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TECHNICAL NOTES

The stability of Prandtl–Darcy convection in a vertical porous layer

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1. INTRODUCTION

Recent years have seen a rapid increase of interest in the influence of non-Darcy effects on convection in fluid-saturated porous media (see, e.g. refs. [1–7]). These studies have been motivated by a desire to model realistic effects such as inertial drag which are assumed to be absent in the usual Darcy formulation. Forchheimer [8] was the first to present a correction to Darcy's law for high flow rates; this takes the form of a quadratic velocity term and its effect becomes significant when the microscopic Reynolds number is of order unity or higher. A viscous-like term was proposed by Brinkman [9] which allows the satisfaction of the no-slip condition. A third modification, which is the subject of this note, is a time-dependent velocity term. This corresponds to the relaxing of the infinite Prandtl–Darcy number assumption, and the presence of this term means that the flow field no longer adjusts instantaneously to changes in the temperature field or an externally applied pressure gradient.

In this note the stability of the flow generated by heating a vertical layer of fluid-saturated porous material from the side is considered. Such a configuration is of considerable importance in insulation engineering, e.g. since the onset of an instability will greatly enhance the heat transferred through the layer. The layer is assumed to be infinite in extent in order to simplify the analysis, but it should be noted that the presence of endwalls in a finite cavity could modify the present results. The infinite layer Darcy-flow case was treated by Gill [10] who showed that the basic flow profiles are stable to all infinitesimal disturbances. In a recent paper Georgiadis and Catton [6] sought to investigate the effects of the above three modifications to Darcy's law. Their conclusion was that a finite value of the Prandtl–Darcy number is sufficient to guarantee the linear instability of the basic flow. However, there is an error in their analysis which renders this conclusion invalid, as shown later. It is the task of this note, therefore, to consider the effect of a finite Prandtl–Darcy number on Gill's result. Although the Forchheimer and Brinkman terms are neglected here, one has been unable to furnish an analytical proof of stability/instability. One has therefore resorted to numerical methods to solve the perturbation equations arising from a simple linear stability analysis. Although every possible combination of Rayleigh number, wave number and Prandtl–Darcy number cannot be considered, the results presented here indicate that the basic flow is stable for all Prandtl–Darcy numbers.

2. EQUATIONS OF MOTIONS AND LINEAR STABILITY ANALYSIS

The flow field confined between two isothermal, impermeable, vertical walls is considered, as shown in Fig. 1. The governing non-dimensional Boussinesq equations are given by

$$u_x + v_y + w_z = 0 \tag{1}$$

$$\varepsilon u_t + u = -p_x \tag{2a}$$

$$\varepsilon v_t + v = -p_y \tag{2b}$$

$$\varepsilon w_t + w = -p_z + R\theta \tag{2c}$$

$$\theta_t + u\theta_x + v\theta_y + w\theta_z = \nabla^2\theta \tag{3}$$

which are to be solved subject to the boundary conditions

$$u = 0, \quad \theta = \pm 1 \quad \text{on } x = \pm 1 \tag{4}$$

together with the appropriate periodicity conditions in the y- and z-directions. For the purposes of this note the equations have been nondimensionalized as in Gupta and Joseph [11] except that the reference length and temperature are half the channel width and temperature drop, respectively. In equations (2) the inverse Prandtl–Darcy number is given by $\varepsilon = \frac{1}{4}(\kappa/\nu)(K/d^2)$, typical values of which are fluid and medium dependent but are usually considered to be negligibly small. Equations (1)–(3) are readily solved to give the basic flow profile

$$u = v = 0, \quad w = Rx, \quad p = 0, \quad \theta = x. \tag{5}$$

In order to analyse the linear stability of this flow infinitesimal disturbances are introduced by setting

$$u = u^*, \quad v = v^*, \quad w = Rx + w^*, \quad p = p^*, \quad \theta = x + \theta^* \tag{6}$$

in equations (1)–(4). After linearization the equations governing the evolution of the perturbations become

