Z. angew. Math. Phys. 53 (2002) 48–52
0044-2275/02/010048-10 $ 1.50+0.20/0

Zeitschrift für angewandte Mathematik und Physik ZAMP

Natural convection of fluid with variable viscosity from a heated vertical wavy surface

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Abstract. In the present paper, the effect of a temperature dependent viscosity on natural convection flow of viscous incompressible fluid from a vertical wavy surface has been investigated using an implicit finite difference method. Here we have focused our attention on the evaluation of the local skin-friction and the local Nusselt number. The governing parameters are the Prandtl number, Pr, ranging from 1 to 100, the amplitude of the waviness of the surface, α, ranging from 0.0 to 0.4 and the viscosity variation parameter, ε, ranging from 0.0 to 6.

Keywords. Natural-convection, variable-viscosity, wavy-surface.

1. Introduction

The objective of the present paper is to study the natural convection flow from a vertical wavy surface for a fluid with temperature dependent viscosity. Roughened surfaces are encountered in several heat transfer devices such as flat plate solar collectors and flat plate condensers in refrigerators. Larger scale surface nonuniformities are encountered in cavity wall insulating systems and grain storage containers, for example. The only papers to date which study the effects of such nonuniformities on the thermal boundary layer flow of Newtonian fluid are those of Yao (1983) and Moulic and Yao (1989). Hossain and Pop (1996) investigated the magnetohydrodynamic boundary layer flow and heat transfer from a continuous moving wavy surface and the problem of free-convection flow from a wavy vertical surface in presence of a transverse magnetic field was studied by Hossain et al. (1997). On the other hand, Rees and Pop (1995) investigated the free convection induced by a horizontal surface exhibiting small-amplitude waves and which is embedded in a porous medium. Recently, Hossain and Rees (1999) have investigated the combined effect of thermal and mass diffusion on the natural convection flow of a viscous incompressible fluid from a vertical wavy surface. The effect of waviness of the surface on the heat-flux and mass flux had been investigated in combination with the species concentration for a fluid with Prandtl number equal to 0.7.

In all the above studies the viscosity of the fluid had been assumed to be constant. However, it is known that this physical property may change significantly with temperature. For instance, the viscosity of water decreases by almost 60 percent when the temperature increases from 100^0C ($\mu = 0.0131 g/$cms) to 500^0C ($\mu = 0.00548 g/$cms). It is necessary to take into account this variation of viscosity in order to predict accurately the flow behaviour. Recently Gary et al. (1982), and Mehta and Sood (1992) have shown that when this effect is included, the flow characteristics may be changed substantially compared with the constant viscosity case. Recently, Kafoussias and Williams (1997) and Kafoussias and Rees (1998) investigated the effect of temperature dependent viscosity on the mixed convection flow from a vertical flat plate in the region near the leading edge using the local nonsimilarity method. Very recently, Hossain and Munir (2000) have investigated the same problem in the regions both near and far from the leading edge using perturbation methods and in the intermediate region by means of a finite difference method. From the above investigations it is found that variation of viscosity with temperature is an interesting macroscopic physical phenomenon in fluid mechanics.

In this investigation the focus is on the boundary-layer regime caused by a uniform temperature wavy surface immersed in a fluid with a temperature dependent viscosity. The transformed boundary-layer equations are solved numerically using the Keller-box technique (Keller 1973), an implicit finite difference method. Consideration is given to the situation where the buoyancy forces assist the natural convection flow for various values of the viscosity-variation parameter,ε, with the Prandtl number Pr ranging between 1 to 100. The results allow us to predict the different behavior that can be observed when the relevant parameters are varied.

2. Formulation of the problem

Consider the steady laminar free convective boundary layer flow of a viscous incompressible fluid along a semi-infinite wavy vertical plate where the viscosity of the fluid depends on its temperature. The surface of the plate may be described by

$$\hat{y} = \hat{\sigma}(\hat{x})$$

and the temperature of the surface is held uniform at T_w which is higher than the ambient temperature T_∞. The boundary layer analysis outlined below allows σ to be arbitrary, but our detailed numerical work will assume that the surface exhibits sinusoidal deformations.

The dimensionless governing equations are the Navier-Stokes equation, the energy equation and the continuity equation in two-dimensional Cartesian coordinates (see Fig. 1).

