Pergamon

Int. J. Heat Mass Transfer. Vol. 41, No. 12, pp. 1833–1843, 1998

Printed in Great Britain
0017 9310/98 \$19.00+0.00

PII: S0017–9310(97)00219–6

The influence of higher-order effects on the linear instability of thermal boundary layer flow in porous media

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(*Received* 22 *January* 1997 *and in final form* 16 *July* 1997)

Abstract—We examine the instability of free convective boundary layer flows in porous media. The medium is bounded by two semi-infinite plane surfaces forming a wedge of angle α. One of the surfaces is heated uniformly and the other is either cold or insulated. The basic flow used in the analysis is the most accurate obtainable by means of higher order boundary layer theory. In general, the critical distance from the leading edge beyond which disturbances grow is found to be strongly dependent on the outer flow field. The only exception to this arises when the heated surface is close to the vertical and if the wedge is not too close to either 0° or 360°. The main implication of this paper is that instability occurs too close to the leading edge for the basic flow to be represented adequately either by the leading order boundary layer theory used in previous papers, or by even the most accurate higher order theory obtained using matched asymptotic expansions. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

This paper describes an investigation of the instability of free convective boundary layers in porous media. The main difference between the present work and previously published analyses is that we do not rely on the leading order boundary layer flow as a sufficiently accurate approximation to the basic steady state flow. Rather, we make use of the most accurate steady boundary layer flow obtainable by means of asymptotic methods, and investigate whether or not the external flowfield generated by the leading order boundary layer has an appreciable effect on the stability criterion.

The first papers to appear dealing with steady thermal boundary layer flows in porous media were by Cheng and Chang [1] and Cheng and Minkowycz [2]. In these papers, certain geothermal formations are modelled by assuming that they are represented adequately by semi-infinite surfaces which are horizontal and vertical, respectively. Cheng and co-workers further assumed that the boundary layer approximation is valid and analysed the flow and heat transfer by determining the leading-order boundary layer flow. The use of higher order boundary layer theory, which requires the use of the method of matched asymptotic expansions, enables the determination of a more accurate value of the rate of heat transfer into the medium. Such work has been undertaken by Chang and Cheng [3], Daniels and Simpkins [4], Cheng and Hsu [5], Riley and Rees [6] and Hsu and Cheng [7].

However, if the ultimate aim of such analyses is to obtain accurate rates of heat transfer, it is essential to investigate the possible instability of these flows, for the presence of an instability serves to modify both the velocity and temperature fields and hence the temperature gradient on the heated surface. To date there are three papers which cover this aspect for an isothermal surface: Hsu, *et al.* [8], Hsu and Cheng [9] and Rees and Bassom [10]. The paper by Hsu and Cheng [9] determines the critical distance from the leading edge of an inclined heated surface beyond which disturbances in the form of vortices grow. The other two consider the corresponding horizontal thermal boundary layer. The results of [10] indicate that the critical distance for wave disturbances is 28.90, which should be compared with 33.47 for vortices in ref. [8]. Given that streamwise vortices were assumed in ref. [8] to constitute the preferred mode of instability, it is clear that further work is essential in order to understand fully this problem. Furthermore, it is not clear from these papers whether there are any discernable effects of the outer flow field on the stability criterion.

All three papers cited in the above paragraph assume that the basic flow examined for stability is

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NOMENCLATURE

- a constant
- b growth rate for vortices
- c wavespeed
- $f_0, f_1, \bar{f}_2, f_2$ coefficient functions for the streamfunction
- $g_0, g_1, \bar{g}_2, g_2$ coefficient functions for the temperature
- p pressure
- t time
- u, v, w seepage velocities in the x, y and z directions, respectively
- x, y, z streamwise, cross-stream and spanwise Cartesian coordinates.

Greek symbols

- α wedge angle
- δ inclination of heated surface
- η similarity variable
- θ, Θ temperatures
- ϕ angular coordinate
- ψ, Ψ streamfunctions
- $\bar{\psi}, \bar{\theta}$ infinitesimal disturbances.

Subscripts

- x, y, z, t differentiation with respect to x, y, z and t, respectively
- c critical value.