$$u_x^* + v_y^* + w_z^* = 0 \tag{7}$$

$$\varepsilon u_t^* + u^* = -p_x^* \tag{8a}$$

$$\varepsilon v_t^* + v^* = -p_y^* \tag{8b}$$

$$\varepsilon w_t^* + w^* = -p_z^* + R\theta^* \tag{8c}$$

$$\theta_t^* = \nabla^2\theta^* - u^* - Rx\theta_z^* \tag{9}$$

which are to be solved subject to the boundary conditions

$$u^* = \theta^* = p_x^* = 0 \quad \text{on } x = \pm 1 \tag{10}$$

where the solutions are assumed periodic in y and z. On eliminating v^*, w^* and p^* from equations (7) to (10) and setting

$$u^* = \mathrm{i}\alpha f(x)\,\mathrm{e}^{\mathrm{i}(ly+kz)+\lambda t}$$

$$\theta^* = g(x)\,\mathrm{e}^{\mathrm{i}(ly+kz)+\lambda t}$$

one obtains

$$(1+\varepsilon\lambda)(f'' - \alpha^2 f) + Sg' = 0 \tag{11}$$

$$g'' - (\alpha^2 + \lambda)g - \mathrm{i}\alpha(f + Sxg) = 0 \tag{12}$$

which are to be solved subject to

$$f(\pm 1) = g(\pm 1) = 0 \tag{13}$$

where $\alpha = (l^2 + k^2)^{1/2}$ is the roll wave number and $Rk = S\alpha$. Note that the number of parameters has been reduced from four (R, l, k, ε) to three (S, α, ε) and hence Squire's theorem is valid. Another important point to note is that any incipient

NOMENCLATURE

A, B, C, D	$N \times N$ matrices as defined in the text
d	half-width of the layer
f, g	spatial forms of the disturbances u^*, θ^*
f, **g**	N-vectors containing the Fourier coefficients of f and g
$\hat{g}$	acceleration due to gravity
Gr_{crit}	critical Grashof number (see ref. [6])
I	$N \times N$ identity matrix
K	permeability
l, k	wave numbers
N	truncation level of the Galerkin expansion
p	non-dimensional pressure
R	Rayleigh number, $\hat{g}\beta K d \Delta T / \nu\kappa$
S	modified Rayleigh number, Rk/α
t	non-dimensional time
ΔT	half the temperature difference across the layer
u, v, w	non-dimensional velocity components
x, y, z	non-dimensional coordinates.

Greek symbols

α	wave number
β	coefficient of thermal expansion of the fluid
ε	inverse Prandtl–Darcy number, $\frac{1}{4}\kappa K/\nu d^2$
θ	non-dimensional temperature
κ	thermal diffusivity of the saturated medium
λ	exponential growth rate of the disturbances
ν	fluid viscosity
ϕ	$N \times N$ zero matrix.

Other symbol

0	zero vector of length N.

Superscripts

*	infinitesimal disturbances
T	transpose.

instability must take the form of two travelling modes corresponding to a complex pair of eigenvalues, λ, since setting $\lambda = 0$ in equations (11) and (12) gives identical equations to the Darcy case.

In only one case can one proceed analytically, namely for vertically orientated rolls for which $k = 0$ and therefore $S = 0$. From equation (11) it is easily deduced that $f = 0$ and that $\lambda = -\alpha^2 - \frac{1}{4}n^2\pi^2$ for integer n. Hence disturbances of the form of vertical rolls decay for all Rayleigh numbers, wave numbers and values of ε. For all other roll orientations

