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CONVECTIVE PLUMES IN POROUS MEDIA: THE EFFECT OF ASYMMETRICALLY PLACED BOUNDARIES

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ABSTRACT

A boundary layer analysis is performed for the free convective plume induced by a uniform line source which is placed at the intersection of two semi-infinite planes. Particular emphasis is placed on how the presence of the planes, which are not placed symmetrically with respect to the vertical, affects the direction of the plume. To this end, a two-term boundary layer expansion together and a one-term approximation of the flow in the outer regions either side of the plume are performed using the method of matched asymptotic expansions. A closed-form solution is deduced which determines the orientation of the plume in terms of the inclinations of the bounding planes. We find that symmetrically placed planes result in a vertical plume, but that the plume is aligned differently otherwise.

Introduction

The free convective flow induced by a uniform line source of heat in a porous medium is one of the fundamental problems in fluid mechanics. Wooding [1] was able to show that, subject to the boundary layer approximation, the resulting plume may be determined analytically in terms of hyperbolic functions. In particular, the temperature field may be written in terms of a sech^2 profile. Ingham [2] extended this analysis to the case of (form-drag) inertia-dominated flows in porous media. Once more a self-similar flow results which may be solved analytically, although the temperature field now exhibits a sech^4 profile.

In common with many other boundary layer analyses Afzal [3] considered the detailed effects of the external flow generated by entrainment from the region outside the boundary layer. He performed a high-order asymptotic analysis of this situation and found corrections to the leading order flow first studied in [1]. However, the work contained in [3] is restricted to cases where the external domain is symmetric with respect to the vertical. The aim of this present paper is to extend Afzal's analysis to cases where the external domain is not symmetric. A closed-form equation relating the orientation of the plume to the orientations of the bounding surfaces is obtained. We confirm Afzal's assumption that the plume rises vertically when the domain is symmetric, but find that in all other circumstances the plume is affected by the external flow to such an extent that its centreline lies at an angle to the vertical.

Mathematical Formulation

The present work deals with natural convective flow arising from a horizontal line source of heat which is embedded in an isotropic porous medium. A Cartesian frame of reference is chosen where the x-axis is in the upward vertical direction and the y- and z-axes are both horizontal, and aligned normal to and along the line source, respectively. We consider two-dimensional steady flow and assume that Darcy's law and the Boussinesq approximation are both valid. Therefore the full governing equations take the form

$$\psi_{xx} + \psi_{yy} = \theta_y, \tag{1a}$$

$$\theta_{xx} + \theta_{yy} = \psi_y\theta_x - \psi_x\theta_y, \tag{1b}$$

where ψ is the streamfunction defined in the usual way, and θ is the temperature. All the variables in (1) have been made dimensionless using a lengthscale obtained by setting the appropriate Rayleigh number to unity.

Far away from the heat source the velocity and temperature approach their respective ambient values:

$$\psi_y \to 0, \quad \theta \to 0 \quad \text{as} \quad y \to \infty, \tag{2}$$

although we note that this is correct only from the point of view of the leading order boundary layer; appropriate conditions in the external region are given later. The equation for the global conservation of heat then takes the form,

$$\int_{-\infty}^{\infty} (\psi_y\theta - \theta_x)\, dy = 1, \tag{3}$$

where the unit value on the right hand side is the dimensionless rate of heat production per unit length of the source.

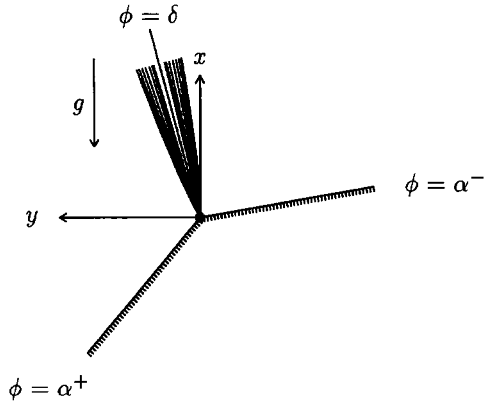

FIG. 1.
Schematic diagram of the flow configuration depicting the two bounding surfaces at $\phi = \alpha^+$ and $\phi = \alpha^-$, and the plume with centreline along $\phi = \delta$. Also shown are the x and y axes ($\phi = 0$ and $\phi = 90°$, respectively) and the direction of gravity.