Superscripts

- $'$ differentiation with respect to η
- $\hat{}$ scaled critical value.

adequately represented by the leading order boundary layer flow. In this paper we relax this assumption and use as the basic flow the higher order solutions given in Riley and Rees [6] and the other papers cited earlier. In line with these papers we shall assume that the porous medium is bounded by a heated surface inclined at an angle δ to the vertical. and a second surface which is either insulated or at the ambient temperature of the medium and which forms a wedge-angle α with the first surface (see Fig. 1). Thus the basic flow is dependent on both α and δ. Daniels and Simpkins [4] show that it is not possible to present explicitly an arbitrary number of terms in the asymptotic series for the boundary layer flow because eigen-solutions with arbitrary amplitudes always arise at some point in the series. Typically we can expect three terms, but no more.

We investigate the effect on stability of varying α and the number of terms in the basic boundary layer flow for any given inclination δ. In general we find that the critical distance is very strongly dependent on both these factors; it is only when the heated surface is near to the vertical that statements about critical distances can be made with any certainty, and only then if the wedge angle is not too close to either 0° or 360°.

In Section 2 we derive the governing equations for thermal boundary layer flow from a generally inclined surface and present the basic boundary layer flow obtained using matched asymptotic expansions. The

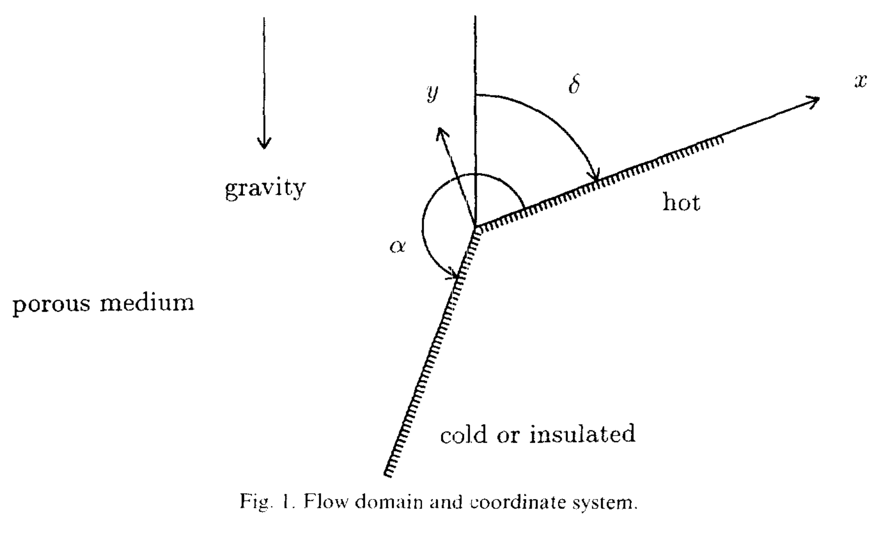

Fig. 1. Flow domain and coordinate system.

equations satisfied by vortex disturbances are given in Section 3 and the results are presented and discussed in Section 4. The equivalent results for both vortex and wave disturbances for a horizontal boundary layer are given in Section 5. We conclude with a brief discussion of the implications of our results in Section 6.

2. GOVERNING EQUATIONS

We consider the instability of free convective boundary layer flow in a fluid-saturated porous medium. The flow is induced by an isothermal upward-facing semi-infinite surface which is at an angle δ to the vertical, where $0^\circ \leqslant \delta \leqslant 90^\circ$. The porous medium is bounded by a second semi-infinite surface which is either insulated or held at the ambient temperature of the medium. The surfaces form a wedge of angle α, where $0^\circ < \alpha < 360^\circ$. A Cartesian frame of reference is chosen where the x-axis is aligned with the heated surface, the y-axis is perpendicular to the heated surface, and the z-axis is in the spanwise direction. The configuration is shown in Fig. 1.

The surface $y = 0$, $x \geqslant 0$, is isothermal and is maintained at the nondimensional temperature $\theta = 1$, whilst the ambient temperature of the porous medium is $\theta = 0$. The other bounding surface is either held at the ambient temperature ($\theta = 0$) or is insulated.

The basic equations for unsteady three-dimensional flow in a porous medium may be written in the form

$$u_x + v_y + w_z = 0 \quad u = -p_x + \theta\cos\delta$$
$$v = -p_y + \theta\sin\delta \quad w = -p_z$$
$$\theta_t + u\theta_x + v\theta_y + w\theta_z = \theta_{xx} + \theta_{yy} + \theta_{zz} \tag{1}$$

where Darcy's law and the Boussinesq approximation have been assumed to be valid. All the variables in equation (1) have been nondimensionalised as in Riley and Rees [6] and Rees and Storesletten [11]. Here u, v and w are the seepage velocities in the x, y and z directions, respectively, p the dynamic pressure, θ the temperature, and t the time. We note that there is no nondimensional parameter in these equations; this is a consequence of there being no natural length scale, but rather the material parameters of the fluid and the porous medium define a macroscopic length scale (cf. equation (6a) in ref. [11]).

