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THE EFFECT OF INERTIA ON FREE CONVECTIVE PLUMES IN POROUS MEDIA

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ABSTRACT

The boundary layer approximation to the flow induced by a line source of heat embedded in a porous medium predicts that seepage velocities become large as the source is approached, and therefore it is expected that inertia (form drag) should become significant. Such an inertia-dominated regime was studied by Ingham [1], but we extend that analysis to intermediate distances from the source by computing the smooth transition between the inertia-dominated and inertia-free regimes.

Introduction

The line source plume has received much less attention in the published literature than has any other thermal boundary layer flow in porous media. Wooding [2] was the first to analyse the plume subject to the boundary layer approximation and showed that the resulting self-similar equations have an analytical solution. Afzal [3] extended this analysis to wedge-shaped domains with symmetry about the vertical direction by employing higher-order boundary layer theory. Very recently the work of Afzal has itself been extended to include the effects of (a) a nonsymmetrical domain [4] and (b) anisotropy [5]. In these last two cases the convective plume was shown not to rise vertically, in general.

Of more interest here is the fact that the solution given by Wooding [2] implies that the fluid flux velocities become large as the leading edge is approached. In the context of boundary layer flows this usually means that the boundary layer approximation itself breaks down, but sometimes this is not true. In Hossain and Rees [6] the present authors reconsidered and extended the classical boundary layer analysis of the flow induced by an upward-facing semi-infinite horizontal surface embedded in a porous medium, a configuration which was first studied by Cheng and Chang [7], by including inertia effects. It was found that there is in fact a region relatively close to the leading edge where inertia effects dominate but within which the boundary layer approximation still applies. In the present paper we demonstrate that a similar situation arises for the line-source. Thus our aim is to extend the inertia-dominated analysis of Ingham [1] by considering the smooth transition between the inertia-dominated and inertia-free regimes. To this end the governing non-similar boundary layer equations are solved numerically and compared with the analytical solutions which are valid asymptotically close to and far away from the source.

Governing Equations

The steady dimensional equations for the boundary layer flow induced by a line heat source are given by

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0, \tag{1}$$

$$\overline{u} + \frac{K^*}{\nu}\overline{u}^2 = \frac{g\beta K}{\nu}\Big(T - T_\infty\Big), \tag{2}$$

$$\overline{u}\frac{\partial T}{\partial \overline{x}} + \overline{v}\frac{\partial T}{\partial \overline{y}} = \alpha\frac{\partial^2 T}{\partial \overline{y}^2}, \tag{3}$$

$$\rho C\int_{-\infty}^{\infty} \overline{u}(T - T_\infty)\, d\overline{y} = q'. \tag{4}$$

In these equations all the terms take their familiar meanings in the porous medium context: K is the permeability, K^* an inertia parameter, g gravity, ν the fluid viscosity, α the thermal diffusivity of the saturated medium and β the coefficient of cubical expansion. We have taken $\overline{x}$ to be the vertical coordinate and $\overline{y}$ the horizontal coordinate, with $\overline{u}$ and $\overline{v}$ the respective fluid flux velocities. T is the temperature of the medium while T_∞ is the ambient temperature which is also taken as the reference temperature, and q' is the prescribed rate of heat flux per unit length of the line source.

Eqs. (1) to (4) may be simplified by the introduction of the streamfunction, ψ according to $\overline{u} = \overline{\psi}_{\overline{y}}$ and $\overline{v} = -\overline{\psi}_{\overline{y}}$, and nondimensionalised using the scalings,

$$\overline{\psi} = \Big(\frac{q'}{\rho C}\Big)\Big(\frac{g\beta KK^*}{\nu^2}\Big)\psi, \qquad T = T_\infty + \Big(\frac{\nu^2}{g\beta KK^*}\Big)\theta, \tag{5,6}$$

$$\overline{y} = \Big(\frac{q'}{\rho C}\Big)\Big(\frac{g\beta KK^*}{\nu^2}\Big)\Big(\frac{K^*}{\nu}\Big)y, \qquad \overline{x} = \frac{1}{\alpha}\Big(\frac{q'}{\rho C}\Big)^2\Big(\frac{g\beta KK^*}{\nu^2}\Big)^2\Big(\frac{K^*}{\nu}\Big)x. \tag{7,8}$$

We note that these scalings look very different from those used by Ingham [1], but in his analysis an undefined lengthscale, l, was used; if that lengthscale were to be set equal to

$$l = \frac{1}{\alpha}\Big(\frac{q'}{\rho C}\Big)^2\Big(\frac{g\beta KK^*}{\nu^2}\Big)^2\Big(\frac{K^*}{\nu}\Big), \tag{9}$$

then Ingham's scalings reduce to those used here.