Under the usual Boussinesq approximation, the flow is governed by the follow-

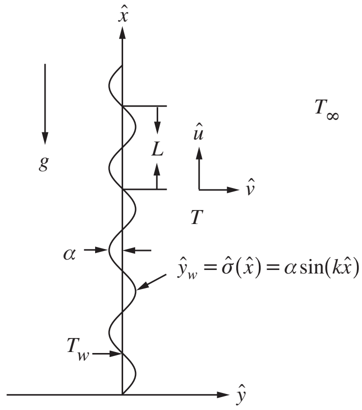

Figure 1.
Physical model and coordinate system.

ing boundary layer equations:

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0 \tag{1}$$

$$\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{1}{\rho}\nabla.(\mu\nabla\hat{u}) + g\beta(T - T_\infty) \tag{2}$$

$$\hat{u}\frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{y}} = -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \hat{y}} + \frac{1}{\rho}\nabla.(\mu\nabla\hat{v}) \tag{3}$$

$$\hat{u}\frac{\partial T}{\partial \hat{x}} + \hat{v}\frac{\partial T}{\partial \hat{y}} = \frac{k}{\rho C_p}\nabla^2 T \tag{4}$$

where $\hat{u}$ and $\hat{v}$ are the velocity components parallel to $\hat{x}$ and $\hat{y}$, ∇^2 $(= \partial^2/\partial\hat{x}^2 + \partial^2/\partial\hat{y}^2)$ is Laplacian, g is the acceleration due to gravity, ρ is the density, κ is the thermal conductivity, C_p is the specific heat at constant pressure, and $\mu(T)$ is the viscosity of the fluid depending on the temperature T of the fluid in the boundary layer region.

Out of the many forms of viscosity variation which are available in the literature we have considered only the form introduced by Ling and Dybbs (1987) and which was also used by Kafoussias and Rees (1998):

$$\mu = \mu_0\,[1 + \gamma(T - T_\infty)] \tag{5}$$

Here γ is a constant and μ_0 is the viscosity of the ambient fluid. The boundary conditions for the present problem are

$$\begin{aligned} &\hat{u} = 0, \quad \hat{v} = 0, \quad T = T_w \quad \text{at} \quad \hat{y} = \hat{y}_w = \sigma(\hat{x}) \\ &\hat{u} \to 0, \quad T \to T_\infty, \quad p \to p_\infty \quad \text{as} \quad \hat{y} \to \infty \end{aligned} \tag{6}$$

where p_∞ is constant pressure of the ambient fluid.

Now we introduce the following nondimensional dependent and independent variables:

$$x = \frac{\hat{x}}{L}, \quad y = \frac{\hat{y} - \hat{\sigma}}{L} Gr^{1/4}, \quad p = \frac{L^2}{\rho \nu_0^2} Gr^{-3/4} (\hat{p} - p_\infty)$$

$$u = \frac{\rho L}{\mu_0} Gr^{-1/2} \hat{u}, \quad v = \frac{\rho L}{\mu_0} Gr^{-1/4} (\hat{v} - \sigma_x \hat{u})$$

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}, \quad \sigma_x = \frac{d\hat{\sigma}}{d\hat{x}} = \frac{d\sigma}{dx}, \quad Gr = \frac{g\beta(T_w - T_\infty)}{\nu^2} L^3 \tag{7}$$

where L is the characteristic length associated with the wavy surface, and ν_0 $=(\mu_0/\rho)$ is the reference kinematic viscosity. Introducing the above transformations into the Eqs. (1)-(6) and after ignoring terms of small orders in the Grashof number Gr, we obtain the following equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + Gr^{1/4}\sigma_x \frac{\partial p}{\partial y} + \frac{1+\sigma_x^2}{1+\varepsilon\theta}\frac{\partial^2 u}{\partial y^2} - \frac{\varepsilon}{1+\varepsilon\theta}\frac{\partial u}{\partial y}\frac{\partial \theta}{\partial y} + \theta \tag{9}$$

$$\sigma_x\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \ \sigma_{xx}u^2 = -Gr^{1/4}\frac{\partial p}{\partial y} + \frac{\sigma_x(1+\sigma_x^2)}{1+\varepsilon\theta}\frac{\partial^2 u}{\partial y^2} \tag{10}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\mathrm{Pr}}(1+\sigma_x^2)\frac{\partial^2 \theta}{\partial y^2} \tag{11}$$

The boundary conditions for the present problem are

$$\begin{aligned} u = 0, \quad v = 0, \quad \theta = 1 \quad &\text{at } y = 0 \\ u \to 0, \quad \theta \to 0, \quad p \to 0 \quad &\text{as } y \to \infty \end{aligned} \tag{12}$$

In Eq. (7) $\varepsilon = \gamma(T_w - T_\infty)$ is the viscosity-variation parameter. From (12) it can be seen clearly that the dimensionless viscosity μ/μ_0 lies in the range $1/(1+\varepsilon)$ and 1; its value decreasing with increasing temperature when $\varepsilon > 0$.