The undisturbed basic flow, which we denote by the subscript zero (i.e. as u_0, v_0, w_0, p_0 and θ_0), is steady and two-dimensional and satisfies the equations.

$$\frac{\partial^2\psi_0}{\partial x^2} + \frac{\partial^2\psi_0}{\partial y^2} = \frac{\partial\theta_0}{\partial y}\cos\delta - \frac{\partial\theta_0}{\partial x}\sin\delta \tag{2a}$$

$$\frac{\partial^2\theta_0}{\partial x^2} + \frac{\partial^2\theta_0}{\partial y^2} = \frac{\partial\psi_0}{\partial y}\frac{\partial\theta_0}{\partial x} - \frac{\partial\psi_0}{\partial x}\frac{\partial\theta_0}{\partial y} \tag{2b}$$

where

$$u_0 = \frac{\partial\psi_0}{\partial y} \quad v_0 = -\frac{\partial\psi_0}{\partial x} \quad w_0 = 0 \tag{3}$$

and ψ_0 is the basic flow streamfunction. Equations (2a) and (2b) are to be solved subject to the boundary conditions:

$$\psi_0 = 0 \quad \theta_0 = 1 \quad \text{on} \quad \phi = 0 \tag{4a}$$

$$\psi_0 = 0 \quad \theta_0 = 0 \quad \text{or} \quad \frac{\partial\theta_0}{\partial\phi} = 0 \quad \text{on} \quad \phi = \alpha \tag{4b}$$

where ϕ is the azimuthal angle measured from the heated surface (cf. Fig. 1).

The solution of this basic flow problem has been presented in Riley and Rees [6] and in more detail in Rees [12]; these authors used the method of matched asymptotic expansions to determine a series solution for the flow and temperature fields within both the main boundary layer and the outer flow field. In the boundary layer region the solution takes the following form as $x \to \infty$:

$$\psi_0 = x^{1/2}f_0(\eta) + f_1(\eta) + x^{-1/2}\ln x\bar{f}_2(\eta) + x^{-1/2}f_2(\eta) + \cdots \tag{5a}$$

$$\theta_0 = g_0(\eta) + x^{-1/2}g_1(\eta) + x^{-1}\ln x\bar{g}_2(\eta) + x^{-1}g_2(\eta) + \cdots \tag{5b}$$

where the similarity variable η is given by

$$\eta = \frac{y}{x^{1/2}}. \tag{6}$$

The functions f_0, f_1, $\bar{f}_2$, f_2, g_0, g_1, $\bar{g}_2$ and g_2 which appear in equation (5) satisfy the ordinary differential equations:

$$f_0'' - g_0'\cos\delta = 0 \tag{7a}$$

$$g_0'' + \tfrac{1}{2}f_0g_0' = 0 \tag{7b}$$

$$f_1'' - g_1'\cos\delta = \tfrac{1}{2}\eta g_0'\sin\delta \tag{7c}$$

$$g_1'' + \tfrac{1}{2}(f_0g_1' + f_0'g_1) = 0 \tag{7d}$$

$$\bar{f}_2'' - \bar{g}_2'\cos\delta = 0 \tag{7e}$$

$$\bar{g}_2'' + \tfrac{1}{2}f_0\bar{g}_2' + f_0'\bar{g}_2 - \tfrac{1}{2}\bar{f}_2g_0' = 0 \tag{7f}$$

$$f_2'' - g_2'\cos\delta = \tfrac{1}{2}(g_1 + \eta g_1')\sin\delta + \tfrac{1}{4}(f_0 - \eta f_0' - \eta^2 f_0'') \tag{7g}$$

$$g_2'' + \tfrac{1}{2}f_0g_2' + f_0'g_2 - \tfrac{1}{2}f_2g_0' = -\tfrac{1}{2}f_1'g_1 - \tfrac{3}{4}\eta g_0' - \tfrac{1}{4}\eta^2 g_0'' + f_0'\bar{g}_2 - \bar{f}_2g_0'. \tag{7h}$$