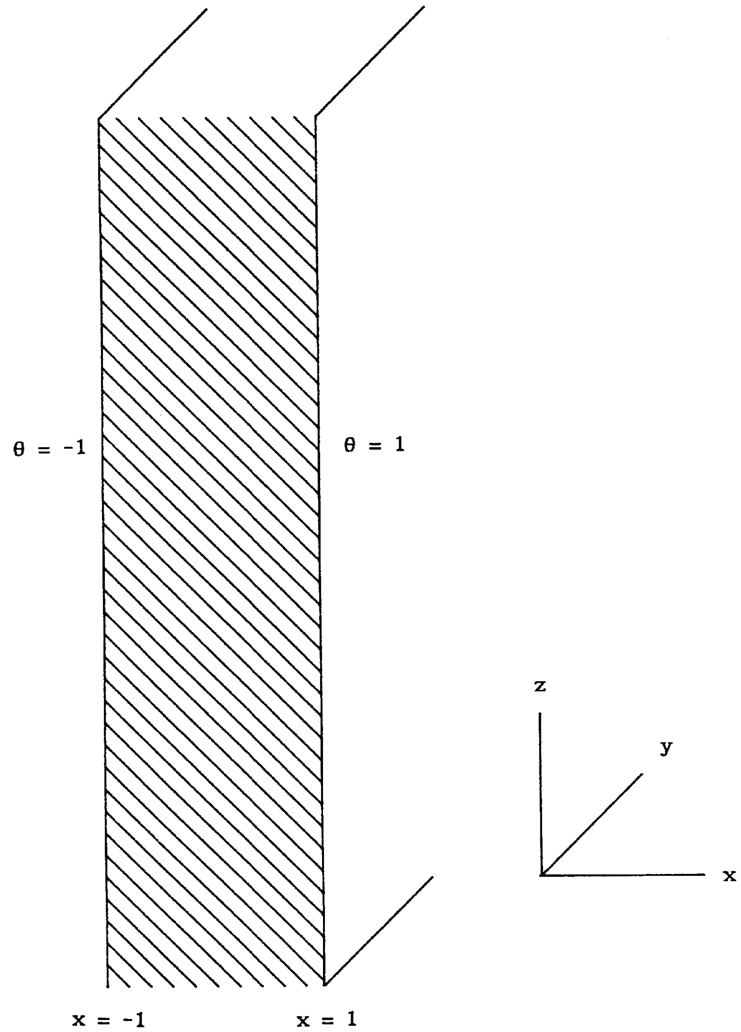


FIG. 1. Flow regime and coordinate system.

$S \neq 0$, and equations (11)–(13) must be solved numerically since Gill's method [11] cannot be applied here. To facilitate the solution of equations (11)–(13) a simple Galerkin expansion is used as follows:

$$(f,g) = \sum_{n=1}^{N} (f_n, g_n) \sin(\tfrac{1}{2}n\pi(x+1)) \quad (14)$$

where N is the truncation level. Since the single derivative in equation (11) gives a cosine series each term is re-expanded in a sine series. A sine series is also obtained for the xg term in equation (12) by using a cosine series for x. Hence one obtains the eigenvalue problem

$$\left[\begin{pmatrix} -(1/\varepsilon)I & -(1/\varepsilon)A^{-1}B \\ \mathrm{i}\alpha I & -A+\mathrm{i}C \end{pmatrix} - \lambda \begin{pmatrix} I & \phi \\ \phi & I \end{pmatrix}\right]\begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = \mathbf{0} \quad (15)$$

for the growth rate, λ, where $\mathbf{f} = (f_1, f_2, \ldots, f_N)^{\mathrm{T}}$, $\mathbf{g} = (g_1, g_2, \ldots, g_N)^{\mathrm{T}}$, I is the $N \times N$ identity matrix, ϕ the $N \times N$ zero matrix, $A = \alpha^2 I + \frac{1}{4}\pi^2 \operatorname{diag}(1^2, 2^2, \ldots, N^2)$ and elements b_{pq}, c_{pq} of B and C are given by

$$b_{pq} = \begin{cases} 2Spq/(p^2-q^2) & (p+q \text{ odd}) \\ 0 & (p+q \text{ even}) \end{cases} \quad (16)$$

$$c_{pq} = \begin{cases} (16/\pi^2)S\alpha pq/(p^2-q^2)^2 & (p+q \text{ odd}) \\ 0 & (p+q \text{ even}). \end{cases} \quad (17)$$

