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HOSSAIN, M. A., HUSSAIN, S., REES, D. A. S.

Influence of Fluctuating Surface Temperature and Concentration on Natural Convection Flow from a Vertical Flat Plate

A linearized theory is used to investigate how a free double-diffusive boundary layer flow is affected by small amplitude temporal variations in the surface temperature and species concentration. The mean temperature and the mean species concentration are assumed to vary as a power n of the distance measured from the leading edge. Three distinct methods, namely, a perturbation method for low frequencies, an asymptotic series expansion for high frequencies, and a finite difference method for intermediate frequencies, are used. Calculations have been carried out for a wide range of parameters in order to examine the results obtained from the three methods. Comparisons are made in terms of the amplitudes and phases of the surface heat transfer and surface mass transfer. It has been found that the amplitudes and phase angles predicted by perturbation theory and the asymptotic method are in good agreement with the finite difference computations.

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1. Introduction

The study of laminar boundary layers in the presence of an oscillatory flow with a steady mean component was first undertaken by LIGHTHILL [1]. He considered the effects of small fluctuations in the free stream velocity on the skin-friction and heat transfer for plates and cylinders by employing the Karman-Pohlhausen approximate integral method. NANDA and SHARMA [2] and ESHGHY et al. [3] later extended Lighthill's theory for free convective flows. MUHURI and MAITI [4] and SINGH et al. [5] investigated the free-convection flow and heat transfer along a semi-infinite horizontal plate with a small amplitude surface temperature oscillation about a non-zero mean by using the approximate integral method. On the other hand, KELLEHER and YANG [6] investigated the heat transfer response of laminar free convection boundary layers along vertical heated plates to surface-temperature oscillations. Solutions of the governing nonsimilar equations for the fluctuating flow and heat transfer were obtained for both small and large frequencies using suitable perturbation techniques. With an oscillating surface heat flux, the problem of natural convective flow from a vertical surface has been investigated by HOSSAIN et al. [7]. The additional effects of a transverse magnetic field on the problem posed by KELLEHER and YANG, has more recently been investigated by HOSSAIN et al. [8]. In all the above studies solutions were obtained for both small and large frequencies by using suitable perturbation techniques. For intermediate frequencies the equations governing the fluctuating flow and heat transfer were obtained by using both an implicit finite difference method and the local non-similarity method of MINKOWYCZ and SPARROW [9]. In these investigations involving small-amplitude oscillations, it has been found that in the low-frequency range, the oscillating component of the skin-friction always lags behind the plate temperature oscillations while the rate of heat transfer has a phase lead. In the high-frequency range, the velocity and temperature in the boundary layer are of 'shear-wave' type, predicting a phase lead of 45° in the rate of heat transfer fluctuation and an equivalent phase lag in the skin-friction fluctuation.

There are many transport processes in industry and in the environment in which buoyancy forces arise for both thermal and mass diffusion as a result of the co-existence of thermal and concentration gradients. In free convection these may either hinder or aid one another. SOMERS [10], WILCOX [11], and GILL et al. [12] studied the effects of mass transfer on free convection. Similarity analysis of the natural convection flows adjacent to both vertical and horizontal surfaces which result from the combined buoyancy effects of thermal and mass diffusion was first investigated by GEBHART and PERA [13] and PERA and GEBHART [14]. The combined effect of buoyancy forces from thermal and mass diffusion on forced convection was studied by CHEN et al. [15]. In this latter paper attention is directed to forced convection along vertical and inclined plates for which the plate is either maintained at a uniform temperature and concentration or subjected to a uniform surface heat-flux and mass-flux. Both local nonsimilarity and finite difference methods for the fluid having Prandtl numbers 0.7 and 7.0 gave solutions of the transformed conservation equations. HOSSAIN [16] also investigated the effect of transpiration along with the combined effect of buoyancy forces from thermal and mass diffusion on forced convective heat and mass transfer from a vertical plate. But all the above studies are confined to steady flows. Very much less attention has been given to the effects of combined heat and mass transfer on unsteady flow. Among those few investigators HOSSAIN and BEGUM [17] studied the combined effect of heat and mass transfer to unsteady convection flow from a doubly infinite vertical permeable surface with small-amplitude fluctuations in the free stream, the surface temperature, and the surface species concentration. Recently, HOSSAIN [18] investigated the effect of a fluctuating surface temperature on the unsteady two dimensional natural convection flow with uniform species concentration at the surface by the use of the Karman-Pohlhausen approximate integral method.

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In the present paper we investigate the effect of the buoyancy force arising from both thermal and mass diffusion in the unsteady natural convection flow from a vertical plate. The surface is subjected to small-amplitude temporal oscillations in both its temperature and species concentration with non-zero means. In this study we also neglect cross-diffusional effects as modelled by the Soret and Dufour terms. In the present analysis, the linearised theory used by Kelleher and Yang [6] and Hossain et al. [7, 8] has been adopted. The implicit finite difference method used by Hossain et al. [7, 8] has also been employed in obtaining the solutions for intermediate frequencies. Numerical results are expressed in terms of amplitude and phase of skin-friction, the rate of heat transfer and the rate of mass transfer at the surface and these are obtained for a wide range of the frequency of oscillation. It is found that in the low-frequency range, the oscillating component of the skin-friction always lags behind the oscillations in surface temperature and surface concentration, while the rate of heat transfer and the rate of mass transfer have phase lead in presence of foreign species.

2. Formulation of the problem

A two-dimensional unsteady free convective flow of a viscous incompressible fluid along a vertical flat in presence of a soluble species is considered. We assume that both the surface temperature and the surface species concentration exhibit small amplitude oscillations in time about a steady non-zero mean temperature and concentration, respectively. The coordinate system and the flow configuration are shown in Fig. 1.

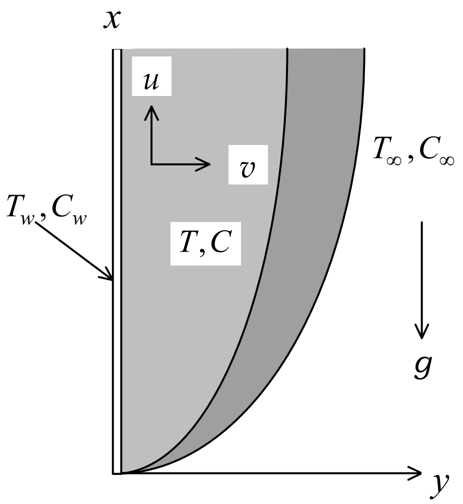

Fig. 1. Flow configuration and the coordinate system

Given the validity of the Boussinesq approximation, the flow, the heat transfer, and the mass transfer processes are governed by the following equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = \nu\,\frac{\partial^2 u}{\partial y^2} + g(\beta_T \theta + \beta_C \phi)\,, \tag{2}$$

$$\frac{\partial \theta}{\partial t} + u\,\frac{\partial \theta}{\partial x} + v\,\frac{\partial \theta}{\partial y} = \alpha\,\frac{\partial^2 \theta}{\partial y^2}\,, \tag{3}$$

$$\frac{\partial \phi}{\partial t} + u\,\frac{\partial \phi}{\partial x} + v\,\frac{\partial \phi}{\partial y} = D\,\frac{\partial^2 \phi}{\partial y^2}\,, \tag{4}$$

where u and v are the x- and y-components of the velocity field, respectively, g is the acceleration due to gravity, β_T and β_C are the volumetric expansion coefficients for temperature and concentration, respectively, α is the thermal diffusivity, and D is the molecular diffusivity of the species. Further, $\theta = T - T_\infty$ where T and T_∞ are the temperature of the fluid and the ambient temperature, respectively, and $\phi = C - C_\infty$ where C and C_∞ are the species concentration and the ambient concentration.