The convection induced by the wavy surface is described in equations (8)-(11). Equation (10) indicates that the pressure gradient along the y direction is $O(Gr^{-1/4})$, which implies that lowest order pressure gradient along x direction can be determined from the inviscid-flow solution. For the present problem this pressure gradient is zero since there is no externally induced free stream. Equation (9) also shows that $Gr^{1/4}\partial p/\partial y$ is $O(1)$ and is determined by the left hand side of this equation. Thus, the elimination of $\partial p/\partial y$ between Eqs. (9) and (10) leads to

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1+\sigma_x^2}{1+\varepsilon\theta}\frac{\partial^2 u}{\partial y^2} + \frac{\sigma_x\sigma_{xx}}{1+\sigma_x^2}u^2 - \frac{\varepsilon}{(1+\varepsilon\theta)^2}\frac{\partial u}{\partial y}\frac{\partial \theta}{\partial y} + \left(\frac{1}{1+\sigma_x^2}\right)\theta \tag{13}$$

Now we introduce the further transformations into the Eqs. (13), (8)-(9) as described below:

$$\psi = x^{3/4} f(x, y), \quad \eta = y x^{-1/4}, \quad \theta = \theta(x, y)$$

where ψ is the stream function which satisfies the equation of continuity (6), and which is defined according to $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. The boundary layer equations now become

$$\frac{1+\sigma_x^2}{1+\varepsilon\theta} f''' + \frac{3}{4} f f'' - \left(\frac{1}{2} + \frac{x\sigma_x\sigma_{xx}}{1+\sigma_x^2}\right) f'^2 - \frac{\varepsilon}{(1+\varepsilon\theta)^2}\theta' f'' + \left(\frac{1}{1+\sigma_x^2}\right)\theta$$

$$= x\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right) \tag{14}$$

$$\frac{1}{\mathrm{Pr}}\left(1+\sigma_x^2\right)\theta'' + \frac{3}{4} f\theta' = x\left(f'\frac{\partial \theta}{\partial x} - \theta'\frac{\partial f}{\partial x}\right) \tag{15}$$

The boundary conditions to be satisfied are that

$$\begin{aligned} f = f', \quad \theta = 1 \quad &\text{at} \quad \eta = 0 \\ f' = 0, \quad \theta = 0 \quad &\text{as} \quad \eta \to \infty \end{aligned} \tag{16}$$

The quantities of physical interest are the surface shear-stress and rate of heat transfer which may be described in terms of the skin-friction, c_f and the local Nusselt number, Nu, respectively, from the relations given below:

$$C_f (Gr/4)^{3/4} = \left(\frac{1+\sigma_x^2}{1+\varepsilon}\right) f''(x,0) \tag{17}$$

$$Nu (Gr/4)^{-1/4} = -\left(1+\sigma_x^2\right)^{1/3} \theta'(x,0) \tag{18}$$

Here we integrate the locally nonsimilar partial differential equations (14)-(15) subject to the boundary conditions (16) by means of the Keller box method.

3. Solution methodology

Equations (15) and (16) along with the boundary conditions (17) have been discretized with a simple implicit finite difference scheme, very similar to that used by Keller and Cebeci (1972). In this method, the present system of equations are reduced to one consisting of equations which are first order in η. Central difference approximations based halfway along both η and x intervals are made and the resulting set of nonlinear difference equations are solved by using the Newton-Raphson quasi-linearization method. The Jacobian matrix has a block-tridiagonal structure and the difference equations are solved using a block-matrix version of the Thomas algorithm. For a given value of x, the iterative procedure is stopped when the maximum change between successive iterates is less than 10^{-6}. A uniform grid of 801 points was used in the x-direction with $\Delta x = 0.025$, while a non-uniform grid of 301 points lying between $\eta = 0.0$ and 12.05 was chosen. Grid points were concentrated towards the heated surface in order to improve resolution and the accuracy of the computed values of the surface shear stress and rate of heat transfer. The same method has also been used by Hossain et al. (1996,1997, 1999) to study the problem of mixed convection flow of a fluid with variable viscosity from a vertical surface. To initiate the process at $x = 0$, we prescribe the profiles for the functions f, θ and their derivatives from the solutions of the equations:

