Pergamon

Int. Comm. Heat Mass Transfer, Vol. 28, No. 5, pp. 641–650, 2001

0735-1933/01/$–see front matter

PII: S0735-1933(01)00268-8

THE ONSET OF CONVECTION IN AN ANISOTROPIC POROUS LAYER INCLINED AT A SMALL ANGLE FROM THE HORIZONTAL

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(Communicated by O.G. Martynenko)

ABSTRACT

We consider the onset of convection in a porous layer heated from below. The layer is anisotropic with respect to both its permeability and diffusivity and is inclined at a small angle to the horizontal. The aim of this work is to determine analytically by how much the critical Rayleigh number varies when the layer first ceases to be horizontal. For nearly isotropic layers we show that rolls may be either longitudinal or transverse depending on the nondimensional parameters; this is unlike the case of $O(1)$ inclinations the results of which are presented in a companion paper.

Introduction

One of the most widely studied and fundamental processes in the study of convection in fluid-saturated porous media is the onset and development of convection in layers heated from below. The pioneering works in the field were carried out by Horton and Rogers [1] and Lapwood [2], and the general problem has become known as the Horton-Rogers-Lapwood or Darcy-Bénard problem. A fairly comprehensive account of the current state-of-the-art may be found in Rees [3]. In its classical formulation a porous medium is sandwiched between two uniform temperature plane surfaces and is heated from below.

While there are many different extensions to Darcy's law, we focus solely on the effects of introducing anisotropy where the principal axes of the permeability and diffusivity tensors coincide with the coordinate directions. The first work to deal with such problems was undertaken by Castinel and Combarnous [4], and detailed reviews of anisotropic convection may be found in Storesletten [5] and Vasseur and Robillard [6].

Here we investigate how small angles of tilt of the layer away from the horizontal direction affects the criterion for the onset of anisotropic convection. For isotropic layers it is well-known from theoretical investigations that longitudinal vortices with axes directly up the layer are favoured; see Rees and Bassom [7], for example, who give a very detailed analysis of tilted isotropic layers.

To date, only one published paper deals with the combined effects of anisotropy and layer inclination. Storesletten and Tveitereid [8] assume that the preferred mode of convection at onset is either a longitudinal vortex, or a transverse roll which has its axis perpendicular to the direction of the basic flow. In that paper the authors give conditions under which transverse modes may be preferred to longitudinal vortices at low but $O(1)$ inclinations. One of the aims of this paper and its sequel [9] is to determine whether the assumption of [8] is valid. Indeed, for $O(1)$ inclinations we demonstrate in [9] that it is invalid. However, for small inclinations the assumption is indeed correct. A second aim of the present paper is to determine by how much the critical Rayleigh number varies when the inclination from the horizontal is small, since it is highly likely that experimental devices would be slightly misaligned with respect to the direction of gravity. In fact, we show that the magnitude of this $O(\alpha^2)$ correction (where α is the inclination) can achieve very large values.

Equations of Motion

We consider the onset of convection in a tilted anisotropic porous layer heated from below. The layer, of thickness h, is inclined at an angle α to the horizontal. The horizontal z-axis forms the direction about which the layer has been rotated, the y-axis is perpendicular to the bounding surfaces, and the x-axis lies in the lower surface pointing up the plane. We assume that Darcy's law and the Boussinesq approximation hold, that the solid matrix and the saturating fluid are in local thermal equilibrium. Therefore the flow and heat transfer are taken to be modelled by the continuity equation, Darcy's law and the equation of conservation of energy:

$$\nabla.\underline{u} = 0, \tag{1}$$

$$\mu\underline{u} + \underline{\underline{K}}.(\nabla p + \rho_0\beta(T - T_m)\underline{g}) = 0, \tag{2}$$

$$\sigma\frac{\partial T}{\partial t} + \underline{u}.\nabla T = \nabla.(\underline{\underline{D}}.\nabla T), \tag{3}$$

where the permeability and diffusivity tensors have their principal axes in the coordinate directions and are given by

$$\underline{\underline{K}} = K_1\underline{i}\,\underline{i} + K_2\underline{j}\,\underline{j} + K_3\underline{k}\,\underline{k}, \qquad \underline{\underline{D}} = D_1\underline{i}\,\underline{i} + D_2\underline{j}\,\underline{j} + D_3\underline{k}\,\underline{k}. \tag{4}$$