As suggested by Gebhart and Pera [13], the cross diffusion effects (i.e. Soret and Dufour effects) are assumed to be negligible compared with the direct effects modelled by Fourier's law and Fick's law; these are often neglected in heat and mass transfer processes.

The boundary conditions to be satisfied by eqs. (1)–(4) are

$$\left.\begin{aligned} &u(x,0,t) = v(x,0,t) = 0\,, \\ &\theta(x,0,t) = \theta_x(x)\,\{1+\varepsilon\exp(i\omega t)\}\,, \\ &\phi(x,0,t) = \phi_x(x)\,\{1+\varepsilon\exp(i\omega t)\}\,, \end{aligned}\right\} \tag{5}$$
$$u(x,\infty,t) = 0\,, \qquad \theta(x,\infty,t) = 0\,, \qquad \phi(x,\infty,t) = 0$$

where ε is real and much less than unity so that amplitude of oscillation is small. Here ω is the frequency of oscillation and $\theta_w(x)$ and $\phi_w(x)$, are, respectively, the mean temperature and mean concentration as given by GEBHART and PERA [13]:

$$\theta_x = \theta_0 x^n \quad \text{and} \quad \phi_x = \phi_0 x^n\,. \tag{6}$$

θ_0 and ϕ_0 are, respectively, the scaled form of the temperature and species concentration at the surface of the plate. The exponent n in (6) can be expressed as

$$n = \frac{d\ln\theta_w(x)}{d\ln x} = \frac{d\ln\phi_w(x)}{d\ln x} \tag{7}$$

and may be considered as the temperature gradient as well as the concentration gradient at the surface of the plate.

3. Transformation of the equations

The surface temperature and the concentration given above in boundary conditions (5) suggest the following group of transformations:

$$\begin{aligned} &\psi(x,y,t) = \nu\,\mathrm{Gr}_x^{1/4}[f(\eta) + \varepsilon\exp(i\omega t)\,F(\xi,\eta)]\,, \\ &\theta(x,y) = \theta_w(x)\,[g(\eta) + \varepsilon\exp(i\omega t)\,G(\xi,\eta)]\,, \\ &\phi(x,y) = \phi_w(x)\,[h(\eta) + \varepsilon\exp(i\omega t)\,H(\xi,\eta)]\,, \\ &\eta = \frac{y}{x}\,\mathrm{Gr}_x^{1/4}\,, \qquad \xi = \frac{\omega}{\nu}\,\mathrm{Gr}_x^{-1/4}x \end{aligned} \tag{8}$$

where ψ is the stream function, defined by $u = \dfrac{\partial\psi}{\partial y}$ and $v = -\dfrac{\partial\psi}{\partial x}$ which satisfies the continuity equation (1). Here f, g, h and F, G, H are, respectively, the steady parts and the unsteady parts of the stream-function, the temperature, and concentration functions. η is the pseudo-similarity variable and ξ is the frequency parameter that measures the frequency of the surface oscillations; ξ may also be interpreted as being a scaled streamwise variable. Therefore large values of ξ correspond either to large values of x at fixed frequencies or to large frequencies at fixed values of x. We define

$$\mathrm{Gr}_x = \frac{g[\beta_T\theta_w + \beta_C\phi_x]}{\nu^2}\,x^3 = \mathrm{Gr}_{x,T} + \mathrm{Gr}_{x,C} \tag{9}$$

as the modified local Grashof number, with $\mathrm{Gr}_{x,T}$ being the local Grashof number for thermal diffusion and $\mathrm{Gr}_{x,C}$ the local Grashof number for mass diffusion.

Now, substituting the above group of transformations given in (8) into (2)–(4) and taking the terms upto $O(\varepsilon)$, one first obtains the following similarity equations for the steady flow:

$$f''' + \frac{n+3}{4}\,ff'' - \frac{n+1}{2}\,f'^2 + (1-w)\,g + wh = 0\,, \tag{10}$$

$$\frac{1}{\mathrm{Pr}}\,g'' + \frac{n+3}{4}\,fg' - nf'g = 0\,, \tag{11}$$

$$\frac{1}{\mathrm{Sc}}\,h'' + \frac{n+3}{4}\,fh' - nf'h = 0\,, \tag{12}$$

$$f(0) = f'(0) = 0\,, \qquad g(0) = 1\,, \qquad h(0) = 1\,, \qquad f'(\infty) = h(\infty) = g(\infty) = h(\infty) = 0\,. \tag{13}$$

In these equations, primes denote differentiation of the functions with respect to η, as is traditional in boundary layer theory.

In eqs. (10) and (14), $w = N/(1+N)$, where $N = \mathrm{Gr}_{x,C}/\mathrm{Gr}_{x,T}$ and w measures the relative importance of solutal and thermal diffusion in causing the density changes which drive the flow. It is to be noted that $N = 0$ corresponds to no species diffusion and infinity to no thermal diffusion. Positive values of N correspond to both effects combining to drive the flow, whereas negative values correspond to opposing effects from the two diffusing components. We further see that when $N = 0$, $w = 0$ and as $N \to \infty$, $w \to 1$.

Solutions of the eqs. (10)–(12) are already available in the open literature for wide ranges of values of the governing parameters, Sc, the Schmidt number and n, the temperature-concentration exponent for values of the Prandtl number $\mathrm{Pr} = 7.0$ and 0.7 which represent, respectively, the water and air at 20 °C at 1 atmosphere.