$$\frac{1}{1+\varepsilon\theta}f''' + \frac{3}{4}ff'' - \frac{1}{2}f'^2 - \frac{\varepsilon}{(1+\varepsilon\theta)^2}\theta' f'' + \theta = 0 \tag{19}$$

$$\frac{1}{\mathrm{Pr}}\theta'' + \frac{3}{4}f\theta' = 0 \tag{20}$$

satisfying the boundary conditions

$$\begin{aligned} f = f', \quad \theta = 1 \quad &\text{at } \eta = 0 \\ f' = 0, \quad \theta = 0 \quad &\text{as } \eta \to \infty \end{aligned} \tag{21}$$

The solution of the above equations, which governs the natural convective flow of a viscous incompressible fluid with temperature dependent viscosity from a vertical flat plate, are obtained using the Newton-Raphson iteration technique and these are entered in Table 1 for different values of the governing parameters; since these results are not readily available in the literature.

In the present investigation we considered the wavy surface to have the sinusoidal profile given by $\sigma_x = \alpha \sin(x)$.

4. Results and discussion

Although there are three parameters of interest in the present problem, our main aim is to determine the effects of varying ε, the parameter measuring the strength of temperature dependence of the viscosity.

Table 1 displays the numerical values of $f''(0)$ and $-\theta'(0)$ for different combinations of the Prandtl numbers (Pr = 1.0, 10.0, 25.0, 50.0 and 100.0) and the viscosity variation parameter (ε = 0.0, 1.0, 2.0, 3.0, 4.0, 5.0 and 6.0) for natural convection flow of a variable viscosity fluid from a plane vertical flat plate. In this table the values for $\varepsilon = 0.0$ are those for the case of a fluid with a constant viscosity, and which was studied by Sparrow and Gregg (1958). However from this table it is observed that an increasing value of the viscosity parameter leads to increase in the values of both the skin-friction and surface heat-flux coefficients.

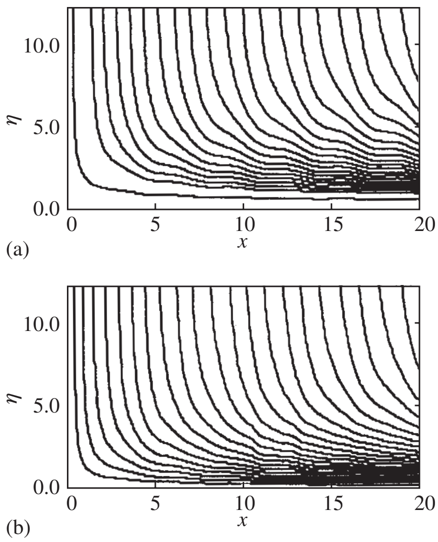

Figure 2.
Dimensionless stream-lines for Pr = 10.0 and $\alpha = 0.3$: (a) $\varepsilon = 0.0$, (b) $\varepsilon = 4.0$

This qualitative effect arises because the fluid is able to move more easily close to the heated surface since its viscosity is lower relative to the constant viscosity case. This results in thinner momentum and thermal boundary layers, and hence increased surface coefficients.

Table 1. Numerical values of and for different Prandtl number against ε

	$f''(0)$					$-\theta'(0)$				
ε/ Pr	1.0	10.0	25.0	50.0	100.0	1.0	10.0	25.0	50.0	100.0
0.0	0.908	0.591	0.485	0.485	0.352	0.401	0.825	1.066	1.066	1.542
1.0	1.439	0.975	0.810	0.810	0.598	0.436	0.931	1.212	1.212	1.771
2.0	1.864	1.288	1.079	1.079	0.804	0.457	0.997	1.307	1.307	1.921
3.0	2.232	1.563	1.315	1.315	0.987	0.471	1.045	1.376	1.376	2.035
4.0	2.562	1.813	1.530	1.530	1.154	0.481	1.084	1.432	1.432	2.126
5.0	2.864	2.045	1.729	1.729	1.309	0.490	1.116	1.478	1.478	2.203
6.0	3.145	2.264	1.916	1.916	1.453	0.496	1.143	1.518	1.518	2.267

The results obtained from the solutions of the Eqs. (15)-(17) are summarized briefly in Figures 2, 3 and 4. Figure 2 displays the streamlines in $\eta - x$ space of a Pr = 10 fluid with surface undulations of amplitude, $\alpha = 0.3$, for two values of ε: 0 and 4. When $\varepsilon = 0$ we recover the problem discussed by Yao (1983), where the fluid viscosity is independent of temperature. In Figure 2 we see that an increase in the value of ε causes the effects of the wavy surface to be attenuated and the boundary layer becomes thinner. It is clear from a close examination of these figures that the strength of the flow also increases as ε increases, since