The vectors, $\underline{i}$, $\underline{j}$ and $\underline{k}$ are the unit vectors in the x, y and z directions, respectively. Terms in (1) to (3) have their familiar meanings in the porous medium context: $\underline{u} = (u, v, w)$ is the velocity flux vector, p is the pressure, T the temperature, t the time, ρ_0 a reference density, β the coefficient of cubical expansion, $\underline{g}$ the gravity vector, μ the fluid viscosity and σ the ratio of the heat capacities of the saturated medium and the fluid. The lower surface at $y = 0$ is impermeable and is held at the temperature T_h, while the impermeable upper surface is held at $T = T_c$ where $T_c < T_h$. The mean temperature is $T_m = \frac{1}{2}(T_c + T_h)$.

These equations may be nondimensionalised using the following substitutions,

$$(x, y, z) = h(x^*, y^*, z^*), \quad \underline{u} = D_2 h \underline{u}^*, \quad t = \frac{\sigma h^2}{D_2} t^*, \quad p = \frac{D_2 \mu}{K_2} p^*, \quad T = T_c + (T_h - T_c)\theta. \tag{5}$$

We obtain

$$\nabla . \underline{u} = 0, \tag{6}$$

$$u = -\frac{K_1}{K_2}\Big(\frac{\partial p}{\partial x} + R(\theta - \tfrac{1}{2})\sin\alpha\Big), \qquad v = -\Big(\frac{\partial p}{\partial y} + R(\theta - \tfrac{1}{2})\cos\alpha\Big), \tag{7,8}$$

$$w = -\frac{K_3}{K_2}\frac{\partial p}{\partial z}, \qquad \frac{\partial \theta}{\partial t} + \underline{u}.\nabla\theta = \frac{D_1}{D_2}\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{D_3}{D_1}\frac{\partial^2\theta}{\partial z^2}, \tag{9,10}$$

where asterisk superscripts have been omitted for clarity. In this paper we consider both two-dimensional and three-dimensional modes of instability; in the former case it proves convenient to define a streamfunction, ψ, using

$$u = -\frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x} \qquad w = 0, \tag{11}$$

whereas in the latter case we eliminate $\underline{u}$ to obtain a system of equations in terms of pressure and temperature. For two-dimensional disturbances we consider the equations

$$\xi_1\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = R\xi_1\Big(\frac{\partial\theta}{\partial x}\cos\alpha - \frac{\partial\theta}{\partial y}\sin\alpha\Big), \tag{12}$$

$$\frac{\partial\theta}{\partial t} + \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} = \eta_1\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}, \tag{13}$$

subject to

$$\psi = 0, \quad \theta = 1 \quad \text{on} \quad y = 0 \qquad \text{and} \qquad \psi = 0, \quad \theta = 0 \quad \text{on} \quad y = 1, \tag{14}$$

and for three-dimensional disturbances we consider

$$\xi_1\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \xi_3\frac{\partial^2 p}{\partial z^2} = R\Big[\xi_1\frac{\partial\theta}{\partial x}\sin\alpha + \frac{\partial\theta}{\partial y}\cos\alpha\Big] \tag{15}$$

$$\frac{\partial\theta}{\partial t} + R\Big[\xi_1\frac{\partial\theta}{\partial x}\sin\alpha + \frac{\partial\theta}{\partial y}\cos\alpha\Big]\theta - \xi_1\frac{\partial p}{\partial x}\frac{\partial\theta}{\partial x} - \frac{\partial p}{\partial y}\frac{\partial\theta}{\partial y} - \xi_3\frac{\partial p}{\partial z}\frac{\partial\theta}{\partial z} = \eta_1\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \eta_3\frac{\partial^2\theta}{\partial z^2}, \tag{16}$$

subject to

$$\frac{\partial p}{\partial y} = \tfrac{1}{2}\cos\alpha, \quad \theta = 1 \quad \text{on} \quad y = 0 \qquad \text{and} \qquad \frac{\partial p}{\partial y} = -\tfrac{1}{2}\cos\alpha, \quad \theta = 0 \quad \text{on} \quad y = 1. \tag{17}$$