The equations for the fluctuating flow are

$$F''' + \frac{n+3}{4}\,(fF'' + f''F) - (n+1)\,f'F' + (1-w)\,G + wH - i\xi F' = \frac{(1-n)}{2}\,\xi\left(f'\,\frac{\partial F'}{\partial \xi} - f''\,\frac{\partial F}{\partial \xi}\right), \tag{14}$$

$$\frac{1}{\mathrm{Pr}}\,G'' + \frac{n+3}{4}\,(fG' + g'F) - n(f'G + gF') - i\xi G = \frac{(1-n)}{2}\,\xi\left(f'\,\frac{\partial G}{\partial \xi} - g'\,\frac{\partial F}{\partial \xi}\right), \tag{15}$$

$$\frac{1}{\mathrm{Sc}}\,H'' + \frac{n+3}{4}\,(fH' + h'F) - n(f'H + fF') - i\xi H = \frac{(1-n)}{2}\,\xi\left(f'\,\frac{\partial H}{\partial \xi} - h'\,\frac{\partial F}{\partial \xi}\right). \tag{16}$$

The boundary conditions for the above equations are

$$\begin{aligned} &F(\xi,0) = F'(\xi,0) = 0\,, \qquad G(\xi,0) = 1\,, \qquad H(\xi,0) = 1\,,\\ &F'(\xi,\infty) = H(\xi,\infty) = G(\xi,\infty) = G(\xi,\infty) = 0\,. \end{aligned} \tag{17}$$

Here we again follow the usual practice of denoting derivatives with respect to the similarity variable by primes, even though F, G, and H are functions of two variables. At $\xi = 0$ the system (14)–(17) reduces to ordinary differential form, and therefore the solutions of these equations necessarily form the initial condition.

Once the solutions of eqs. (14)–(17) are known, one quickly obtains the values of the shear-stress, the surface heat-flux and the surface mass-flux, all of which are important from the experimental point of view. Here we propose to present the available solutions in terms of amplitudes and phases of the surface shear stress and rate of heat transfer according to the following relations:

$$A_\tau = \sqrt{\tau_r^2 + \tau_i^2}\,, \qquad A_q = \sqrt{q_r^2 + q_i^2}\,, \qquad A_m = \sqrt{m_r^2 + m_i^2} \tag{18}$$

and

$$\varphi_\tau = \tan^{-1}(\tau_i/\tau_r)\,, \qquad \varphi_q = \tan^{-1}(q_i/q_r)\,, \qquad \varphi_m = \tan^{-1}(m_i/m_r) \tag{19}$$

where A_τ, A_T, A_C and Φ_τ, Φ_T, Φ_C are the respective amplitude and phase of the shear-stress, the surface heat-flux, and the surface mass-flux; (τ_r, τ_i), (q_r, q_i), and (m_r, m_i) are the corresponding real and imaginary parts of $F''(\xi, 0)$, $G''(\xi, 0)$, and $H''(\xi, 0)$.

In the following section, we describe the integration of the eqs. (14) to (17) with the boundary conditions (18) using the above-mentioned three distinct methods.

4. Methods of solution

Solutions of the set of similarity equations (10)–(12) for the steady flow are obtained by the Newton-Raphson iteration technique together with the sixth order implicit Runge-Kutta-Butcher initial value problem solver for different values of the governing parameters. From the present situation one can find the equations developed by Gebhart and Pera [13] by simply converting w to N. Solutions obtained by the aforementioned authors for the skin-friction, the Nusselt number and Sherwood number may be reproduced by multiplying the present values of those quantities by the factor $\sqrt{2}$. To avoid repetition of the results presented by earlier authors, we have omitted discussion of the solutions of the steady-state flow. However, we will obtain the solutions of the eqs. (14)–(17) which govern the small-amplitude fluctuating part of unsteady flow by employing the perturbation method for both low and high frequencies.

Solutions at intermediate frequencies are obtained using an implicit finite difference method. It should be noted here that when $n = 1$ eqs. (14)–(17) reduce to local similarity form in which ξ can be treated as a parameter. Solutions of these equations may be obtained easily either by the linear shooting method or by the method of superposition, since these are linear ordinary differential equations.

Table 1: Numerical values of quasi-steady heat flux and mass-flux coefficients for different Sc and w for $\mathrm{Pr} = 0.7$ and $n = 0.5$

w	0.0		0.5		1.0	
Sc	$-g_0'(0)$	$-h_0'(0)$	$-g_0'(0)$	$-h_0'(0)$	$-g_0'(0)$	$-h_0'(0)$
1.76	0.57462	0.86448	0.54435	0.82774	0.50162	0.78065
0.94	0.57462	0.65883	0.56409	0.64795	0.55214	0.63568
0.60	0.57461	0.53335	0.58037	0.53907	0.58576	0.54442
0.22	0.57492	0.31691	0.61619	0.35159	0.64791	0.37524

Table 2: Comparison of the solutions obtained by perturbation method and finite difference method in terms of the amplitude and phase (degrees) of the local heat-flux number and the local mass flux against the local frequency parameter ξ for different Sc while Pr = 0.7, $w = 0.5$, and $n = 0.5$

ξ	Heat-flux				Mass-flux			
	Amplitude (A_T)		Phase (Φ_T)		Amplitude (A_C)		Phase (Φ_C)	
	Perturbation	Finite Diff.	Perturbation	Finite Diff.	Perturbation	Finite Diff.	Perturbation	Finite Diff.
	Sc = 0.94							
0.0	0.5640	0.5641	0.0000	0.0000	0.6480	0.6481	0.0000	0.0000
0.2	0.5757	0.5763	9.2678	9.4097	0.6604	0.6610	9.2313	9.3660
0.4	0.6081	0.6101	17.5368	17.7448	0.6960	0.6984	17.6510	17.8531
0.6	0.6569	0.6599	24.2586	24.5915	0.7501	0.7559	24.9432	24.9056
0.8	0.7208	0.7214	28.5085	29.9354	0.8189	0.8283	32.1972	30.3828
1.0	0.8002	0.7907	27.2244	33.8894	0.9238	0.9103	43.0117	34.3845
2.0		1.1372		41.5582		1.3172		41.8794
3.0	1.4159	1.4157	43.6382	43.2498	1.6460	1.6413	43.8286	43.4448
4.0	1.6483	1.6475	44.1227	43.8914	1.9140	1.9102	44.2445	44.0239
5.0	1.8507	1.8497	44.3750	44.2149	2.1479	2.1447	44.4614	44.3121
6.0	2.0326	2.0316	44.5258	44.4054	2.3581	2.3554	44.5912	44.4806
7.0	2.1992	2.1982	44.6243	44.5290	2.5508	2.5484	44.6761	44.5893
8.0	2.3538	2.3528	44.6929	44.6147	2.7297	2.7276	44.7351	44.6645
9.0	2.4988	2.4978	44.7428	44.6771	2.8974	2.8955	44.7782	44.7191
10.0	2.6357	2.6347	44.7806	44.7242	3.0559	3.0541	44.8107	44.7602
	Sc = 0.60							
0.0	0.5805	0.5806	0.0000	0.0000	0.5391	0.5391	0.0000	0.0000
0.2	0.5904	0.5909	8.6410	8.7709	0.5488	0.5493	8.7077	8.8428
0.4	0.6187	0.6205	16.5824	16.7862	0.5762	0.5779	16.5937	16.8010
0.6	0.6619	0.6662	23.4895	23.5689	0.6177	0.6207	23.1899	23.4698
0.8	0.7165	0.7242	29.9406	28.9847	0.6716	0.6741	27.9493	28.7983
1.0	0.7897	0.7906	38.1269	33.0859	0.7365	0.7350	29.1706	32.8558
2.0		1.1338		41.3247		1.0503		41.1463
3.0	1.4147	1.4132	43.5895	43.1534	1.3073	1.3082	43.4735	43.0407
4.0	1.6474	1.6456	44.0916	43.8379	1.5233	1.5231	44.0175	43.7601
5.0	1.8501	1.8483	44.3529	44.1800	1.7113	1.7107	44.3004	44.1224
6.0	2.0321	2.0304	44.5090	44.3803	1.8800	1.8793	44.4693	44.3356
7.0	2.1987	2.1972	44.6111	44.5098	2.0345	2.0337	44.5797	44.4738
8.0	2.3534	2.3520	44.6821	44.5994	2.1779	2.1770	44.6564	44.5696
9.0	2.4984	2.4970	44.7338	44.6645	2.3122	2.3114	44.7123	44.6393
10.0	2.6353	2.6340	44.7729	44.7136	2.4391	2.4382	44.7546	44.6920