When $\varepsilon = 0$ the numerical scheme breaks down, but for this case it is straightforward to find $\mathbf{f}$ in terms of $\mathbf{g}$ and obtain the simplified system of equations

$$[(-A+\mathrm{i}C+\mathrm{i}D)-\lambda I]\mathbf{g} = \mathbf{0} \quad (18)$$

to replace equation (15), where elements d_{pq} of D are given by

$$d_{pq} = \begin{cases} 8S\alpha pq\left[\dfrac{\pi}{p^2\pi^2+4\alpha^2} + \dfrac{1}{q^2-p^2}\right] & (p+q \text{ odd}) \\ 0 & (p+q \text{ even}). \end{cases} \quad (19)$$

Eigenvalue problems (15) and (18) were solved using the NAG routine F02AJF for various values of S, α and ε, to find the corresponding exponential growth rates, λ. The values obtained in this way were checked against those found by solving equations (11) and (12) using the boundary value problem solver D02HAF. It was found that they were extremely accurate and required only a very small fraction of the time taken by D02HAF.

3. RESULTS AND DISCUSSION

The eigenvalue spectrum for the Darcy flow case, $\varepsilon = 0$, is displayed in Fig. 2 which confirms the well-known result of Gill [10] that the basic flow is linearly stable. It is interesting to note that the decay rate of the most slowly decaying mode increases as (S or R) increases. Note also that the vertical roll ($S = 0$) constitutes the most unstable mode even though the mode decays, and, in general, the most unstable mode corresponds to the one with zero wave number.

For non-zero values of ε, the general behaviour of λ increases in complexity since there is now available the third parameter, ε. In Fig. 3 a plot of $-\mathrm{Re}(\lambda)$ is displayed as a function of ε for the case $\alpha = 0.25$, $S = 10$. For small values