The line source is placed at the intersection of two bounding planes, see Fig. 1, and we will be concentrating on the behaviour of the plume at large distances from the source for which the boundary layer approximation applies. The flow will be divided into three regions, a boundary layer (inner) region which is thin relative to the flow domain, and two outer regions either side of the plume.

Analysis of the Inner Region

In solving Eqs. (1), (2) and (3) we use the method of matched asymptotic expansions to determine series solutions for ψ and θ. A similar method is used by Afzal [3] in the case of symmetrically placed bounding surfaces. A straightforward application of Afzal's method fails when trying to solve for the first correction to the leading order boundary layer flow, as it becomes impossible to apply all the boundary conditions when the plume is assumed to remain vertically orientated. The key to the analysis is to rotate the coordinate axes through some angle, δ, which is to be determined from the analysis. Relative to polar coordinates (r, ϕ) the positive x-axis corresponds to $\phi = 0$, the y-axis to $\phi = 90°$, and the bounding surfaces to $\phi = \alpha^+$ and $\phi = \alpha^-$, as shown in Fig. 1. The new coordinates, X and Y, obtained by rotation, are given by

$$X = x\cos\delta + y\sin\delta \qquad Y = -x\sin\delta + y\cos\delta\,. \tag{4}$$

The transformed governing equations are then

$$\psi_{XX} + \psi_{YY} = \theta_X \sin\delta + \theta_Y \cos\delta, \qquad \theta_{XX} + \theta_{YY} = \psi_X\theta_Y - \psi_Y\theta_X, \tag{5}$$

and the energy constraint becomes

$$\int_{-\infty}^{\infty}\Big[\psi_X\theta\sin\delta\;+\psi_Y\theta\cos\delta\;-\theta_X\cos\delta\;+\theta_Y\sin\delta\Big]\,dY=\cos\delta\,. \tag{6}$$

In the inner, or plume boundary layer, region the solution takes the following form as $X\to\infty$:

$$\psi=X^{1/3}f_0(\eta)+f_1(\eta)+\cdots,\qquad \theta=X^{-1/3}g_0(\eta)+X^{-2/3}g_1(\eta)+\cdots, \tag{7a, b}$$

where the similarity variable is

$$\eta=Y/X^{2/3}. \tag{7c}$$

This means that the plume is taken to lie around $Y=0$ and is aligned at an angle, δ, to the vertical. After substitution of (7) into (5) and (6) the boundary layer approximation is invoked ($X\gg Y$) and like powers of X are equated to obtain,

$$f_0''-g_0'\cos\delta\;=0,\qquad g_0''+\tfrac{1}{3}(f_0g_0)'=0, \tag{8a, b}$$

$$f_1''-g_1'\cos\delta\;=-\tfrac{1}{3}\sin\delta\,(g_0+2\eta g_0'),\qquad g_1''+\tfrac{1}{3}f_0g_1'+\tfrac{2}{3}f_0'+\tfrac{1}{3}f_1'g_0=0, \tag{8c, d}$$

and the heat flux conditions are:

$$\int_{-\infty}^{\infty}f_0'g_0\,d\eta=1, \tag{9a}$$

$$\int_{-\infty}^{\infty}\Big[f_0'g_1+f_1'g_0+[\tfrac{1}{3}(f_0-2\eta f_0')+g_0']\tan\delta\Big]\,d\eta=0, \tag{9b}$$

The boundary conditions are that

$$f_0(0)=g_0'(0)=0\qquad\text{and}\qquad f_0',g_0,g_1\to0\quad\text{as}\quad\eta\to\pm\infty. \tag{10}$$

The other boundary conditions necessary to solve Eqs. (8) must be determined by asymptotic matching with the solutions from the two outer regions.