4.1 Perturbation solutions for small frequency

The effect of free convection on the flow near the leading edge or, equivalently, for small frequencies of oscillation, may be found by expanding the functions F, G, and H in powers of ξ as given below:

$$F(\xi,\eta) = \sum_{k=0}^{\infty} \xi^k F_k(\eta)\,, \qquad G(\xi,\eta) = \sum_{k=0}^{\infty} \xi^k G_k(\eta)\,, \qquad H(\xi,\eta) = \sum_{k=0}^{\infty} \xi^k H_k(\eta)\,. \tag{20}$$

On substituting these into eqs. (14)–(16) and equating the terms of like powers of ξ to zero the following sets of equations are obtained:

$$F_0''' + \frac{n+3}{4}\,(fF_0'' + f''F_0) - (n+1)\,f'F_0' + (1-w)\,G_0 + wH_0 = 0\,, \tag{21}$$

$$\frac{1}{\mathrm{Pr}}\,G_0'' + \frac{n+3}{4}\,(fG_0' + g'F_0) - n(f'G_0 + gF_0') = 0\,, \tag{22}$$

$$\frac{1}{\mathrm{Sc}}\,H_0'' + \frac{n+3}{4}\,(fH_0' + h'F_0) - n(f'H_0 + fF_0') = 0\,, \tag{23}$$

$$F_0(0) = F_0'(0) = 0\,, \qquad G_0(0) = 1\,, \qquad H_0(0) = 1\,, \qquad F_0'(\infty) = G_0(\infty) = H_0(\infty) = 0 \tag{24}$$

and

$$F_k''' + \frac{n+3}{4}\,fF_k'' + \left\{\frac{n+3}{4} + \frac{1}{2}\,(n-1)\,k\right\} f''F_k - \left[(n+1) + \frac{1}{2}\,(1-n)\,k\right] f'F_k' + (1-w)\,G_k + wH_k = F_{k-1}\,, \tag{25}$$

$$\frac{1}{\mathrm{Pr}}\, G_k'' + \frac{n+3}{4}\, f G_k' + \left\{\frac{n+3}{4} + \frac{1}{2}\,(n-1)\,k\right\} g' F_k - \left[n + \frac{1}{2}\,(n-1)\,k\right] f' G_k + n g F_k' = G_{k-1}\,, \tag{26}$$

$$\frac{1}{\mathrm{Sc}}\, H_k'' + \frac{n+3}{4}\, f H_k' + \left\{\frac{n+3}{4} + \frac{1}{2}\,(n-1)\,k\right\} h' F_k - \left[n + \frac{1}{2}\,(n-1)\,k\right] f' H_k + n f F_k' = H_{k-1}\,, \tag{27}$$

$$F_k(0) = F_k'(0) = 0\,, \qquad G_k(0) = H_k(0) = 0\,, \qquad F_k'(\infty) = G_k(\infty) = H_k(\infty) = 0 \quad \text{for} \quad k = 1, 2, 3, \ldots\,. \tag{28}$$

It may be seen that eqs. (21)–(24) have exact integrals which, according to Kelleher and Yang [6], may be written in the form

$$F_0 = \tfrac{1}{4}(f + \eta f')\,, \qquad G_0 = g + \tfrac{1}{4}\eta g'\,, \qquad H_0 = h + \tfrac{1}{4}\eta h'\,. \tag{29}$$

Physically, when the frequency of oscillation is small, the boundary layer behavior should be predicted well by the steady-state theory based on instantaneous temperature and concentration differences. It may also be seen that the zero-th order solutions, as presented by eqs. (29), represent the quasi-steady solutions. As the frequency increases, deviations from these solutions occur, and hence more terms in the series (20) must be taken into account. The exact solutions obtained from the eqs. (29) are entered in Table 1 for different values of the Schmidt number and the exponent n while $\mathrm{Pr} = 0.7$ and $w = 0.5$. From this it can be seen that the presence of a variable species concentration causes the quasi-steady heat-flux coefficient to decrease and the mass-flux coefficient to increase given decreasing values of the Schmidt number. On the other hand increases in the combined buoyancy effects lead to decreases in both the quasi-steady heat-flux and the mass-flux.

It can be seen that eqs. (21)–(27) for $k = 1, 2, 3, \ldots$ are linear, but coupled, and may be solved independently setwise one after another. In the present analysis, the implicit Runge-Kutta-Butcher initial value solver together with Nachtsheim-Swigert iteration scheme is employed to solve the system of equations up to $O(\xi^{10})$. Computed results are shown in Table 2 and Figs. 2–5 in order to be compared with the finite difference solutions. We note that these small-ξ solutions are asymptotic solutions and will be valid and accurate for sufficiently small values of ξ; as with all boundary layer analyses of this type we have no guarantee that the series converge.

4.2 Asymptotic solutions for large frequency

In this section attention is given to the solutions of eqs. (14) and (17) when ξ is large. Actually when the frequency of oscillation of the surface temperature and surface mass concentration is very high, the boundary layer response should be confined to a thin region adjacent to the surface. Thus as the frequency approaches infinity the solutions tend to be independent of the distance from the leading edge to downstream, similar to the shear wave solution in the corresponding forced flow problem. Here we again seek a series solution in the high frequency range, utilizing the limiting solution as the zeroth-order approximation. For this region following transformations are introduced:

$$Y = \xi^{1/2}\eta\,, \qquad M(\xi, Y) = \xi^{3/2} F(\xi, \eta)\,, \qquad G(\xi, Y) = G(\xi, \eta)\,, \qquad H(\xi, Y) = H(\xi, \eta)\,. \tag{30}$$