Therefore Eqs. (1) to (4) reduce to

$$\frac{\partial\psi}{\partial y} + \left(\frac{\partial\psi}{\partial y}\right)^2 = \theta, \qquad \frac{\partial^2\theta}{\partial y^2} = \frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}, \tag{9,10}$$

$$\int_{-\infty}^{\infty} \frac{\partial\psi}{\partial y}\theta\, dy = 1. \tag{11}$$

This latter integral constraint may be recast into differential form using the definition of ϕ, as follows,

$$\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial y}\theta. \tag{12}$$

As the plume displays symmetry about its centreline, we may allow for this by applying suitable boundary conditions at $y = 0$. Therefore we will solve Eqs. (9), (10) and (12) subject to

$$y = 0: \quad \psi = \frac{\partial\theta}{\partial y} = \phi = 0, \qquad y \to \infty: \quad \theta \to 0 \quad \phi \to \tfrac{1}{2}. \tag{13}$$

It is in the nature of a plume that the greatest velocities occur near the line source and therefore inertial effects will dominate there. At large distances the induced velocities are relatively small, and therefore Darcy flow will be re-established. In view of these observations we will employ different boundary layer transformations for small and for large values of x. When x is small we set

$$\psi = X^2 f(\zeta, X), \qquad \theta = X^{-2} g(\zeta, X), \qquad \phi = h(\zeta, X) \tag{14}$$

where

$$\zeta = y/x^{3/5} \quad \text{and} \quad X = x^{1/5}. \tag{15}$$

In this regime the equations become,

$$f'(X + f') = g, \qquad g'' + \tfrac{2}{5}(f'g + fg') = \tfrac{1}{5}X[f'g_X - f_X g'], \tag{16,17}$$

$$h' = f'g, \tag{18}$$

and the boundary conditions are

$$f(0) = g'(0) = h(0) = 0 \qquad h'(\zeta) \to \tfrac{1}{2} \quad \text{as} \quad \zeta \to \infty. \tag{19}$$

When $X = 0$ we recover the equations solved by Ingham [1]. For large values of x the appropriate transformation is given by

$$\psi = \xi F(\eta, \xi), \qquad \theta = \xi^{-1} G(\eta, \xi), \qquad \phi = H(\eta, \xi) \tag{20}$$

where

$$\eta = y/x^{2/3} \quad \text{and} \quad \xi = x^{1/3}. \tag{21}$$

The equations are now

$$F'(1+\xi^{-1}F') = G, \qquad G'' + \tfrac{1}{3}(F'G + FG') = \tfrac{1}{3}\xi[F'G_\xi - F_\xi G'], \tag{22,23}$$

$$H' = F'G, \tag{24}$$

subject to boundary conditions identical to those given in (19).

Analytical Solutions

Before describing the numerical methodology it is valuable to state for reference the exact solutions which exist at $x = 0$ and at asymptotically large distances from the leading edge. In the inertia-dominated regime Ingham [1] showed that the solution takes the form

$$f = C_1 \tanh(\tfrac{1}{10}C_1\zeta), \qquad g = \tfrac{1}{100}C_1^4 \mathrm{sech}^4(\tfrac{1}{10}C_1\zeta) \tag{25}$$

where

$$C_1 = \left[\frac{375}{4}\right]^{1/4}. \tag{26}$$

On the other hand, the solution at asymptotically large distances is unaffected by inertia and is given by

$$F = C_2 \tanh(\tfrac{1}{2}C_2\eta), \qquad G = \tfrac{1}{6}C_2^2 \mathrm{sech}^2(\tfrac{1}{6}C_2\eta) \tag{27}$$

where

$$C_2 = \left[\frac{9}{2}\right]^{1/3}. \tag{28}$$

Numerical Solutions

Both systems (16-18) and (22-24) have been solved using a standard Keller-box implementation. The switch between the respective systems takes place at $x = X = \xi = 1$ at which point corresponding variables (such as f and F or ζ and η) are equal. We used a nonuniform grid of 80 points in the ζ or η direction in the range $0 \le \zeta, \eta \le 20$, and a nonuniform grid of 401 values of either X or ξ from $X = 0$ to $\xi = 100$, i.e. from $x = 0$ to $x = 10^6$. As there are no parameters to vary, only one numerical simulation was undertaken, subject to numerical checking of the accuracy of the overall computation. For example, we find numerically that the centreline temperature at $X = 0$ is 0.37801 which should be compared with $\frac{1}{100}(375/4)^{4/5} \simeq 0.37807$, which is the analytical value given in (25) and (26). At $X = 100$ ($x = 10^6$) we have the centreline temperature 0.45425 which is to be compared with $\frac{1}{6}(9/2)^{2/3} \simeq 0.45428$ given in (27) and (28).