The nondimensional parameter, $R = \rho_0 g\beta K_2(T_h - T_c)h/\mu D_2$, is the Darcy-Rayleigh number based upon the permeability and diffusivity in the y-direction, and

$$\xi_1 = \frac{K_1}{K_2}, \quad \xi_3 = \frac{K_3}{K_2}, \quad \eta_1 = \frac{D_1}{D_2} \quad \text{and} \quad \eta_3 = \frac{D_3}{D_2} \tag{18}$$

are permeability and diffusivity ratios.

As we are concerned with linear stability characteristics, we linearise equations (12) and (13) and equations (15) and (16) about the basic flow profiles by means of the substitutions,

$$\psi = -\tfrac{1}{2}R\xi_1(y - y^2)\sin\alpha + \Psi, \qquad \theta = 1 - y + \Theta, \qquad p = \tfrac{1}{2}R(y - y^2)\cos\alpha + P, \tag{19}$$

where the magnitudes of Ψ, Θ and P are assumed to be infinitesimally small. We obtain the linearised perturbation equations,

$$\xi_1\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = R\xi_1\Big(\frac{\partial\Theta}{\partial x}\cos\alpha - \frac{\partial\Theta}{\partial y}\sin\alpha\Big), \tag{20}$$

$$\frac{\partial\Theta}{\partial t} = \eta_1\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} + \frac{\partial\Psi}{\partial x} + R\xi_1\sin\alpha\,(y - \tfrac{1}{2})\frac{\partial\Theta}{\partial x}, \tag{21}$$

and

$$\xi_1\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \xi_3\frac{\partial^2 P}{\partial z^2} = R\Big[\xi_1\frac{\partial\Theta}{\partial x}\sin\alpha + \frac{\partial\Theta}{\partial y}\cos\alpha\Big], \tag{22}$$

$$\frac{\partial\Theta}{\partial t} = \eta_1\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} + \eta_3\frac{\partial^2\Theta}{\partial z^2} + R\cos\alpha\,\Theta + R\xi_1\sin\alpha\,(y - \tfrac{1}{2})\frac{\partial\Theta}{\partial x} - \frac{\partial P}{\partial y}, \tag{23}$$

subject to

$$\Psi = \Theta = \frac{\partial P}{\partial y} = 0 \quad \text{on both} \quad y = 0 \text{ and } \quad y = 1. \tag{24}$$

Finally we Fourier-decompose the disturbances in the x and z directions which will reduce the perturbation equations to ordinary differential eigenvalue form. Thus we substitute

$$\Psi = if(y)e^{ikx+\lambda t} \qquad \Theta = g(y)e^{ikx+\lambda t}, \tag{25}$$

into (20) and (21) to obtain

$$f'' - k^2\xi_1 f = R\xi_1[(k\cos\alpha)g + (i\sin\alpha)g'], \tag{26}$$

$$g'' - k^2\eta_1 g = kf - (Rik\xi_1\sin\alpha)(y - \tfrac{1}{2})g + \lambda g, \tag{27}$$

subject to $f = g = 0$ at $y = 0, 1$. Here k is the wavenumber of the disturbance and λ is the exponential growth rate. For three-dimensional disturbances we substitute

$$P = q(y)e^{ik(z\cos\phi + x\sin\phi)+\lambda t}, \qquad \Theta = g(y)e^{ik(z\cos\phi+x\sin\phi)+\lambda t}, \tag{28}$$

into equations (22) and (23) to obtain

$$q'' - k^2(\xi_1\sin^2\phi + \xi_3\cos^2\phi)q = R\Big[(\cos\alpha)g' + (ik\xi_1\sin\phi\sin\alpha)g\Big] \tag{29}$$