Eqs. (14) and (17) then become

$$\begin{aligned} &M''' + \frac{n+3}{4}\,\xi^{-1/2} f M'' + \frac{3n+1}{4}\,\xi^{-3/2} f' - \frac{n+3}{4}\,\xi^{-1} f' M' - iM' + (1-w)\,\theta + w\phi \\ &\quad = \frac{(1-n)}{2}\left[f'\left(M_{\xi 1}' + \frac{1}{2}\,Y\xi^{-1} M''\right) - \xi^{-1/2} f''\left(M_\xi + \frac{1}{2}\,Y\xi^{-1} M'\right)\right], \end{aligned} \tag{31}$$

$$\frac{1}{\mathrm{Pr}}\, G'' + \frac{n+3}{4}\,\xi^{-1/2} f G' + n\xi^{-5/2}\theta M' - iG = \frac{(1-n)}{2}\left[f'\left(G_\xi + \frac{1}{2}\,Y\xi^{-1} G'\right) - \xi^{-3/2} g'\left(M_\xi + \frac{1}{2}\,Y\xi^{-1} M'\right)\right], \tag{32}$$

$$\frac{1}{\mathrm{Sc}}\, H'' + \frac{n+3}{4}\,\xi^{-1/2} f H' + n\xi^{-5/2} h M' - iH = \frac{(1-n)}{2}\left[f'(H_\xi + \frac{1}{2}\,Y\xi^{-1} H') - \xi^{-3/2} h'\left(M_\xi + \frac{1}{2}\,Y\xi^{-1} M'\right)\right]. \tag{33}$$

The high frequencies affect only the region immediately next to the surface. Consequently the functions f, θ, and ϕ in this region can be represented well by the following power series:

$$\begin{aligned} f &= a_2\eta^2 + a_3\eta^3 + a_4\eta^4 + \cdots\,, \\ \theta &= 1 + b_1\eta + b_2\eta^2 + b_3\eta^3 + \cdots\,, \\ \phi &= 1 + c_1\eta + c_2\eta^2 + c_3\eta^3 + \cdots \end{aligned} \tag{34}$$

where according to eqs. (10)–(13) we get

$$\begin{aligned} &a_2 = \tfrac{1}{2} f''(0)\,, \qquad a_3 = -\tfrac{1}{6}\,, \qquad a_4 = -\tfrac{1}{24}\{(1-w)\,g'(0) + wh'(0)\}\,,\\ &b_1 = g'(0), \ldots\,, \qquad c_1 = h'(0), \ldots\,. \end{aligned} \tag{35}$$

Based on the above expansions, the solutions to eqs. (40)–(42) may be obtained in the following forms:

$$M(\xi, Y) = \sum_{m=0}^{\infty} \xi^{-m/2} E_m(Y)\,, \qquad G(\xi, Y) = \sum_{m=0}^{\infty} \xi^{-m/2} L_m(Y)\,, \qquad H(\xi, Y) = \sum_{m=0}^{\infty} \xi^{-m/2} S_m(Y)\,. \tag{36}$$

When eqs. (36) are substituted into (31)–(33) and terms of like powers of ξ are collected, one obtains

$$E_0''' - iE_0' = -(1-w)\,L_0 - wS_0\,, \tag{37}$$

$$E_1''' - iE_1' = -(1-w)\,L_1 - wS_1\,, \tag{38}$$

$$E_2''' - iE_2' = -(1-w)\,L_2 - wS_2\,, \tag{39}$$

$$E_3''' - iE_3' = -(1-w)\,L_3 - wS_3 - a_2(((3n+1)/4)\,Y^2 E_0'' + ((3n+5)/2)\,YE_0' - ((3n+1)/2)\,E_0)\,, \tag{40}$$

$$E_k(0) = E_k'(0) = 0\,, \qquad E_k'(\infty) = 0\,, \qquad k = 0,1,2,3\,, \tag{41}$$

$$(1/\mathrm{Pr})\,L_0'' - iL_0 = 0\,, \tag{42}$$

$$(1/\mathrm{Pr})\,L_1'' - iL_1 = 0\,, \tag{43}$$

$$(1/\mathrm{Pr})\,L_2'' - iL_2 = 0\,, \tag{44}$$

$$(1/\mathrm{Pr})\,L_3'' - iL_3 = -a_2(((3n+1)/4)\,Y^2 L_0' + 2nYL_0)\,, \tag{45}$$

$$L_0(0) = 1\,, \qquad L_k(0) = 0\,, \qquad L_k(\infty) = 0\,, \qquad k = 1,2,3\,, \tag{46}$$

$$(1/\mathrm{Sc})\,S_0'' - iS_0 = 0\,, \tag{47}$$

$$(1/\mathrm{Sc})\,S_1 - iS_1 = 0\,, \tag{48}$$

$$(1/\mathrm{Sc})\,S_2'' - iS_2 = 0\,, \tag{49}$$

$$(1/\mathrm{Sc})\,S_3'' - iS_3 = -a_2(((3n+1)/4)\,Y^2 S_0' + 2nYS_0)\,, \tag{50}$$

$$S_0(0) = 1\,, \qquad S_k(0) = 0\,, \qquad S_k(\infty) = 0\,, \qquad k = 1,2,3\,. \tag{51}$$

Solutions of the above equations yield the following relations:

$$F''(0) = E_0''(0)\,\xi^{-1/2} + E_3''(0)\,\xi^{-3/2} + O(\xi^{-5/2}) \tag{52}$$

where

$$\begin{aligned} E_0''(0) &= \frac{1}{\sqrt{i}}\left\{\frac{1-w}{1+\sqrt{\mathrm{Pr}}} + \frac{w}{1+\sqrt{\mathrm{Sc}}}\right\},\\ E_3''(0) &= \frac{(1-w)\,i}{(\mathrm{Pr}-1)}\left\{\frac{3a_2(5n-1)}{16\sqrt{i\,\mathrm{Pr}}\,(\mathrm{Pr}-1)}\,\frac{(1+3\sqrt{\mathrm{Pr}})}{\sqrt{\mathrm{Pr}}-1} - \frac{a_2(5n-1)\,i}{4}\left(1+\frac{2\sqrt{\mathrm{Pr}}}{\mathrm{Pr}-1}\right)\left(\frac{1}{4i} - \frac{\mathrm{Pr}}{\mathrm{Pr}-1}\right)\right\}\\ &\quad + \frac{wi}{(\mathrm{Sc}-1)}\left\{\frac{3a_2(5n-1)}{16\sqrt{i\,\mathrm{Sc}}\,(\mathrm{Sc}-1)}\,\frac{(1+3\sqrt{\mathrm{Sc}})}{\sqrt{\mathrm{Sc}}-1} - \frac{a_2(5n-1)\,i}{4}\left(1+\frac{2\sqrt{\mathrm{Sc}}}{\mathrm{Sc}-1}\right)\left(\frac{1}{4i} - \frac{\mathrm{Sc}}{\mathrm{Sc}-1}\right)\right\}\\ &\quad + a_2\,\frac{3n+1}{2}\left[\left\{\frac{1-w}{\mathrm{Pr}-1}\left(\frac{1}{8} + \frac{\sqrt{\mathrm{Pr}}\,(3\sqrt{\mathrm{Pr}}-1)}{(\mathrm{Pr}-1)^2}\right) + \frac{w}{\mathrm{Sc}-1}\left(\frac{1}{8} + \frac{\sqrt{\mathrm{Sc}}\,(3\sqrt{\mathrm{Sc}}-1)}{(\mathrm{Sc}-1)^2}\right)\right\}\right.\\ &\qquad\qquad \left. + \frac{1-w}{2(\sqrt{\mathrm{Pr}}+1)^2} + \frac{w}{2\,(\sqrt{\mathrm{Sc}}+1)^2}\right]\\ &\quad + a_2\,\frac{3n+5}{2}\left\{\frac{1-w}{\mathrm{Pr}-1}\left(\frac{1}{(\sqrt{\mathrm{Pr}}+1)^2} - \frac{1}{4}\right) + \frac{w}{\mathrm{Sc}-1}\left(\frac{1}{(\sqrt{\mathrm{Sc}}+1)^2} - \frac{1}{4}\right)\right\}, \end{aligned}$$