Following Afzal [3] there exist exact closed-form solutions of the leading and first order Eqs. (8a) to (8d) which satisfy the heat flux relations (9). We introduce a scaled independent variable

$$\zeta=\tfrac{1}{6}a_0\eta \tag{11}$$

and the new coefficient functions, $\overline{f}_0$, $\overline{g}_0$, $\overline{f}_1$ and $\overline{g}_1$ by setting

$$f_0(\eta)=a_0\overline{f}_0(\zeta),\qquad g_0(\eta)=b_0\overline{g}_0(\zeta), \tag{12a}$$

$$f_1(\eta)=a_1\overline{f}_1(\zeta),\qquad g_1(\eta)=b_1\overline{g}_1(\zeta), \tag{12b}$$

where

$$a_0 = \left(\frac{9\cos\delta}{2}\right)^{1/3}, \qquad b_0 = \left(\frac{3}{32\cos\delta}\right)^{1/3}, \tag{13a}$$

$$a_1 = 4, \qquad b_1 = \frac{2a_0}{3\cos\delta}. \tag{13b}$$

Substitution of (11), (12) and (13) into Eqs. (8) and (9) yields the following equations,

$$\overline{f}_0'' - \overline{g}_0' = 0, \qquad \overline{g}_0'' + 2(\overline{f}_0\overline{g}_0)' = 0, \tag{14a, b}$$

$$\overline{f}_1'' - \overline{g}_1' = B(\overline{f}_0' + \zeta\overline{f}_0''), \qquad \overline{g}_1'' + 2\overline{f}_0\overline{g}_1' + 4\overline{f}_0'\overline{g}_1 + 2\overline{f}_1'\overline{g}_0 = 0, \tag{14c, d}$$

where

$$B = -\frac{3b_0\sin\delta}{a_0^2}, \tag{14e}$$

and the constraints,

$$\int_{-\infty}^{\infty}\overline{f}_0'\overline{g}_0\,d\zeta = \tfrac{4}{3}, \qquad \int_{-\infty}^{\infty}(\overline{f}_0'\overline{g}_1 + \overline{f}_1'\overline{g}_0)\,d\zeta = 0. \tag{15}$$

The closed-form solutions are

$$\overline{f}_0 = \tanh\zeta, \qquad \overline{g}_0 = \operatorname{sech}^2\zeta, \tag{16}$$

$$\overline{f}_1 = B\Big[\zeta\overline{f}_0 - 2\ln(\cosh\zeta)\Big] + C_1 + C_2\overline{f}_0' + C_3(\zeta\overline{f}_0' + \overline{f}_0 - 2\zeta), \tag{17a}$$

$$\overline{g}_1 = B(-\zeta\overline{f}_0') + C_2\overline{f}_0'' + C_3(\zeta\overline{f}_0'' + 2\overline{f}_0'), \tag{17b}$$

were we have imposed exponential decay on the temperature as $\zeta \to \pm\infty$. The values, C_1, C_2 and C_3 are constants of integration.

Note that $\overline{f}_1 \sim -(\pm B + 2C_3)\zeta$ as $\zeta \to \pm\infty$, which provides a symmetric and antisymmetric parts for the outer region matching conditions. As B is dependent on the unknown direction of the plume, δ, via (14e) and (13a), the satisfaction of the matching conditions will yield this angle. But matching with the outer regions requires the use of polar coordinates. In terms of polar coordinates we have $\eta = r^{1/3}\sin\Phi/(\cos\Phi)^{2/3}$ where $\Phi = \phi - \delta$ is the angle relative to the direction of the plume. If we expand for small values of Φ, then the behaviour of ψ as $\eta \to \pm\infty$ is

$$\psi \sim r^{1/3}\Big[a_0 - \tfrac{1}{6}a_0a_1(B + 2C_3)\Phi\Big] \quad \text{as } \eta \to \infty, \tag{18a}$$

$$\psi \sim r^{1/3}\Big[-a_0 + \tfrac{1}{6}a_0a_1(B - 2C_3)\Phi\Big] \quad \text{as } \eta \to -\infty. \tag{18b}$$

These expressions will be used for matching purposes.