At $\eta = 0$ these functions satisfy the boundary conditions:

$$f_0 = f_1 = \bar{f}_2 = f_2 = 0 \tag{8a}$$

$$g_0 = 1 \quad g_1 = \bar{g}_2 = g_2 = 0 \tag{8b}$$

whilst the appropriate conditions which match with the outer flow are that

$$f'_0, \bar{f}'_2, g_0, g_1, \bar{g}_2, g_2 \to 0$$

$$f'_1 \to -\tfrac{1}{2} a_0 \cot\left(\tfrac{1}{2}\alpha\right)$$

$$f'_2 \to \tfrac{1}{4}\eta a_0 - a_1/\alpha \tag{8c}$$

as $\eta \to \infty$, where the constants a_0 and a_1 are given by the limiting forms:

$$f_0 \to a_0 \quad \text{and} \quad f_1 - \eta f'_1 \to a_1 \quad \text{as} \quad \eta \to \infty. \tag{8d}$$

We note that equations (7a)–(7d) can be solved explicitly whereas the solution to equations (7e) and (7f) involves an eigensolution with an unknown amplitude. The presence of eigensolutions in the expansion at this point results from what is known as the 'leading-edge shift' which reflects a small uncertainty in the location of the leading edge in an asymptotic theory for large distances from the leading edge. This constant amplitude can be found by insisting that equations (7g) and (7h) can be solved, but the resulting solutions for f_2 and g_2 also contain a component of the form of the same eigensolution with an arbitrary amplitude. This second constant cannot be obtained except by comparing the asymptotic solution with a solution of the full elliptic equations of motion. We are therefore restricted to using only the first three terms in the asymptotic series equations (5a) and (5b), although we have to consider the first four terms in order to obtain the third explicitly. Further details on the derivation of these boundary conditions, the form of the outer flow field and the argument concerning the solutions of equations (7) are given in ref. [12].

3. LINEAR STABILITY ANALYSIS

In this section we develop the linear stability equations for the basic flow given above. It is well known that the primary mode of instability for upward-facing inclined surfaces takes the form of stationary streamwise vortices, and therefore we shall concentrate on small disturbances of this form. We shall restrict attention to disturbances which are locally independent of x; such an assumption was also made in a recent paper by Rees and Bassom [10] who considered the wave instability of horizontal thermal boundary layer flow. Therefore, we shall set

$$
\begin{aligned}
u &= u_0(x, y) + u_1(y, z, t) \\
v &= v_0(x, y) + v_1(y, z, t) \\
w &= w_0(x, y) + w_1(y, z, t) \\
p &= p_0(x, y) + p_1(y, z, t) \\
\theta &= \theta_0(x, y) + \theta_1(y, z, t)
\end{aligned} \tag{9}
$$

where the basic flow is denoted by the '0' subscript and the infinitesimal disturbances by the '1' subscript. The substitution of equation (9) into the full governing equations (1) gives the following set of linearised disturbance equations:

$$u_1 = \theta_1 \cos\delta \tag{10a}$$

$$\frac{\partial^2 \psi_1}{\partial y^2} + \frac{\partial^2 \psi_1}{\partial z^2} = -\frac{\partial \theta_1}{\partial z}\sin\delta \tag{10b}$$

$$\frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2} = \frac{\partial \theta_0}{\partial x}\theta_1\cos\delta - \frac{\partial \psi_0}{\partial x}\frac{\partial \theta_1}{\partial y} - \frac{\partial \theta_0}{\partial y}\frac{\partial \psi_1}{\partial z} + \frac{\partial \theta_1}{\partial t} \tag{10c}$$

where

$$v_1 = -\frac{\partial \psi_1}{\partial z} \quad \text{and} \quad w_1 = \frac{\partial \psi_1}{\partial y}. \tag{11}$$

The appropriate boundary conditions for the disturbances are

$$\psi_1 = 0 \quad \theta_1 = 0 \quad \text{on both} \quad \phi = 0 \quad \text{and} \quad \phi = \alpha. \tag{12}$$