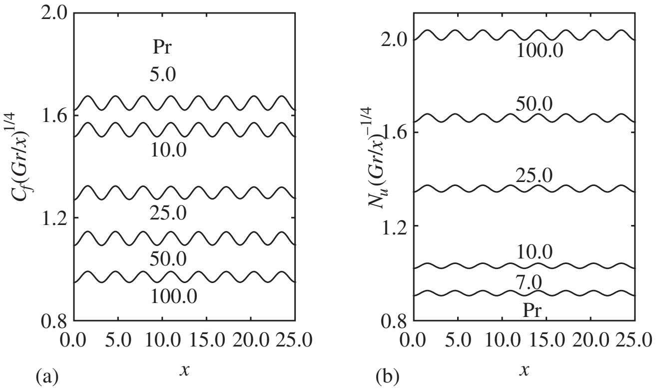

Figure 3.
Local (a) skin friction and (b) rate of heat transfer for different values of ε while $\alpha = 0.2$ and Pr $= 7.0$

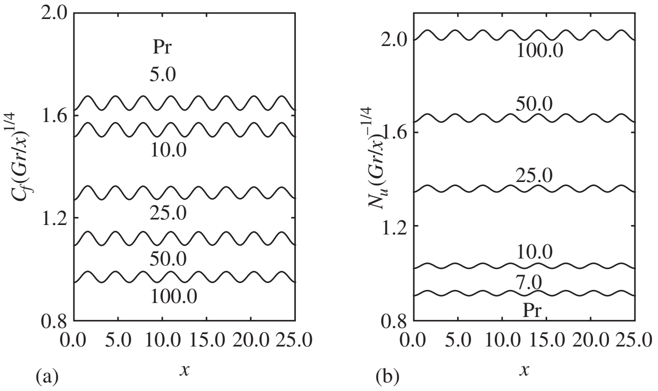

Figure 4. Local (a) skin friction and (b) rate of heat transfer for different values of Pr for $\alpha = 0.3$ and $\varepsilon = 3.0$.

the streamlines, which are plotted at identical intervals in the two cases, become increasingly close.

We omit streamline plots for other cases as no new phenomenological changes are apparent except for variations in the boundary layer thickness as Pr changes, or the strength of boundary effects as α changes. Therefore we summarize other cases in Figures 3 and 4 by displaying variations in the local skin friction coefficient. Figure 3 shows the effects of varying α for a Pr $= 7$ fluid with ($\alpha = 0.2$. Although the vigour of the flow increases as ε increases, as displayed in Figure 2, the skin friction coefficient, as defined by Eq. (18), and as shown in Figure 3a, decreases. The increasing strength of the flow is matched by an increasing rate of heat transfer, and this is reflected by the behavior of the curves in Figure 3b.

Figure 4 shows how variations in the Prandtl number affect the surface coefficients when $\alpha = 0.3$ and $\varepsilon = 3$. Increases in the value of Pr speed up the decay of the temperature field away from the heated surface with a consequent increase in the rate of heat transfer and a reduction in the thermal boundary layer thickness. On the other hand, the skin-friction coefficient decreases owing to an increase in the value of the Prandtl number; this is consistent with the data given in Table 1.

5. Conclusions

In this paper we have sought to investigate the effect of a temperature-dependent viscosity on the vertical free convective boundary layer flow from a uniform temperature wavy surface. The degree of dependence of the viscosity on the temperature is measured by ε where $\varepsilon = 0$ corresponds to a constant viscosity fluid and $\varepsilon > 0$ to a fluid which is less viscous at relatively high temperatures. The value $\varepsilon = 0$ corresponds to the work undertaken by Yao (1983).

When ε increases from zero, the fluid motion becomes stronger near the heated surface due to the decreased viscosity there. It was also shown that the localized effects of the surface waves penetrate further into the boundary layer as ε increases even though the boundary layer should become thinner. As in Yao (1983) the flow seems to attain a periodic state (in terms of the skin friction and heat transfer parameters) very quickly, and therefore detailed shear stress heat transfer results at large distances from the leading edge may be inferred easily from the present numerical analysis without having to resort to a detailed asymptotic analysis.

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(Received: October 30, 1998; revised July 7, 1999)