Analysis of the Outer Regions

In the outer regions it is convenient to use polar coordinates. As the thermal field decays exponentially out of the boundary layer we may neglect thermal effects. The governing equation becomes

$$\psi_{rr} + \tfrac{1}{r}\psi_r + \tfrac{1}{r^2}\psi_{\Phi\Phi} = 0, \tag{19}$$

where we have used Φ as the angular coordinate for convenience. The corresponding expansion in the outer region is

$$\psi = r^{1/3}F_0(\Phi) + \cdots, \tag{20}$$

where F_0 satisfies

$$F_0'' + \tfrac{1}{9}F_0 = 0. \tag{21}$$

This equation will be solved in the outer region to the left of the plume and the solution denoted by F_0^+, while the solution to the right of the plume will be denoted by F_0^-. As the two bounding plane surfaces meet at the position of the line source, both form streamlines, and therefore we set

$$F_0^+(\alpha^+ - \delta) = F_0^-(\alpha^- - \delta) = 0. \tag{22}$$

Given the form of (18), asymptotic matching with the flow field just outside the plume yields the conditions,

$$F_0^- \to -a_0, \qquad [F_0^-]' \to \tfrac{1}{6}a_0a_1(B - 2C_3), \quad \text{as } \Phi \to 0^-, \tag{23a}$$

$$F_0^+ \to a_0, \qquad [F_0^+]' \to -\tfrac{1}{6}a_0a_1(B + 2C_3), \quad \text{as } \Phi \to 0^+. \tag{23b}$$

Therefore we have a fourth order system of equations to solve with two unknown constants, δ and C_3, and six boundary conditions, (21) to (23), to satisfy. The solution proceeds analytically and easily, and we obtain

$$F_0^+ = a_0\left[\cos\left(\frac{\Phi}{3}\right) - \cot\left(\frac{\alpha^+ - \delta}{3}\right)\sin\left(\frac{\Phi}{3}\right)\right], \tag{24a}$$

$$F_0^- = -a_0\left[\cos\left(\frac{\Phi}{3}\right) + \cot\left(\frac{\delta - \alpha^-}{3}\right)\sin\left(\frac{\Phi}{3}\right)\right], \tag{24b}$$

where δ is given by the transcendental equation,

$$\cot\left(\frac{\alpha^+ - \delta}{3}\right) - \cot\left(\frac{\delta - \alpha^-}{3}\right) = -2\tan\delta. \tag{25}$$

This closed-form equation gives the plume orientation, δ, in terms of the bounding surface inclinations, α^+ and α^-.

Results and Discussion

The focus of attention in this paper lies in determining how a two-dimensional plume reacts to the location of two external impermeable planes meeting at the line source. Intuition would normally dictate that the plume should rise vertically. But some account must be taken of the influence of the flow induced in the regions outside the plume, for fluid must be entrained into the plume as it rises and thickens. Thus if one surface is very much closer to the vertical than is the other, then it is possible that the plume will be attracted towards the nearer surface (such an effect may be seen in the impressive photographs of clear fluid plumes displayed in Gebhart [4]) and which is caused by a lack of available fluid to maintain the required rate of entrainment. Alternatively, the plume will maintain itself in a direction dictated by the results of the analysis of this paper where the flowfield consists of three regions with the plume forming the central region.

Eq. (25), which gives the plume orientation, δ, in terms of the values of α^+ and α^-, may be solved in a fairly straightforward manner using Newton-Raphson iteration since the relation is nonlinear. However, it is much easier to find, say, α^+, in terms of α^- and δ as (25) may be rearranged to give α^+ explicitly. The results of doing this are displayed in Fig. 2 where we confine presentation to positive values of δ.