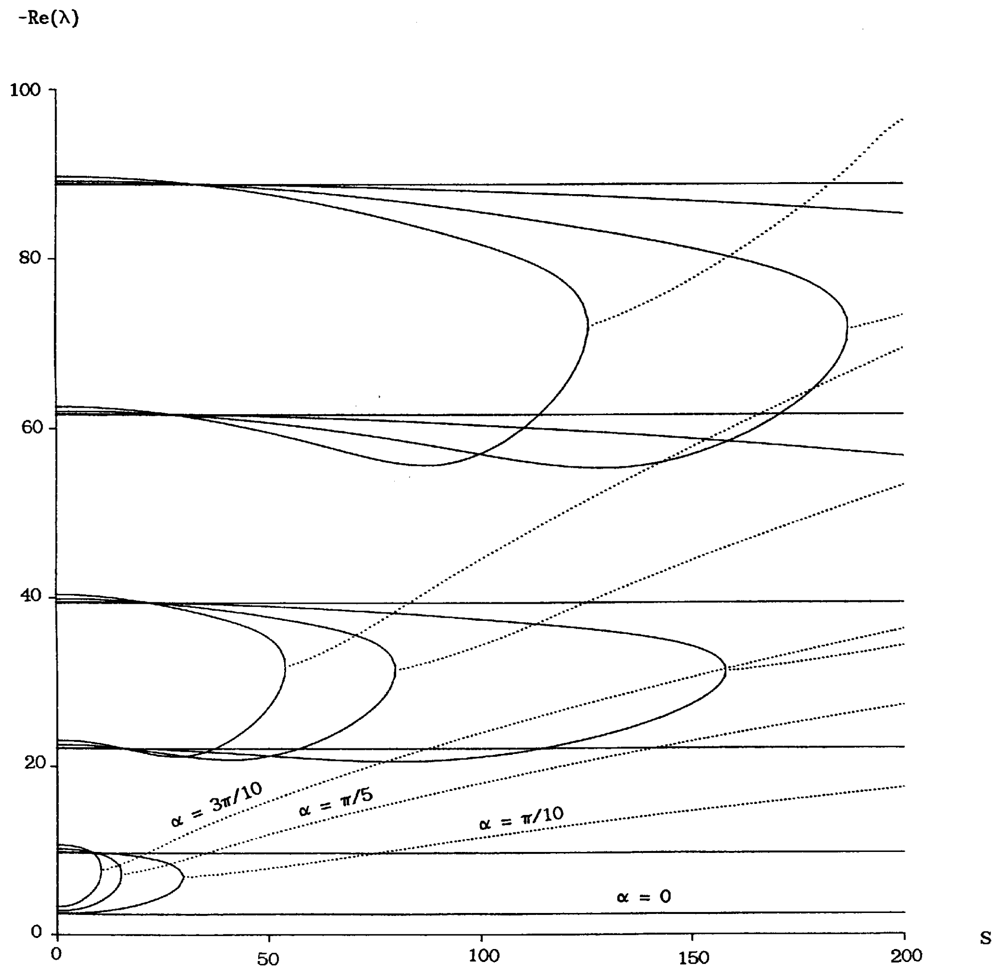


FIG. 2. Values of $-\mathrm{Re}(\lambda)$ for Darcy flow, $\varepsilon = 0$, as a function of R for wave numbers $\alpha = 0$, $\pi/10$, $\pi/5$, $3\pi/10$. The values were calculated with $N = 10$: ——, stationary modes ($\mathrm{Im}(\lambda) = 0$); ·····, travelling modes ($\mathrm{Im}(\lambda) \neq 0$). This convention also applies to Figs. 3 and 4.