For x-independent vortices, the disturbance equations (9) are of the form:

$$\psi_1(y, z, t) = \bar{\psi}(y)e^{iaz+bt} \tag{13a}$$

$$\theta_1(y, z, t) = \bar{\theta}(y)e^{iaz+bt} \tag{13b}$$

where $\bar{\psi}$ and $\bar{\theta}$ are small amplitudes, a is the spanwise wavenumber and b is the temporal growth rate. Equations (13) may now be substituted into equation (10) to give

$$\bar{\psi}_{yy} - a^2\bar{\psi} = -ia\bar{\theta}\sin\delta \tag{14a}$$

$$\bar{\theta}_{yy} - a^2\bar{\theta} = \frac{\partial \theta_0}{\partial x}\bar{\theta}\cos\delta - \frac{\partial \psi_0}{\partial x}\bar{\theta}_y - ia\frac{\partial \theta_0}{\partial y}\bar{\psi} + b\bar{\theta}. \tag{14b}$$

On changing variables from y to η, and expanding the basic flow variables according to equation (5), the disturbance equations become:

$$\bar{\psi}'' - a^2 x\bar{\psi} = -iax\bar{\theta}\sin\delta \tag{15a}$$

$$
\begin{aligned}
\bar{\theta}'' - a^2 x\bar{\theta} = {} & -ia[x^{1/2}g'_0 + g'_1 + x^{-1/2}\ln x\,\bar{g}'_2]\bar{\psi} \\
& -[\tfrac{1}{2}\eta g'_0 + \tfrac{1}{2}x^{-1/2}(g_1 + \eta g'_1) \\
& + x^{-1}(-\bar{g}_2 + \ln x(\bar{g}_2 + \tfrac{1}{2}\eta g'_2))]\bar{\theta}\cos\delta \\
& -[\tfrac{1}{2}(f_0 - \eta f'_0) - \tfrac{1}{2}x^{-1/2}\eta f'_1 \\
& + \tfrac{1}{2}x^{-1}(2\bar{f}_2 - \ln x(\bar{f}_2 + \eta\bar{f}'_2))]\bar{\theta}' + b\bar{\theta}
\end{aligned} \tag{15b}
$$

where dashes represent derivatives with respect to η. The boundary conditions to be satisfied by the disturbances are

$$\bar{\psi} = 0 \quad \bar{\theta} = 0 \quad \text{on} \quad \eta = 0 \tag{16a}$$

and

$$\bar{\psi}, \bar{\theta} \to 0 \quad \text{as} \quad \eta \to \infty. \tag{16b}$$

Given the large-η asymptotic behaviour of the basic flow quantities in equations (5), it is easy to show that equations (15) admit solutions with exponential decay. Thus the disturbance is confined to the boundary layer, and it is not necessary to investigate the form of the disturbance in the outer flow regime, as was the case for the basic flow.

As we are interested in determining where the basic boundary layer becomes unstable, we have to find the critical distance x_c beyond which disturbances grow. The value of x_c, however, is a function of the wavenumber a and it is therefore necessary to minimise x_c with respect to a. In order to do this easily, we supplement equations (15) with a further set which is obtained from equation (15) by differentiating with respect to a and setting $\partial x/\partial a = 0$. If we now define Ψ and Θ according to,

$$\Psi = \frac{\partial \bar{\psi}}{\partial a} \quad \text{and} \quad \Theta = \frac{\partial \bar{\theta}}{\partial a} \tag{17}$$

then Ψ and Θ satisfy

$$\Psi'' - a^2 x \Psi = -ix(a\Theta + \bar{\theta})\sin\delta + 2ax\bar{\psi}, \tag{18a}$$

$$\begin{aligned}\Theta'' - a^2 x\Theta = {} & 2ax\bar{\theta} - i[x^{1/2}g_0' + g_1' + x^{-1/2}\ln x\bar{g}_2'] \\ & \times (a\Psi + \bar{\psi}) - [\tfrac{1}{2}\eta g_0' + \tfrac{1}{2}x^{-1/2}(g_1 + \eta g_1') \\ & + x^{-1}(-\bar{g}_2 + \ln x(\bar{g}_2 + \tfrac{1}{2}\eta\bar{g}_2'))]\Theta\cos\delta \\ & - [\tfrac{1}{2}(f_0 - \eta f_0') - \tfrac{1}{2}x^{-1/2}\eta f_1' \\ & + \tfrac{1}{2}x^{-1}(2\bar{f}_2 - \ln x(\bar{f}_2 + \eta\bar{f}_2'))]\Theta' + b\Theta.\end{aligned} \tag{18b}$$