$$g'' + \Big[R\cos\alpha - k^2(\eta_1\sin^2\phi + \eta_3\cos^2\phi)\Big]g = q' - (Rik\xi_1\sin\phi\sin\alpha)(y - \tfrac{1}{2})g + \lambda g, \tag{30}$$

subject to $q' = g = 0$ at $y = 0, 1$. Here ϕ represents the orientation of the axis of the vortex disturbance relative to the x-direction. The value $\phi = \frac{1}{2}\pi$ represents the two-dimensional case and is termed a transverse roll, $\phi = 0$ represents the longitudinal roll, and rolls of other orientations are called oblique.

In this paper we consider first two-dimensional disturbances, for although there exist parameter sets for which three-dimensional disturbances are more destabilising, it is always possible to eliminate three-dimensionality by restricting the layer sufficiently in the spanwise (or z) direction with impermeable insulating sidewalls.

Two-dimensional Instability

We begin the general analysis of stability by performing a small-α series solution to determine an expression for the leading order change in the critical Rayleigh number for small changes in the inclination of the layer from the horizontal. The following power series is introduced into equations (26) and (27),

$$\begin{pmatrix} f \\ g \\ R \end{pmatrix} = \begin{pmatrix} f_0 \\ g_0 \\ R_0 \end{pmatrix} + \alpha\begin{pmatrix} f_1 \\ g_1 \\ 0 \end{pmatrix} + \alpha^2\begin{pmatrix} f_2 \\ g_2 \\ R_2 \end{pmatrix} + \cdots. \tag{31}$$

At leading order we obtain the following eigenvalue problem for R_0,

$$f_0'' - k^2\xi_1 f_0 - R_0 k\xi_1\, g_0 = 0, \qquad g_0'' - k^2\eta_1 g_0 - kf_0 = \lambda g_0, \tag{32, 33}$$

which has the solution

$$f_0 = -\Big(\frac{\pi^2 + k^2\eta_1}{k}\Big)\sin\pi y, \qquad g_0 = \sin\pi y, \qquad R_0 = \frac{(\pi^2 + k^2\xi_1)(\pi^2 + k^2\eta_1)}{k^2\xi_1}, \tag{34}$$

with $\lambda = 0$. As this is a linear problem we have chosen the normalisation condition to be such that g_0 is a sine with unit amplitude. The value of R_0 varies with the wavenumber and takes its minimum value when

$$k_c = \pi/(\xi_1\eta_1)^{1/4} \tag{35}$$

and the corresponding 'critical' value of R_0 is

$$R_{0c} = \pi^2\Big[1 + \Big(\frac{\eta_1}{\xi_1}\Big)^{1/2}\Big]^2. \tag{36}$$

At $O(\alpha)$ the equations are

$$f_1'' - k^2\xi_1 f_1 - R_0 k\xi_1\, g_1 = R_0 i\xi_1 g_0', \qquad g_1'' - k^2\eta_1 g_1 - kf_1 = -R_0 ik\xi_1(y - \tfrac{1}{2})g_0, \tag{37,38}$$

and their solution may be written in the form

$$f_1 = A(y - y^2)\cos\pi y + B(y - \tfrac{1}{2})\sin\pi y, \tag{39}$$

$$g_1 = C(y - y^2)\cos\pi y + D(y - \tfrac{1}{2})\sin\pi y, \tag{40}$$

where

$$C = -\frac{R_0 ik\xi_1(\pi^2 + k^2\xi_1)}{4\pi(2\pi^2 + k^2\xi_1 + k^2\eta_1)} \equiv -R_0 ik\xi_1\tilde{C}, \tag{41}$$

$$D = -\frac{R_0 ik\xi_1[2\pi^4 + \pi^2k^2(5\xi_1 - \eta_1) + k^4\xi_1(\xi_1 + \eta_1)]}{4\pi^2(2\pi^2 + k^2\xi_1 + k^2\eta_1)^2} \equiv -R_0 ik\xi_1\tilde{D} \tag{42}$$

and which defines the constants $\tilde{C}$ and $\tilde{D}$ which are used below. The values of A and B may be found easily, but they are not required when finding R_2 and therefore we omit their presentation.