$$G'(\xi, 0) = -\sqrt{i\,\mathrm{Pr}} - \frac{ia_2}{16}\,(1+5n)\,\xi^{-1} + O(\xi^{-3/2})\,, \tag{53}$$

$$H'(\xi, 0) = -\sqrt{i\,\mathrm{Sc}} - \frac{ia_2}{16}\,(1+5n)\,\xi^{-1} + O(\xi^{-3/2})\,. \tag{54}$$

The above solutions are valid for $\mathrm{Pr} \neq 1$ and $\mathrm{Sc} \neq 1$, although the limits as either parameter tends to unity are well-behaved. But since most fluids have values of Pr and Sc which are different from unity, solutions for $\mathrm{Pr} = 1$ and $\mathrm{Sc} = 1$

are not presented here. Again, these solutions for large values of ξ are asymptotic solutions and it is unlikely that such a series continued indefinitely and will yield a convergent series. Indeed it is well-known that such large-ξ series are hampered by the appearance of eigensolutions with indeterminate amplitude, and this limits the usefulness of such series.

4.3 Solutions for intermediate frequencies

For intermediate frequencies we integrate the locally nonsimilar partial differential equations (14)−(17) subject to the boundary conditions (18) by using an implicit finite difference method. To begin with, the partial differential equations (14)−(17) are first converted into a system of first order equations in η. The resulting equations are expressed in finite difference form by approximating the functions and their derivatives in terms of central differences. On denoting the mesh points in the (ξ, η)-plane by ξ_i and η_j, where $i = 0, 1, \dots, M$ and $j = 1, 2, \dots, N$, central difference approximations are made, such that those equations involving ξ explicitly are centered at ($\xi_{i-1/2}$, $\eta_{j-1/2}$) and the remainder are approximated at (ξ_i, $\eta_{j-1/2}$), where $\eta_{j-1/2} = (\eta_j + \eta_{j-1})$. This procedure results in a set of nonlinear difference equations for the unknowns at ξ_i in terms of their values at ξ_{i-1}. To solve resulting equations, Newton's iteration technique, known in this context as the Keller box method, is used. Recently this method was discussed in more detail and used efficiently by Hossain et al. (1998) in studying the effect of oscillating surface temperature on the natural convection flow from a vertical flat plate. To initiate the process at $\xi = 0$, we first prescribe the profiles for the functions F, F', F'', G, G', H, and H', which are obtained from the solutions of the eqs. (21)−(23). These profiles are then employed in the Keller box scheme, which has second-order accuracy, to march stepwise along the boundary layer. For any given value of ξ, the iterative procedure is stopped to obtain the final velocity and temperature distributions when the difference in computing the velocity, the temperature, and the species concentration in the latest iteration is less than 10^{-6}, i.e. when $\delta_f^i \le 10^{-6}$, where the superscript i denotes the number of iterations. Throughout the computations a nonuniform grid has been used by setting $\eta = \sinh(j/a)$. Such a grid makes efficient use of computational time and computer memory. In the computations, the maximum η_e ranged from 10.05 to 12.00 as the Schmidt number Sc ranged from 0.2 to 2.0, which represents the diffusing chemical species mostly present in air at 25 °C 1 atm for which the value of Prandtl number, Pr, equals 0.7 and for the frequency parameter ξ ranging from 0.0 to 25.0.

Finally, we note that the Keller box method as described above gives second order accurate solutions in both directions. As the method is related to the Crank Nicholson method no numerical stability problems were encountered even though the governing system is nonlinear.

5. Results and discussion

Natural convection flows driven by a combination of diffusion effects are very important in many applications. The foregoing formulations may be analysed to indicate the nature of the interaction of the various contributions to buoyancy. These may aid or oppose one another and be of different magnitudes characterised by the value of N. When the thermal and solutal effects are opposed, the value of N is negative in order to ensure that the flow is in the positive x direction. For example, Gebhart and Pera [13] used Pr = Sc for which $N = 1$ (i.e. $w = \infty$). The relative physical extent (η) of the two effects in the convection region is governed by the magnitudes of the Prandtl number and Schmidt numbers and by their relative values. For steady state other authors have discussed situations the effects of varying the parameters Pr, Sc, and N on the nature of the fluid flow and heat/mass transport at length. Here we restrict our discussion to the aiding or favorable case only, for fluids with Prandtl number Pr = 0.7 which represents air at 20 °C at 1 atmosphere. Although the diffusing chemical species of most common interest in air has Schmidt numbers in the range from 0.1 to 10.0, the present investigation considers a range from 0.2 to 2.0.

The results obtained by the aforementioned methods are expressed in terms of amplitude and phase of the rates of heat chemical species transfer. We restrict attention to air as the fluid for which the Prandtl number is 0.7. The values of the Schmidt number, Sc, are chosen to represent the presence of the various species Benzene (Sc = 1.76), carbon dioxide (Sc = 0.94), water vapor (Sc = 0.60), and hydrogen (Sc = 0.22) and that of the conjugate buoyancy parameter w equals 0.0, 0.5, and 1.0. Values of the parameter n are chosen to be 0.0, 0.5, and 1.0. It should be noted that from the present solutions, the solutions obtained by Kelleher and Yang [6] are for the case $w = 0.0$, i.e., when the flow is governed solely by natural convection and the presence of other chemical species is deemed passive. Further it should be noted that results obtained by the present method for the case of $w = 0.0$ have been compared by Hossain et al. [7] with those of Kelleher and Yang [6].

The computed values of the amplitude and phase of the local heat flux and the mass-flux at the surface obtained from the above mentioned methods at Sc = 0.94 and 0.60 with $n = 0.5$ and $w = 0.5$ against ξ in [0, 10] are entered in Table 2. The comparison shows that the low and high frequency solutions are in excellent agreement with those of the finite difference solutions. From this table it also observed that an increase in the value of the Schmidt number leads to a decrease in the value of the amplitude of the local heat flux and also leads to an increase in the amplitude of the mass-flux. We further observe that the phase angles of both the heat flux and the mass-flux increase owing to increasing values of the Schmidt number.