Assuming that the temporal growth rate b is zero, this results in a tenth order system of equations where the ninth and tenth equations, are simply,

$$\frac{\partial x}{\partial \eta} = 0 \quad \text{and} \quad \frac{\partial a}{\partial \eta} = 0. \tag{19}$$

As both x and a are found as eigenvalues of this homogeneous system, it is necessary to impose normalising conditions; thus we set

$$\bar{\theta}'(0) = 1 \quad \text{and} \quad \Theta'(0) = 0. \tag{20}$$

Equations (16), (18) and (19) were solved numerically using a fourth order Runge–Kutta scheme coupled with a standard shooting method employing Newton–Raphson iteration. The values of δ and α were varied systematically as was the number of terms used in the basic boundary layer flow in order to determine their separate effects on the critical distance.

4. RESULTS FOR THE INCLINED HEATED SURFACE

It is necessary to note that the critical distances and wavenumbers we shall be presenting in this section are scaled with respect to the inclination angle, δ, with $0^\circ \leqslant \delta < 90^\circ$. Thus the critical distance of wavenumber, denoted x_c and a_c, respectively, are given in terms of the values $\hat{x}_c$ and $\hat{a}_c$ where

$$\hat{x}_c = \frac{x_c \sin^2\delta}{\cos\delta} \quad \text{and} \quad \hat{a}_c = \frac{a_c}{\sin\delta}. \tag{21}$$

The reason for these scalings is simply when the leading order boundary layer flow is used as the basic flow, the values of $\hat{x}_c$ and $\hat{a}_c$ obtained are independent of δ; a similar scaling was incorporated into the nondimensionalisation procedure used by Hsu and Cheng [9].

The value of $\hat{x}_c$ we obtain using the leading order boundary layer flow as the basic flow is 110.7, which is lower than 120.7, the value found by Hsu and Cheng [9]. The disparity between these results can be explained in terms of the precise form chosen for the disturbance: Hsu and Cheng assumed that $\bar{\psi} \propto x^{1/3}$ and $\bar{\theta}$ is independent of x whereas we assumed that both variables are independent of x. Our critical wavenumber is $\hat{a}_c = 0.623$ which compares with 0.636, obtained by Hsu and Cheng. Clearly the issue of the effect of x-dependence on the onset of instability is important, but is outside the scope of the present paper. We will see that even greater variations in the computed values of $\hat{x}_c$ are obtained when the wedge angle α and the number of basic boundary layer terms are varied.

Figure 2 displays the effect on the critical distance of varying both the wedge angle and the number of terms in the approximation for the basic flow for surface inclinations $\delta = 5^\circ$, $\delta = 15^\circ$, $\delta = 30^\circ$ and $\delta = 60^\circ$. In all four cases the computed value of $\hat{x}_c$ shows quite marked variation with α. When $\delta = 5^\circ$ increasing the number of terms in the asymptotic series for the basic flow from one to two shows that the value of α has an effect on the critical distance. Increasing the number of terms to three has an almost imperceptible effect on $\hat{x}_c$, and therefore we have confidence that our results are quantitatively sound at inclinations near to 5°. This qualitative result follows from the fact that, at such small inclinations, the value of x_c is very large (see equation (21)), and therefore the first two terms of the boundary layer approximation are very accurate representations of the exact flow. However, when α is close to either 0° or 360°, $\hat{x}_c$ begins to show large variations. This behaviour reflects the fact that the second term in the asymptotic series for the basic flow is becoming large in either of these limits (see the boundary condition for f_1' in equation (8c)). In these limits the boundary layer approximation breaks down and therefore our present results will have no validity.

At increasingly large inclinations the variation of $\hat{x}_c$ with both α and the number of terms used becomes progressively greater. At $\delta = 60^\circ$ it is very clear that no confidence can be given to any of the numerical results for although there is marked variation in $\hat{x}_c$ with α, the number of terms used in the asymptotic expansion is clearly insufficient to obtain convergence to a single curve. As mentioned earlier, we cannot use more terms than three because of the presence of eigensolutions with an arbitrary amplitude at $O(x^{-1})$ relative to the leading order terms.