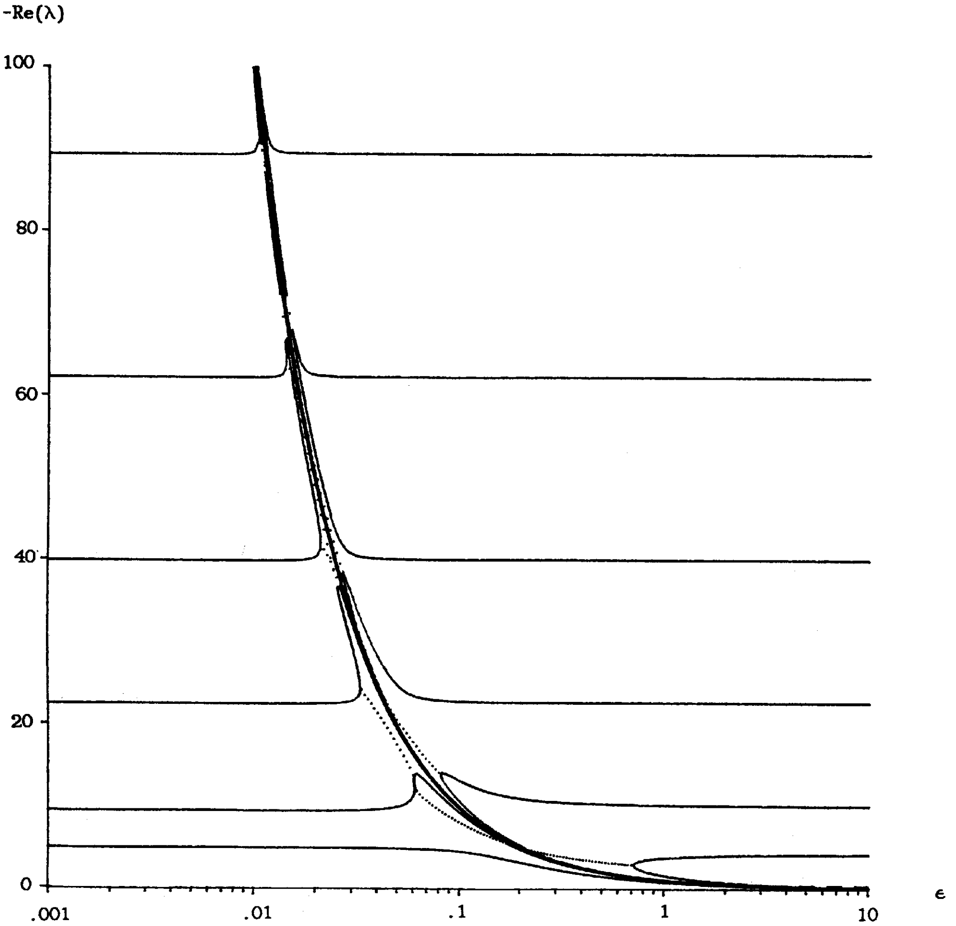
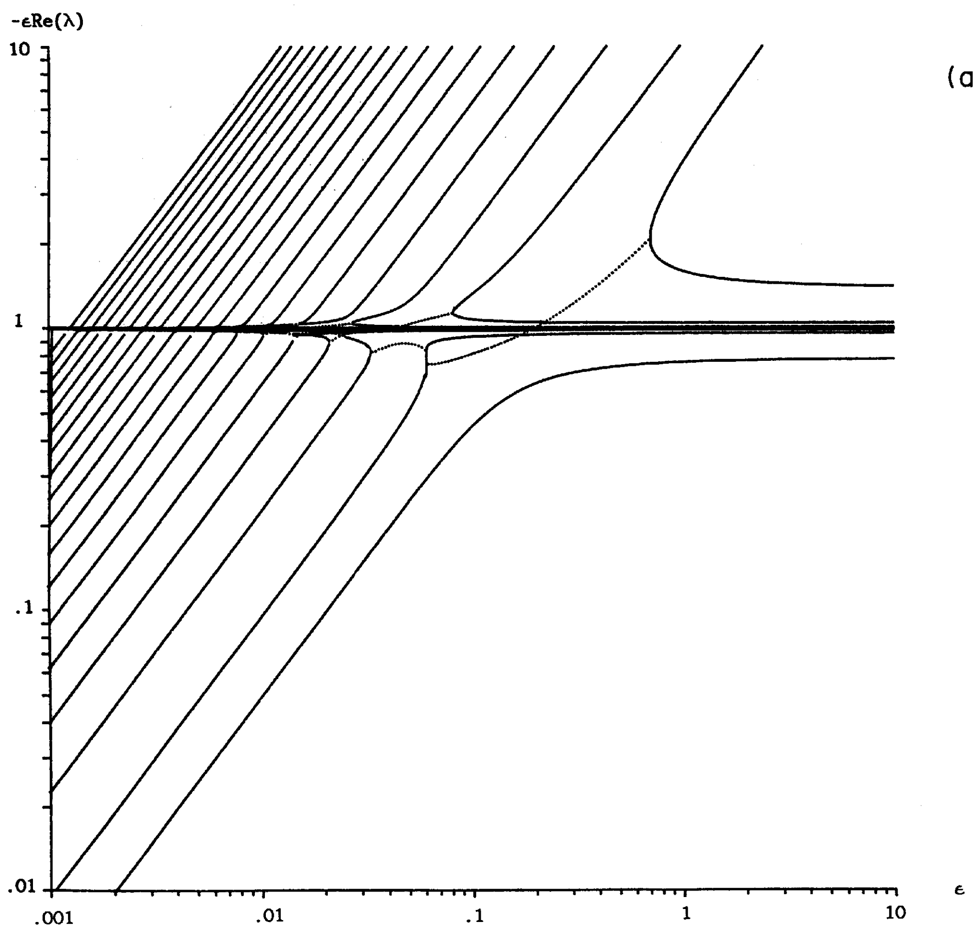


Fig. 3. Values of $-\mathrm{Re}(\lambda)$ for $\alpha = \pi/4$, $S = 10$ as a function of ε (using $N = 18$).