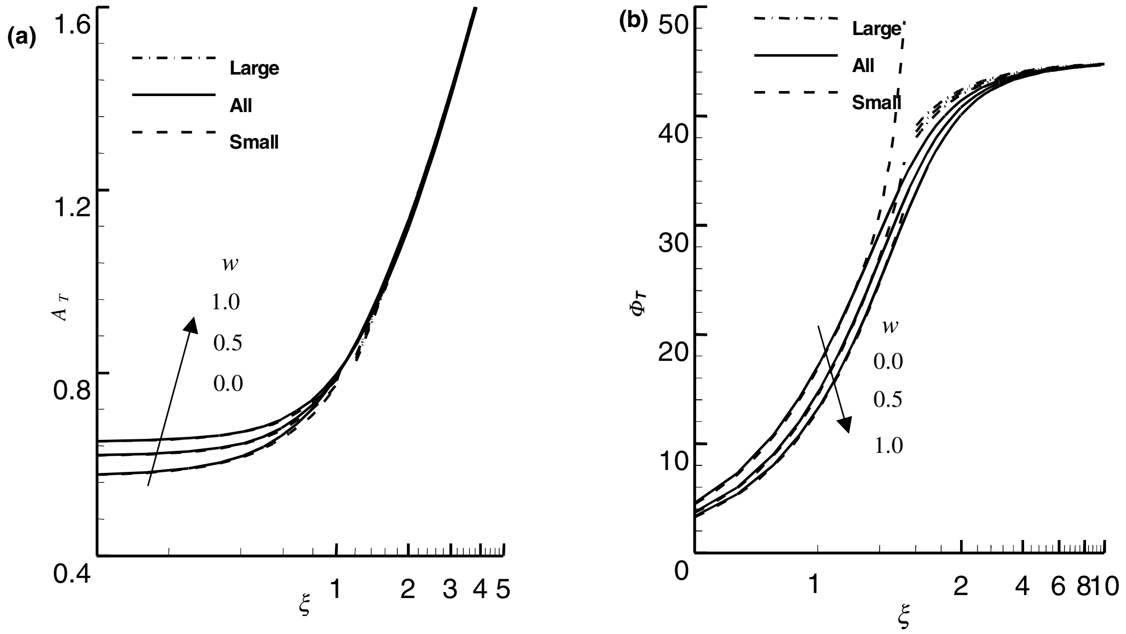

Fig. 2. a) The local amplitude and b) the local phase of heat flux against ξ for Pr = 0.7, Sc = 0.22, and $n = 0.5$ but for different values of the combined buoyancy parameter w

The effect of varying the buoyancy parameter, w, on the amplitude and phase of the rates of heat and mass transfer are depicted in Figs. 2 and 3, respectively, for the case Sc = 0.22 and $n = 0.5$. In these figures the dotted curves and the broken curves represent the solutions obtained for the low frequency and high frequency cases, respectively. From Figs. 2(a) and 3(a) it can be seen that there is an increase in the local amplitudes for both the heat and mass transfer rates due to the increase in the buoyancy parameter, w. This effect is most significant near the leading edge, i.e. in the low-frequency range. As the value of the frequency parameter increases, these values tend toward the asymptotic state analysed above.

It can be seen from Figs. 2(b) and 3(b) that the phase angle of the rate of heat transfer increases, but that of the rate of mass transfer decreases as w increases. These figures also show that for all values of the buoyancy parameter, there is a phase lead for both transfer rates and they both approach 45° at large frequencies. It should be mentioned that, in Figs. 2(a) and 2(b), all the curves for $w = 0$ represent the results obtained for the cases studied by KELLEHER and YANG [6] and HOSSAIN et al. [7] (in absence of a magnetic field).

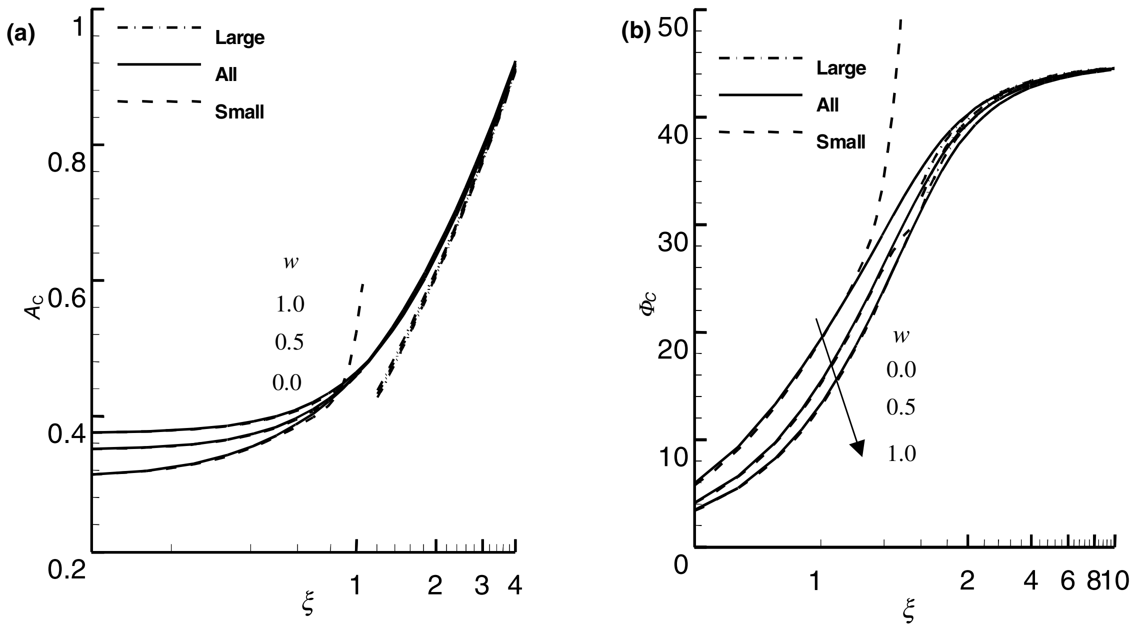

Fig. 3. a) The local amplitude and b) the local phase of mass flux against ξ for Pr = 0.7, Sc = 0.22, and $n = 0.5$ but for different values of the combined buoyancy parameter w