Figure 2 shows which values of α^+ and α^- are required to yield certain inclinations of the plume. The line corresponding to $\delta = 0°$ represents the symmetric case studied by Afzal [3]. Here the line is straight and corresponds precisely to $\alpha^+ + \alpha^- = 0$. The corresponding loci of α^+ and α^- pairs for other values of δ are also given. The termination of each line at the lower end corresponds to where $\alpha^+ = \alpha^- = \delta$, i.e. to when the wedge angle of the porous medium has been reduced to zero. The upper end of each line corresponds to a wedge angle of 360° and is equivalent to the presence of one semi-infinite surface, rather than to two. For example, when an isolated surface is present at $-150°$ (i.e. $\alpha^+ = 210°$ and $\alpha^- = -150°$), then the plume centreline is at $\delta \simeq 9.3°$.

The rightmost line displayed corresponds to $\delta = 85°$; this is a rather extreme case where the plume is almost horizontal. For such a value of δ, the 'right hand' surface (α^-) is just above the plume. We feel that this solution is very likely to be an example of one which is a mathematical possibility, but one which, if it could be set up in practice, would be unstable and which would evolve in time to a state where the plume would travel up the $\phi = \alpha^-$ surface. Such a state would correspond to an asymptotic structure consisting of an inner region and only one outer region, whereas the focus of this paper is on situations where the inner region is embedded between two outer regions. The question of stability and realisability should be raised, but it is not within the scope of the paper to answer that question. However, it is highly likely that the solutions presented in Fig. 2 will be realised when both surfaces lie below the plane of the heat source, i.e. when $\alpha^+ > 90°$ and $\alpha^- < -90°$. When $\alpha^- > 0$ or when $\alpha^+ < 0$ it is also likely that the plume will travel along

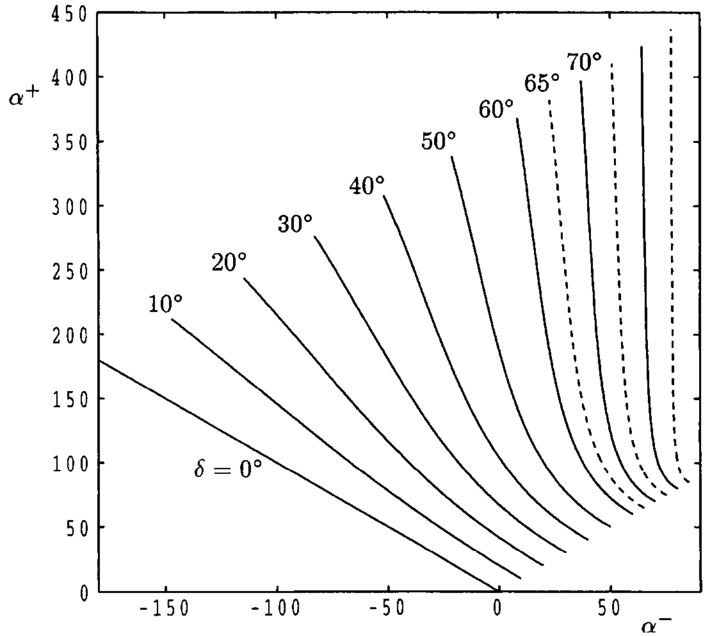

FIG. 2.
The loci of values of the surface angles, α^+ and α^-, which yield plumes of various orientations.

that wall which is closer to the vertical. But the present analysis is not able to indicate, for example, whether the case $\alpha^+ = 10°$ and $\alpha^- = -10°$ will yield a three-region flow of the type considered here, or a two-region flow with the plume firmly attached to one of the walls. This would need to be considered by a numerical simulation of the fully unsteady elliptic equations of motion.

It is clear that asymmetrically placed bounding surfaces affect strongly the predicted trajectory of the free convection plume. In all cases the plume lies above or on the radial line which is precisely half way between the bounding surfaces. The strong dependence shows how important the outer flow is in determining the behaviour of the plume, while its position above that radial line indicates the expected behaviour of a plume which is to try to rise vertically due to buoyancy effects.

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