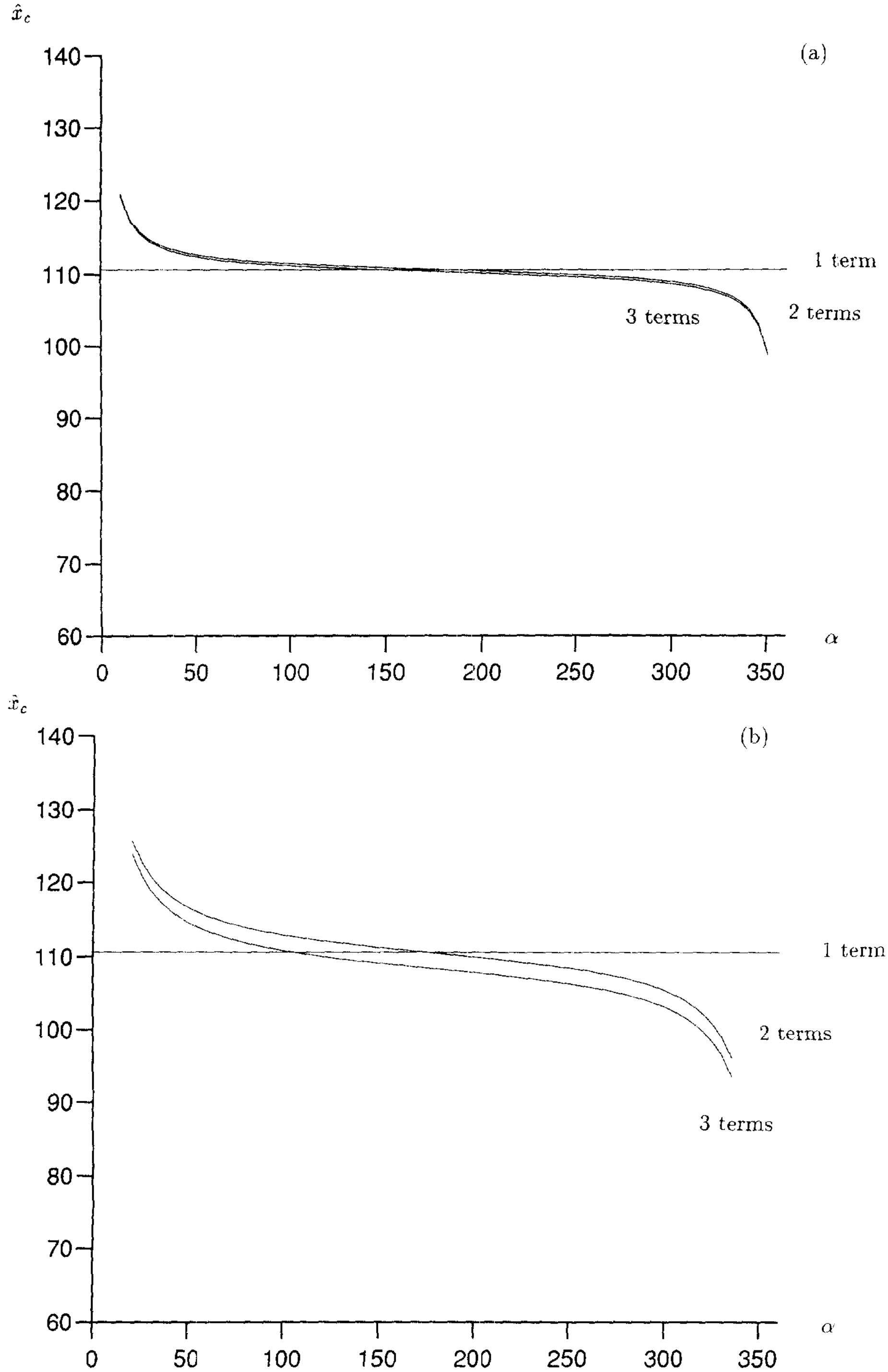

Fig. 2. Variation of the critical distance $\hat{x}_c$ with wedge angle α and the number of terms in the asymptotic series for the basic flow. (a) $\delta = 5^\circ$; (b) $\delta = 15^\circ$; (c) $\delta = 30^\circ$; and (d) $\delta = 60^\circ$.

A different perspective on the above discussion is obtained by the results given in Fig. 3. Here we hold α fixed and vary both the inclination angle and the number of terms used. Three values of α are presented: 90°, 180° and 270°. The progressively worse qualitative nature of the analysis as α increases is readily seen here. For all three cases the three curves corresponding to one, two and three terms in the basic expansion become more widely divergent as δ increases.