(a)

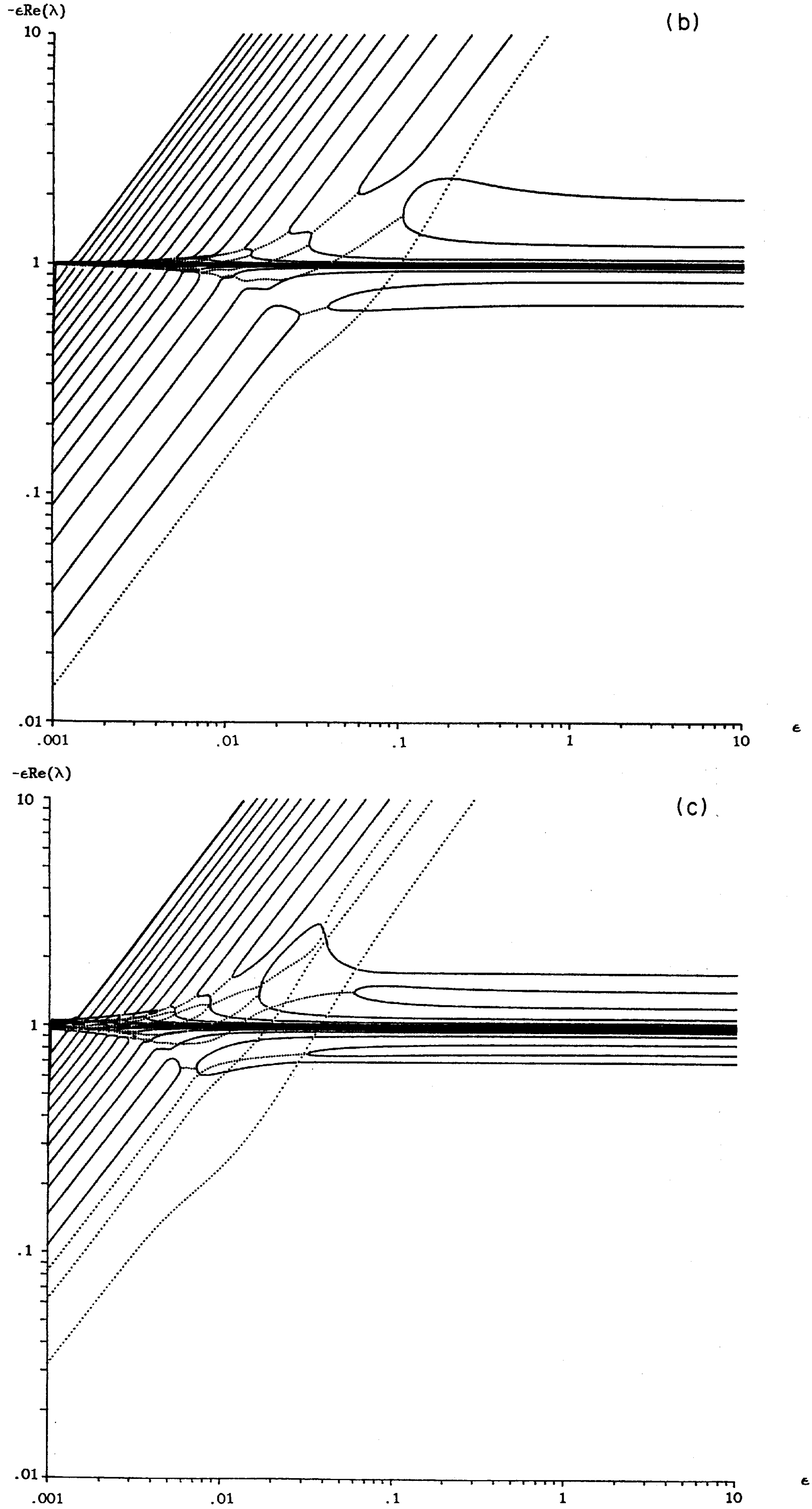


FIG. 4. Graph of $-\varepsilon\,\mathrm{Re}\,(\lambda)$ as a function of ε for $\alpha = \pi/4$: (a) $S = 10$; (b) $S = 50$; (c) $S = 200$. The results were calculated using $N = 16$ which is sufficient to ensure that the first ten modes are accurate as drawn for $S = 10$, and the first six for $S = 200$. The 'first' mode is defined to correspond to the curve closest to the bottom right of the figure and the curve closest to the top right.