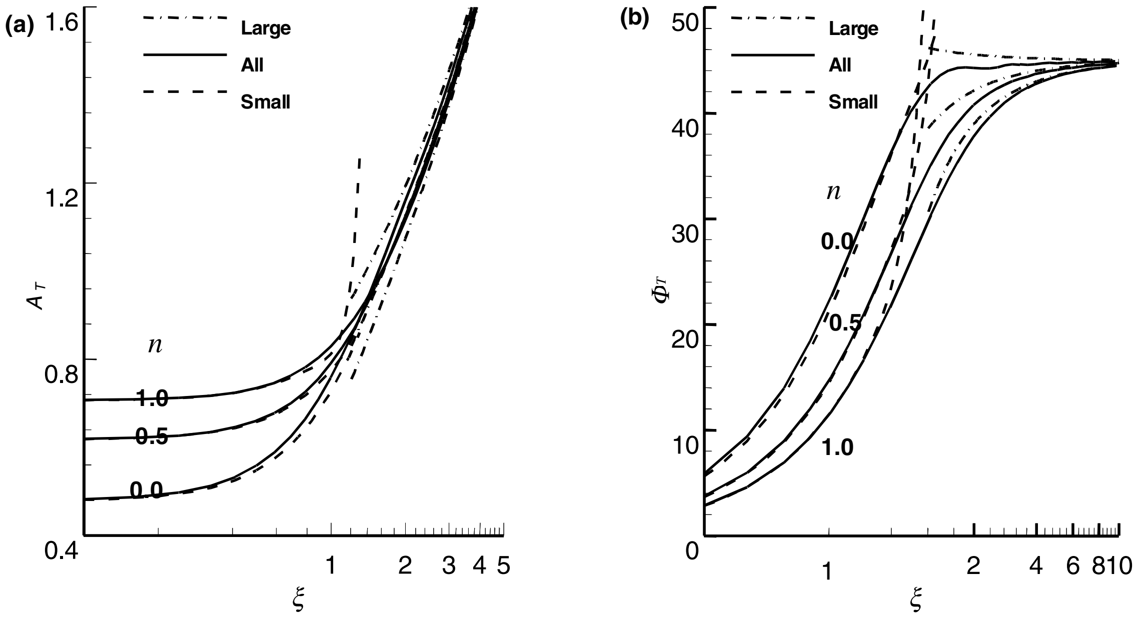

Fig. 4. a) The local amplitude and b) the local phase of heat flux against ξ for $\mathrm{Pr} = 0.7$, $\mathrm{Sc} = 0.22$, and $w = 0.5$ but for different values of the combined buoyancy parameter n

Taking $w = 0.5$ and $\mathrm{Sc} = 0.22$, the effects of varying the temperature-concentration exponent, n, on the amplitude and phase of the surface heat and mass fluxes are depicted, respectively, in Figs. 4 and 5. From these figures we again see excellent agreement between the results obtained for low- and high frequency with those obtained by the finite difference method. It can be seen from Figs. 4(a) and 5(a) that near the leading edge, the amplitudes of both the surface heat-flux and the surface mass-flux increase due to increases in the value of n. On the other hand, from Figs. 4(b) and 5(b) one can see decreases in the phase angles of the surface heat and mass fluxes in the low to moderate frequency region as the temperature exponent, n, increases. Finally, it is observed from the above figures and the tables that, in presence of all the pertinent parameters, there is always a phase lead for both the surface heat-flux and the surface mass-flux, and that these phases tend to 45° at large frequencies.

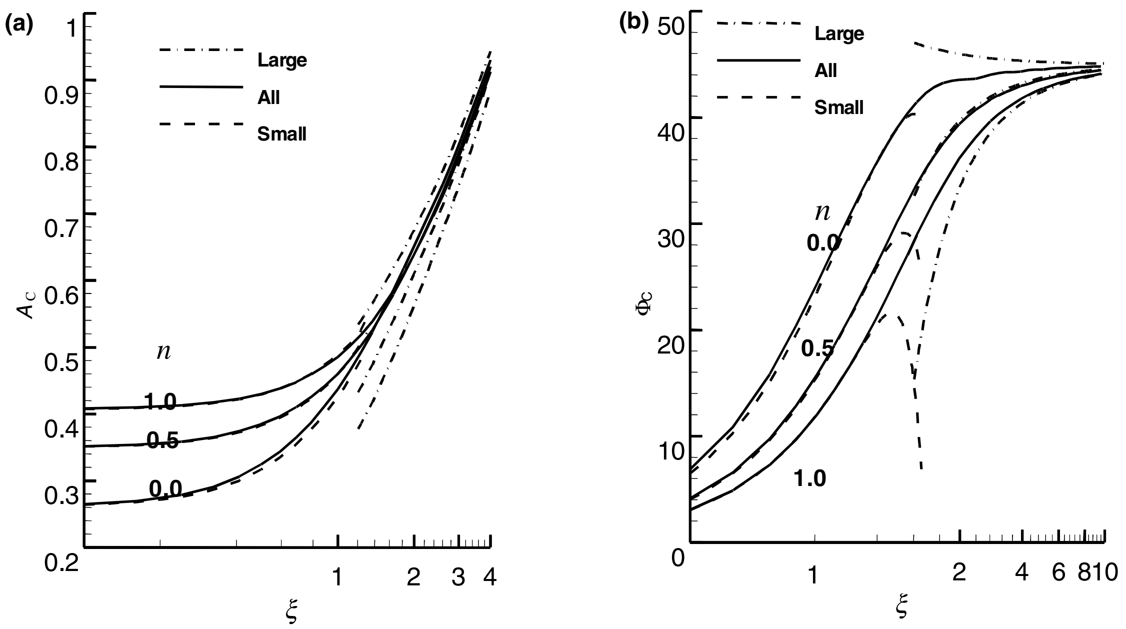

Fig. 5. a) The local amplitude and b) the local phase of mass flux against ξ for $\mathrm{Pr} = 0.7$, $\mathrm{Sc} = 0.22$, and $w = 0.5$ but for different values of the combined buoyancy parameter n

6. Conclusion

A linearised theory has been utilised to study the heat and mass transfer response of the laminar free convective boundary layer flow of viscous incompressible fluid along a vertical impermeable flat plate to an oscillating surface temperature and surface species concentration when both the mean surface temperature and the mean surface concentration vary as a power n of the distance measured from the leading edge. Three distinct methodologies, namely, the perturbation series for the low-frequency range, an asymptotic method for the high-frequency range, and an implicit finite difference method for the intermediate frequencies, have been used. Detailed numerical calculations are carried out for wide range of parameters to examine the results in terms of the surface heat-flux and surface mass-flux responses to surface temperature and surface concentration fluctuations. In particular the amplitudes and phases of the response were considered. The governing parameters were the surface temperature-concentration gradient n, the Schmidt number Sc, and the combined buoyancy parameter w. It has been found that the amplitude and phase angles of the surface heat flux as well as of the surface mass-flux predicted by the perturbation and asymptotic methods are in good agreement with the finite difference solution. From the present investigations, it may also be concluded that the amplitude of the surface heat flux as well as the surface mass flux increase as the frequency increases regardless of the Schmidt number, Sc, the buoyancy parameter, w, and the exponent n. The phase angles for both the surface heat flux and surface mass flux increase monotonically towards the asymptotic values 45° as $\xi \to \infty$ for all values of the governing physical parameters.

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Address: Prof. M. A. Hossain, Ms. S. Hussain, Department of Mathematics, University of Dhaka, Dhaka 1000, Bangladesh;
Dr. D. A. S. Rees, Department of Mechanical Engineering, University of Bath, Bath BA2 7AY, U.K.