ = -\tfrac{3}{5}A_0\frac{\cot(\phi^+ - \delta)}{(\cos\delta)^{1/5}} \quad \text{and} \quad \alpha^- = -\tfrac{3}{5}A_0\frac{\cot(\delta - \phi^-)}{(\cos\delta)^{1/5}}, \tag{59}$$

then they correspond respectively to the positively and negatively large ζ limits of F_1' given in (56) and (57). We define further constants, γ_o and γ_e, as follows:

$$\gamma_o = \tfrac{1}{2}(\alpha^+ + \alpha^-) \qquad \text{and} \qquad \gamma_e = \tfrac{1}{2}(\alpha^+ - \alpha^-). \tag{60}$$

These are the large-ζ limits of F_{1o}' and F_{1e}', respectively.

We consider the odd case first, and, given that the inhomogeneous term in (53) is even, the corresponding equation for the odd component is homogeneous, although the boundary conditions are not. In this case we may integrate (53) once. Given the symmetries of the functions we need only to consider the region from $\zeta = 0$ onwards and therefore F_{1o}, G_{1o} and H_{1o} satisfy

$$F_{1o}''' + G_{1o} + \tfrac{3}{5}F_0F_{1o}'' + \tfrac{1}{5}F_0'F_{1o}' = 0, \tag{61}$$

$$\sigma^{-1}G_{1o}'' + \tfrac{3}{5}F_{1o}'G_0 + \tfrac{6}{5}F_0'G_{1o} + \tfrac{3}{5}F_0G_{1o}' = 0, \tag{62}$$

$$H_{1o}' = F_0'G_{1o} + F_{1o}'G_0, \tag{63}$$

subject to the boundary conditions

$$\zeta = 0: \qquad F_{1o} = 0, \qquad F_{1o}'' = 0, \qquad G_{1o}' = 0, \qquad H_{1o} = 0, \tag{64}$$

$$\zeta \to \infty: \qquad F_{1o}' \to \gamma_o, \qquad G_{1o}, H_{1o} \to 0. \tag{65}$$

Using a standard fourth-order Runge–Kutta code embedded in a shooting method algorithm we found that this system has a solution for every value of γ_o. As the equations are linear, the existence of a solution for any one value of γ_o implies that solutions for other values are the appropriate multiple of the first solution. Therefore this component of F_1 and G_1 has no bearing on the required value of δ.

Turning to the even components, we may split them up into three parts: a particular integral and two complementary functions. The symmetries of F_{1e}, G_{1e} and the leading-order solutions are such that the right-hand side of (55) is even, and hence H_{1e} is an odd function. Therefore, if we set $H_{1e}(-\infty) = 0$ it necessarily implies that $H_{1e}(\infty) = 0$, and hence that the correct boundary conditions for H_{1e} are automatically satisfied because of the symmetries. We write the equations for F_{1e} and G_{1e} in the form

$$F_{1e}'''' + G_{1e}' + \tfrac{3}{5}F_0F_{1e}''' + \tfrac{4}{5}F_0'F_{1e}'' + \tfrac{1}{5}F_0''F_{1e}' = A[\tfrac{3}{5}G_0 + \tfrac{2}{5}\zeta G_0'], \tag{66}$$

$$\sigma^{-1}G_{1e}'' + \tfrac{3}{5}F_{1e}'G_0 + \tfrac{6}{5}F_0'G_{1e} + \tfrac{3}{5}F_0G_{1e}' = 0, \tag{67}$$

where the constant A is $(\tan\delta)/(\cos\delta)^{1/5}$, and where the boundary conditions are

$$\zeta = 0: \qquad F_{1e}' = 0, \quad F_{1e}''' = 0, \quad G_{1e} = 0, \tag{68}$$

$$\zeta \to \infty: \qquad F_{1e}' \to \gamma_e, \quad F_{1e}'', G_{1e} \to 0. \tag{69}$$

Now we define the particular integral, F_{pi}, and the two complementary functions, F_{cf1} and F_{cf2} according to:

$$F_{pi} \text{ is the solution with } F_{1e}(0) = 0,\ F''_{1e}(0) = 0 \text{ and } A = 1, \tag{70}$$

$$F_{cf1} \text{ is the solution with } F_{1e}(0) = 1,\ F''_{1e}(0) = 0 \text{ and } A = 0, \tag{71}$$

$$F_{cf2} \text{ is the solution with } F_{1e}(0) = 0,\ F''_{1e}(0) = 1 \text{ and } A = 0. \tag{72}$$