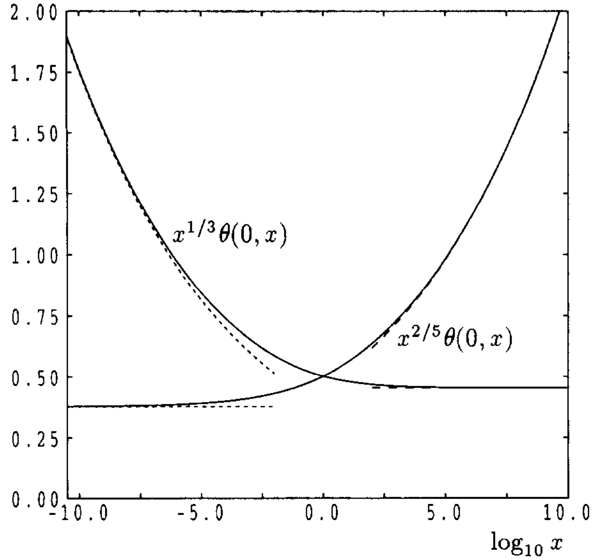

FIG. 1.
Variation with x of the centreline temperature in the forms, $x^{1/3}\theta(0,x)$ and $x^{2/5}\theta(0,x)$. Also shown are the leading order asymptotic forms for small values of x (short dashes) and for large values of x (long dashes).

The transition between the inertia-dominated regime near $x = 0$ and the inertia-free regime at large distances is smooth. But it is necessary to present the variation with x of the centreline temperature in two forms, the first of which is finite at $x = 0$ while the other is finite as $x \to \infty$. In particular we consider the variation with x of $x^{2/5}\theta(0,x)$ and $x^{1/3}\theta(0,x)$, which, when rewritten in terms of the appropriate variables become,

$$x^{2/5}\theta(0,x) = \begin{cases} g(0,X) & x \leq 1, \\ \xi^{1/5}G(0,\xi) & x \geq 1, \end{cases} \tag{29}$$

and

$$x^{1/3}\theta(0,x) = \begin{cases} X^{-1/3}g(0,X) & x \leq 1, \\ G(0,\xi) & x \geq 1. \end{cases} \tag{30}$$

We show the variation of the centreline temperature with $\log_{10} x$ in Figure 1. If we regard a 1% relative error as being the criterion indicating departure from the asymptotic state, then the inertia-dominated regime extends from $x = 0$ to $x = 2 \times 10^{-8}$, while Darcy flow is established as soon as x exceeds 3×10^3. At intermediate values of x the flow may be regarded as transitional and is affected in part by the presence of inertia.

Conclusion

We have extended the analyses of Wooding [2], who considered Darcy (inertia-free) flow, and Ingham [1], who considered inertia-dominated flow, to those intermediate cases between the two regimes. Near the leading edge the flow is controlled by inertia effects, but these decay with distance away from the line source until Darcy-flow is re-established when the induced flow is sufficiently weak. We have shown that this intermediate regime may be taken to lie within the nondimensional range $2 \times 10^{-8} < x < 3 \times 10^{3}$. For values of x which are outside of this range the analytical solutions of Wooding [2] and Ingham [1] may be taken to apply with a relative error of less than 1%.

References

1. D.B. Ingham, *Wärme-Stoffübertrag.* **22**, 125-127 (1988).
2. R.A. Wooding, *J. Fluid Mech.* **15**, 527-544 (1963).
3. N. Afzal, *Int. J. Heat Mass Transfer* **11**, 2029-2041 (1985).
4. A.P. Bassom, D.A.S. Rees and L. Storesletten, *Int. Comm. Heat and Mass Transfer* **28**, 31-38 (2001).
5. D.A.S. Rees, L. Storesletten and A.P. Bassom, "Convective plume paths in anisotropic porous media", To appear in *Transport in Porous Media.*
6. M.A. Hossain and D.A.S. Rees, *Transport in Porous Media* **29**, 309-321 (1998).
7. P. Cheng and I.D. Chang, *Int. J. Heat Mass Transfer* **19**, 1267-1272 (1976).

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