At $O(\alpha^2)$ we have

$$f_2'' - k^2\xi_1 f_2 - R_0 k\xi_1\, g_2 = R_0\xi_1(-\tfrac{1}{2}kg_0 + ig_1') + R_2\xi_1 kg_0 \quad \equiv \quad \mathcal{R}_f, \tag{43}$$

$$g_2'' - k^2\eta_1 g_2 - kf_2 = -R_0 ik\xi_1(y - \tfrac{1}{2})g_1 \quad \equiv \quad \mathcal{R}_g. \tag{44}$$

The forcing terms on the right hand sides of (43) and (44) contain components which are proportional to the eigenfunction of the equivalent homogeneous system and therefore we need to employ an orthogonality condition which will find the value of R_2 for which solutions exist. Such a procedure is quite standard in stability analyses and in this context we require that

$$\int_0^1 \Big[\mathcal{R}_f\, f_0 + \mathcal{R}_g\, R_0\xi_1 g_0\Big]\,dy = 0. \tag{45}$$

Application of this condition yields

$$R_2 = \tfrac{1}{2}R_0 - \frac{k^2\xi_1^2R_0^2}{12\pi^3(\pi^2 + k^2\eta_1)}\Big[\Big(2(\pi^2 + k^2\eta_1)(\pi^2 - 3) - R_0(3 + 2\pi^2)\Big)\tilde{C} + \pi(R_0\pi^2 - 6\pi^2 - 6k^2\eta_1)\tilde{D}\Big]. \tag{46}$$

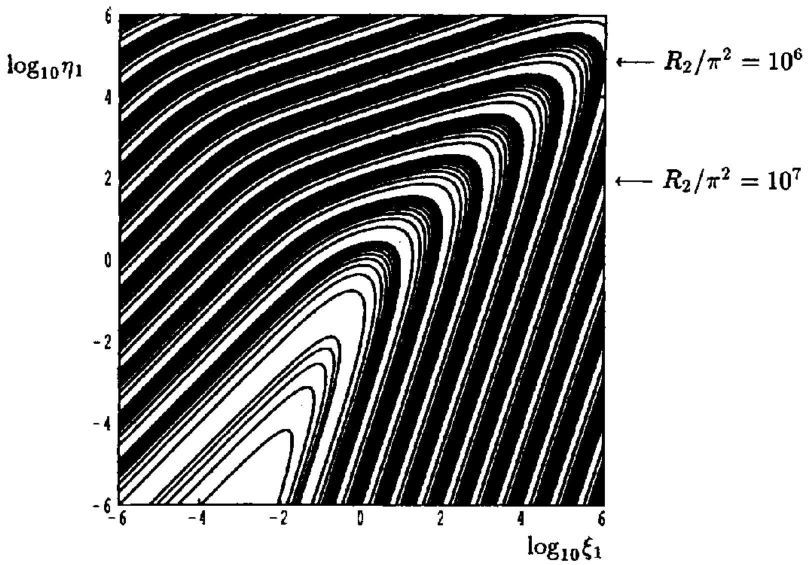

FIG. 1.
The variation with ξ_1 and η_1 of R_2/π^2 for two dimensional convection. We take $R_2/\pi^2 = 0.5, 0.6, 0.7, 0.8, 0.9, 1, 2 \cdots 10, 20 \cdots 100, 200 \cdots 1000$, etc. When $\xi_1 = \eta_1 = 1$, then $R_2 = 5\pi^2$, and when $\xi_1 = 10^6$ and $\eta_1 = 10^{-6}$ then $R_2 \simeq 10^{11}\pi^2$.