It is easily confirmed that $F_{cf1} = 1$ and therefore it too plays no part in determining the value of δ. The final solution for F_{1e} will therefore take the form

$$F_{1e} = \frac{\tan\delta}{(\cos\delta)^{1/5}} F_{pi} + B F_{cf2} + C, \tag{73}$$

where B and C are arbitrary constants. Numerically we find that both F_{pi} and F_{cf2} have solutions which grow quadratically in ζ, and therefore both violate the large-ζ boundary condition on F_{1e} in (69). However, it is possible to combine them to eliminate this quadratic growth and to establish the correct linear growth. Therefore, if we take

$$F_{pi} \sim c_1\zeta^2 + c_2\zeta \qquad \text{and} \qquad F_{cf2} \sim d_1\zeta^2 + d_2\zeta \qquad \text{as} \quad \zeta \to \infty, \tag{74}$$

where the constants c_1, c_2, d_1 and d_2 are obtained numerically, then the overall quadratic growth is suppressed when

$$c_1 \frac{\tan\delta}{(\cos\delta)^{1/5}} + d_1 B = 0, \tag{75}$$

and the correct linear growth is obtained when

$$c_2 \frac{\tan\delta}{(\cos\delta)^{1/5}} + d_2 B = \gamma_e. \tag{76}$$

Given the definition of γ_e in equation (60), the value B may be eliminated from (75) and (76) to obtain the equation

$$\cot \tfrac{3}{5}(\phi^+ - \delta) - \cot \tfrac{3}{5}(\delta - \phi^-) = -\tfrac{10}{3}\Big[\frac{c_2 d_1 - c_1 d_2}{d_1 A_0}\Big] \tan\delta. \tag{77}$$

This equation is very similar to that obtained in (**1**) where the fractions within the cotangents are both $\frac{1}{3}$, and the coefficient of $\tan\delta$ is -2. The important implication of this formula is that we need to compute solutions of the leading-order equations, (61) to (63), and the equations for both F_{pi} and F_{cf2} only once for each value of σ. Having thereby obtained the values of A_0, c_1, c_2, d_1 and d_2 the full dependence of δ may be obtained simply by solving the equation (77).

In our numerical solutions we allowed σ to vary between 0·01 and 10, and, given the wide range of boundary-layer widths which correspond to such extreme values of the Prandtl number, we were very careful to ensure that not only was the steplength sufficiently small to obtain solutions correct to five significant figures, but also that the maximum value of ζ was sufficiently large that the constants given in (77) did not vary upon further increases in $\zeta_{\max}$.

Although (77) cannot be rearranged to give δ explicitly in terms of ϕ^+ and ϕ^-, it may be

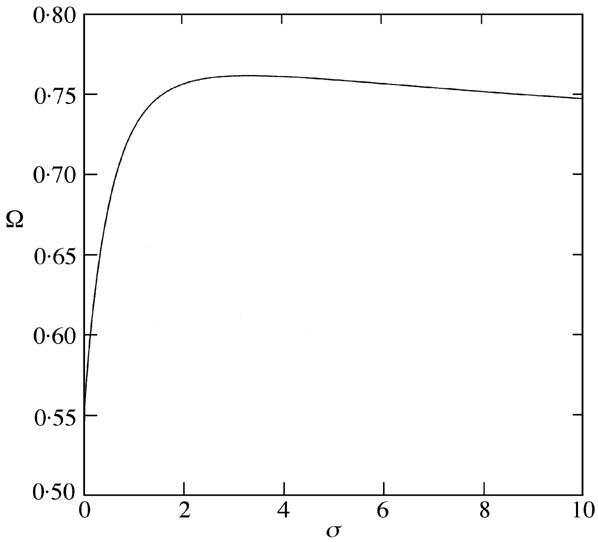

Fig. 2 The value of the parameter Ω defined by equation (78) as a function of the Prandtl number σ

manipulated easily to yield, say, ϕ^+ in terms of δ and ϕ^-. If, for convenience, we define the constant Ω according to

$$\Omega = \tfrac{10}{3}\left[\frac{c_2 d_1 - c_1 d_2}{d_1 A_0}\right], \tag{78}$$

then (77) gives

$$\cot \tfrac{3}{5}(\phi^+ - \delta) - \cot \tfrac{3}{5}(\delta - \phi^-) = -\Omega \tan \delta, \tag{79}$$

from which we obtain

$$\phi^+ = \delta + \tfrac{5}{3} \cot^{-1}\Big[\cot \tfrac{3}{5}(\delta - \phi^-) - \Omega \tan \delta\Big]. \tag{80}$$

Some care is needed to ensure that the correct inverse cotangent is computed. The variation of Ω with σ is shown in Fig. 2. We see that the value of Ω does not vary greatly from the neighbourhood of 0·75 when σ lies in the range $2 < \sigma < 10$, but decreases for $\sigma < 2$. The detailed numerical results do not indicate clearly the limit as $\sigma \to 0$, but this limit is difficult to analyse numerically as the thermal boundary-layer thickness increases indefinitely. An asymptotic analysis of the small-σ limit is outside the scope of this paper.