of ε the values of λ deviate only slightly from those for the Darcy case. As ε increases, these curves approach the line $1+\varepsilon\lambda = 0$ which corresponds to singular points of equations (11) and (12). This critical line arises because the time-dependent term introduced into the Darcy equations causes the term with the highest order spatial derivative to have a coefficient which varies and which is zero on the critical line if λ is real. In this sense the present problem is similar to the inviscid Orr–Sommerfeld problem. Here, however, one has no need to resort to a critical layer analysis as λ becomes complex in order to pass 'around' the critical line.

On fixing S and α, there are only two possible ways for λ (for the most unstable mode) to evolve as ε increases: either λ asymptotes to the curve $\varepsilon\lambda$ = negative constant, or two stationary modes ($\mathrm{Im}(\lambda) = 0$) coalesce to form a pair of travelling modes ($\mathrm{Im}(\lambda) \neq 0$), which can then pass around the critical line, followed by a decoupling of the modes which both eventually asymptote to λ = constant. The former possibility is not evident in Fig. 3 and therefore, for a clearer representation, one rescales the ordinate by plotting $-\varepsilon\,\mathrm{Re}(\lambda)$. On using a log–log scaling, both asymptotic forms are shown as straight lines, the former having unit slope and the latter being horizontal as is the critical line. These may be seen clearly in Fig. 4.

In Fig. 4 the decrement spectrum is displayed for the three cases, $\alpha = 0.25\pi$, $S = 10$, 50 and 200. These figures are typical of all those calculated for different values of α and S. It is a universal feature of the results that when ε is small the effect of increasing either α, S or both is to decrease $\mathrm{Re}(\lambda)$ still further, at least for the most unstable mode; this is similar to the Darcy case as shown in Fig. 2. As ε increases the slope of $\ln(-\varepsilon\,\mathrm{Re}(\lambda))$ for the most unstable mode usually remains positive and $\mathrm{Re}(\lambda)$ is always negative.

To conclude, one has demonstrated that non-zero values of the inverse Prandtl–Darcy number do not induce instabilities of the form of rolls of any orientation. This is also true for unphysically large values of ε. The question which immediately arises is to ask why the results are at variance with those of Georgiadis and Catton [6], who gave an expression for the critical Grashof number (Gr_{crit}; equivalent to our critical Rayleigh number) for instability. A careful examination of that expression and the preceding analysis shows that the defining integrals for Gr_{crit} contain the Grashof number itself, and therefore their expression defines the critical Grashof number implicitly. The results indicate that such a value does not exist and that the flow is linearly stable.

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Surface tension driven flows for a droplet in a micro-gravity environment

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1. INTRODUCTION

WITH THE advent of sustained space flight, studies related to chemical processes in a micro-gravity environment have become important. At low gravity fluid motion is often governed by forces which are often negligible in the earth's gravitational field. One of these forces which is expected to be important in a micro-gravity environment is surface tension. When a gradient in surface tension exists at the interface between two fluid phases, a surface tension driven (Marangoni) flow field may result. The surface tension between two fluids is a function of the temperature and the concentration level of any solute present at the fluid interface. Thus, in the presence of a gradient in the solute concentration or the temperature near a fluid interface, surface tension driven flows may be present. Surface tension (Marangoni) effects on droplets have been studied in several works including: Levan [1], Thompson *et al.* [2], and Rivkind and Sigovtsev [3].

The intent of this work is to demonstrate that irradiant