This expression reduces to that given by Rees and Bassom (2000) for the isotropic layer, $\xi_1 = \eta_1 = 1$, and is

$$R_2 = \frac{(\pi^2 + k^2)^2}{32\pi^4 k^4}(\pi^6 + 23\pi^4 k^2 + 11\pi^2 k^4 + 5k^6) + \frac{(\pi^2 + k^2)^4(\pi^2 - k^2)}{96\pi^2 k^4}. \qquad (47)$$

The dependence of R_2 upon ξ_1 and η_1 in (46) is complicated, and the full expression obtained when the critical values of k and R_0 are substituted is very lengthy. However, a contour plot of R_2 as a function of ξ_1 and η_1 is given in Figure 1 showing by how much the critical value of R varies from R_0 when the inclination is small. The value of R_2 always remains positive, although it may achieve small values when η_1 is small. It is clear that R_2 varies over many orders of magnitude as ξ_1 and η_1 vary. Although it represents a rather extreme example of a porous medium, when $\xi_1 = 10^6$ and $\eta_1 = 10^{-6}$, the critical Rayleigh number is approximately $R_c \sim \pi^2 + 10^{11}\pi^2\alpha^2$. Therefore the inclination must be less than approximately $10^{-11/2}$ for the change in the Rayleigh number to be given accurately by the horizontal value. Such tolerances are unlikely to be achieved in practice.

Three-dimensional Instability

We turn now to the more physically realistic situation of rolls at orientations other than the above two-dimensional cases for which $\phi = \frac{1}{2}\pi$. In this section we sketch briefly the results of the equivalent analysis to that of the last section.

At leading order, which corresponds to a horizontal layer, we obtain

$$q_0 = \left[\frac{\pi^2 + k^2(\eta_1 \sin^2\phi + \eta_3 \cos^2\phi) - R_0}{\pi}\right]\cos\pi y, \qquad g_0 = \sin\pi y, \tag{48, 49}$$

$$R_0 = \frac{[\pi^2 + k^2(\xi_1 \sin^2\phi + \xi_3\cos^2\phi)]\,[\pi^2 + k^2(\eta_1\sin^2\phi + \eta_3\cos^2\phi)]}{k^2(\xi_1\sin^2\phi + \xi_3\cos^2\phi)}. \tag{50}$$

The minimising wavenumber is

$$k_c = \frac{\pi}{(\xi_1\sin^2\phi + \xi_3\cos^2\phi)^{1/4}(\eta_1\sin^2\phi + \eta_3\cos^2\phi)^{1/4}} \tag{51}$$

and the value of R_0 is

$$R_{0c} = \pi^2\left[1 + \left(\frac{\eta_1\sin^2\phi + \eta_3\cos^2\phi}{\xi_1\sin^2\phi + \xi_3\cos^2\phi}\right)^{1/2}\right]^2. \tag{52}$$

Given that the orientation of the roll is at our disposal, it is straightforward to show that extreme values of R_{0c} occur when $\phi = 0$, $\phi = \frac{1}{2}\pi$ or when $(\eta_1/\xi_1) = (\eta_3/\xi_3)$. When $(\eta_1/\xi_1) < (\eta_3/\xi_3)$ then $R_{0c} = \pi^2[1 + (\eta_1/\xi_1)^{1/2}]^2$ with $\phi = \frac{1}{2}\pi$, a transverse roll. When $(\eta_1/\xi_1) > (\eta_3/\xi_3)$ then $R_{0c} = \pi^2[1 + (\eta_3/\xi_3)^{1/2}]^2$ with $\phi = 0$, a longitudinal roll. The transitional case is when $(\eta_1/\xi_1) = (\eta_3/\xi_3) = \gamma$, say, when $R_{0c} = \pi^2[1 + \gamma^{1/2}]^2$ independently of the roll orientation.

At $O(\alpha)$ the solutions are,

$$q_1 = A(y - y^2)\sin\pi y + B(y - \tfrac{1}{2})\cos\pi y, \tag{53}$$

$$g_1 = C(y - y^2)\cos\pi y + D(y - \tfrac{1}{2})\sin\pi y, \tag{54}$$

where A and B are different from those introduced in (39) and where C and D are a generalisation of those in (40). We find that

$$C = -\frac{R_0 ik\xi_1\sin\phi(\pi^2 + k^2\xi)}{4\pi(2\pi^2 + k^2\xi + k^2\eta)} \equiv -R_0 ik\xi_1\sin\phi\,\tilde{C}, \tag{55}$$