5. Results and discussion

For any computed value of Ω, equation (80) yields an analytical solution for ϕ^+ in terms of δ and ϕ^-. In Fig. 3 we use this numerical data to show how ϕ^+ and ϕ^- are related for each chosen value

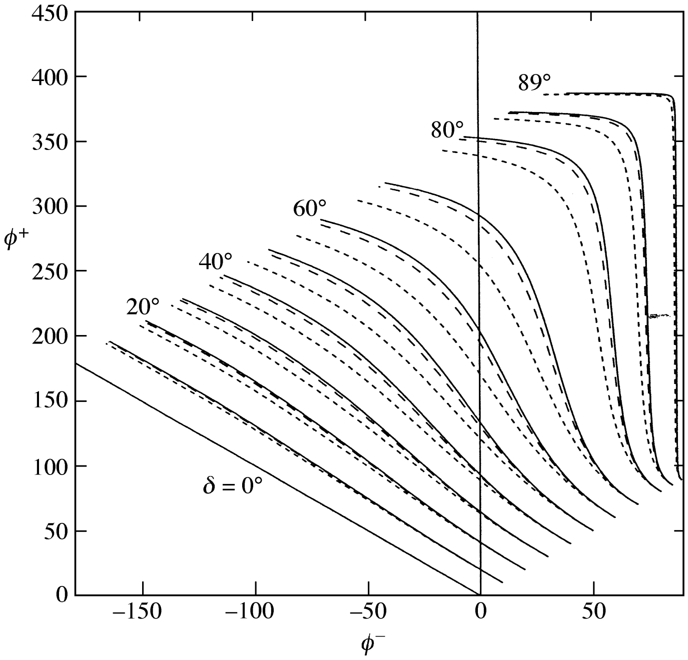

Fig. 3 The loci of values of the surface angles, ϕ^+ and ϕ^-, measured in degrees, which yield plumes of various orientations. Solid curves correspond to fluids with $\sigma = 6{\cdot}7$, long dashes to $\sigma = 0{\cdot}7$ and short dashes to $\sigma = 0{\cdot}01$. The respective values of Ω for these three cases are 0·75485, 0·70327 and 0·54364. The displayed values of δ are 0°, 10°, 20°, ..., 70°, 80°, 85° and 89°. All angles are given in degrees

of δ. We show curves for $\sigma = 6{\cdot}7$ (water), $\sigma = 0{\cdot}7$ (air) and $\sigma = 0{\cdot}01$ (representative of liquid metals).

Before discussing Fig. 3 it is important to note that substitution of the value $\Omega = 0$ into (79) yields

$$\delta = \tfrac{1}{2}(\phi^+ + \phi^-), \tag{81}$$

which means that the plume lies exactly midway between the bounding surfaces. In this case the plume entrains exactly the same amount of fluid from either side and therefore it is dominated completely by the outer region circulations. Therefore increasing positive values of Ω correspond to the increasing influence of buoyancy which would otherwise cause the plume to ascend vertically.

We also note, as in (**1**), that those cases for which $\delta = 0$ correspond to symmetrically placed bounding surfaces so that $\phi^+ + \phi^- = 0$. There are also cases which we suspect strongly of not being physically realizable. These cases almost certainly include those for which ϕ^+ and ϕ^- have the same sign. The present analysis relies on the fact that there are two external regions, whereas these particular cases will have the plume ascending the upper of the two surfaces. Tentatively we would expect that flows for which $\phi^+ > 0$ and $\phi^- < 0$ are realizable, but this may depend on how close to the vertical the bounding surfaces lie; a detailed examination of this aspect lies outside the scope of this paper.

When the surfaces are asymmetrically placed, Fig. 3 shows that plumes do not rise vertically. For example, for $\sigma = 0{\cdot}01$, when $\phi^- = -90°$ and $\phi^+ = 180°$, then the plume deviation is $\delta \simeq 30°$. This value is above the line $\frac{1}{2}(\phi^+ + \phi^-) = 45°$ which demonstrates that, although buoyancy does play a role, its effect is not as great as one might expect. It is clear that deviations from $\frac{1}{2}(\phi^+ + \phi^-) = 45°$ are not large when $\phi^- < 0$ and $\phi^+ > 0$ and therefore the effect of the induced flow external to the plume is particularly strong. This is especially so for liquid metals for which Ω is relatively small. For these fluids thermal diffusion is relatively large and therefore the boundary layer is relatively thick. In turn, the induced velocity along the plume reduces in comparison with fluids of larger Prandtl numbers and therefore the role of buoyancy in determining the direction of the centreline is also reduced relatively.

Acknowledgements

The second author gratefully acknowledges financial support from the Norwegian Research Council during his sabbatical leave at the University of Bath, autumn 2000. The authors thank the referees for their constructive comments.

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