$$D = -\frac{R_0 ik\xi_1\sin\phi[2\pi^4 + \pi^2k^2(5\xi - \eta) + k^4\xi(\xi + \eta)]}{4\pi^2(2\pi^2 + k^2\xi + k^2\eta)^2} \equiv -R_0 ik\xi_1\sin\phi\,\tilde{D} \tag{56}$$

where

$$\xi = \xi_1\sin^2\phi + \xi_3\cos^2\phi \qquad \text{and} \qquad \eta = \eta_1\sin^2\phi + \eta_3\cos^2\phi. \tag{57}$$

The solvability condition at $O(\alpha^2)$ yields the expression,

$$R_2 = \tfrac{1}{2}R_0 - \frac{k^2\xi_1^2R_0^2\sin^2\phi}{12\pi^3(\pi^2 + k^2\eta)}\Big[\Big(2(\pi^2 + k^2\eta)(\pi^2 - 3) - R_0(3 + 2\pi^2)\Big)\tilde{C} + \pi(R_0\pi^2 - 6\pi^2 - 6k^2\eta)\tilde{D}\Big]. \tag{58}$$

In all circumstances $R_2 \geq 0$ indicating that small inclinations from the horizontal causes the critical Rayleigh number to rise above that for the horizontal layer. For the isotropic layer the critical Rayleigh number reduces to

$$R_2 = 2\pi^2 + 3\pi^2 \sin^2 \phi, \tag{59}$$

which shows that longitudinal rolls, for which $\phi = 0$, form the preferred mode of instability in this case.

When the layer is anisotropic, the variation in R_{0c} is typically of $O(1)$ as any one or more of ξ_1, ξ_3, η_1 and η_3 vary, and therefore the $O(\alpha^2)$ correction given in (58) is subdominant. However, when the layer is nearly isotropic, the critical value of $R_0 + \alpha^2 R_2$ may be simplified considerably. If we set $\xi_1 = 1 + \alpha^2 \hat{\xi}_1$, with similar expressions for ξ_3, η_1 and η_3, then (51) becomes

$$k_c = \pi + O(\alpha^3), \tag{60}$$

and the critical Rayleigh number is

$$R_{0c} + \alpha^2 R_{2c} \sim 4\pi^2 + \alpha^2\pi^2 \Big[2 + (3 + \hat{\eta}_1 - 3\hat{\xi}_1)\sin^2\phi + (\hat{\eta}_3 - 3\hat{\xi}_3)\cos^2\phi\Big]. \tag{61}$$

A straightforward analysis is sufficient to show that when $(3 + \hat{\eta}_1 - 3\hat{\xi}_1) < (\hat{\eta}_3 - 3\hat{\xi}_3)$ then transverse rolls ($\phi = \frac{1}{2}\pi$) are preferred, but longitudinal rolls are preferred when the inequality is in the opposite direction. All orientations are equally likely when equality pertains.

Finally, when $(\eta_1/\xi_1) = (\eta_3/\xi_3)$, the value of R_0 is independent of ϕ. A detailed numerical investigation of the value of R_2 in such circumstances shows that $\phi = 0$ always yields the most unstable roll direction.

Conclusion

We have undertaken an analytical study of how the critical Rayleigh number changes when an anisotropic porous layer is inclined at a small angle to the horizontal. In general the most unstable mode is of either longitudinal or transverse form, depending on the relative values of (η_1/ξ_1) and (η_3/ξ_3). When the layer is nearly isotropic a simple analytical condition has been presented which determines which of the two roll directions is preferred. When $(\eta_1/\xi_1) = (\eta_3/\xi_3)$, which corresponds to the case when there is no overall preferred roll direction for a horizontal layer, we have found that the $O(\alpha^2)$ correction favours longitudinal modes.

The present work is currently being extended by Rees & Postelnicu [9] who are considering the effects of large inclinations from the horizontal. Unlike the cases described here, preliminary results indicate that it is possible for oblique modes to become the preferred mode of instability.

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Received February 20, 2001