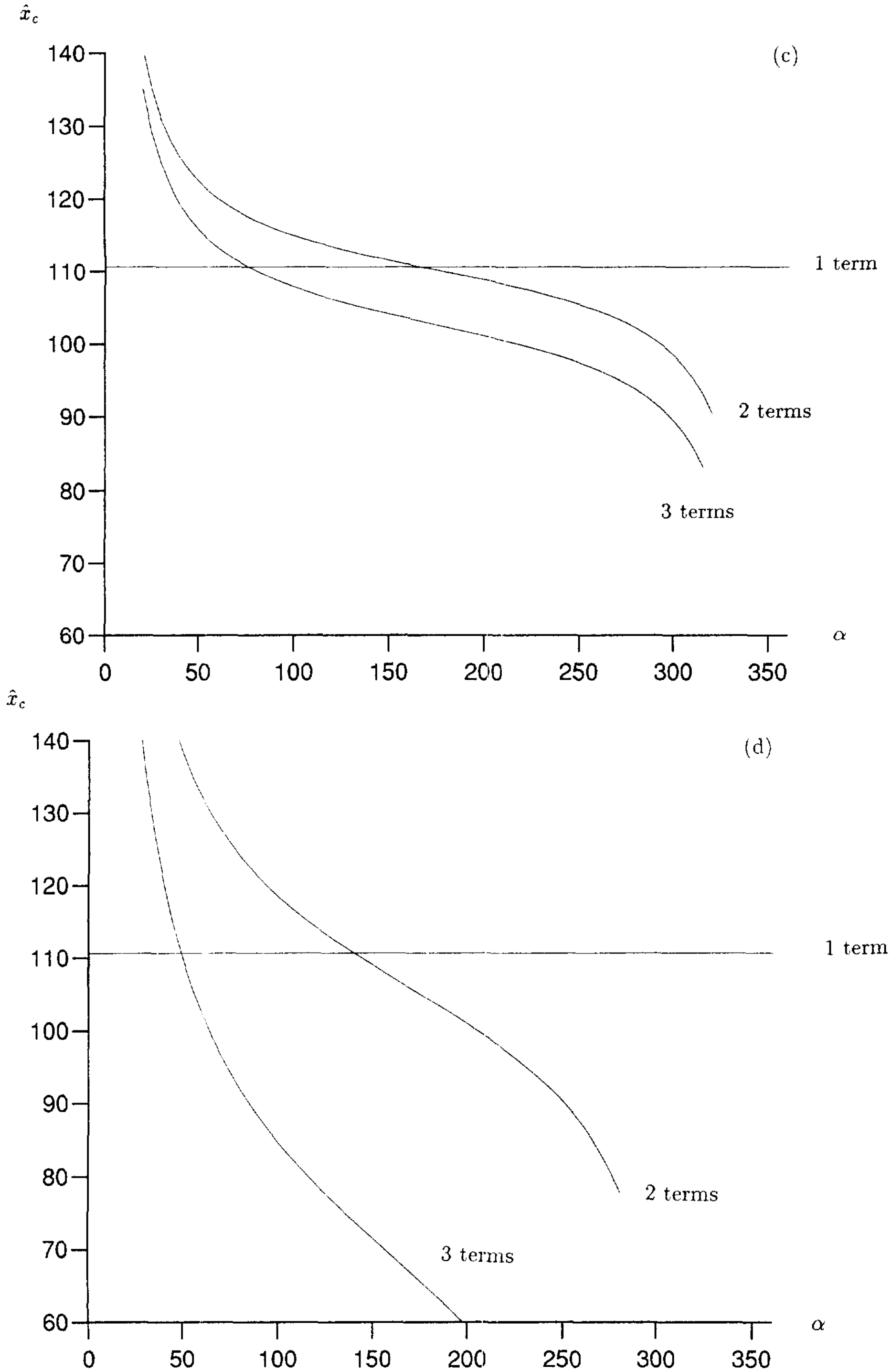

Fig. 2—*continued.*

5. RESULTS FOR THE HORIZONTAL HEATED SURFACE

The above section shows that the stability analysis becomes increasingly unreliable as the heated surface moves closer to the horizontal ($\delta = 90^\circ$). However, when the heated surface is horizontal the basic boundary layer scalings break down and a different similarity variable must be defined (see Hsu, Cheng and Homsy [8] and Rees and Bassom [10]). Therefore, it is necessary to repeat our analysis for the case of a horizontal surface.

The equations for the basic flow are quoted in the appendix, and the equations satisfied by both vortex

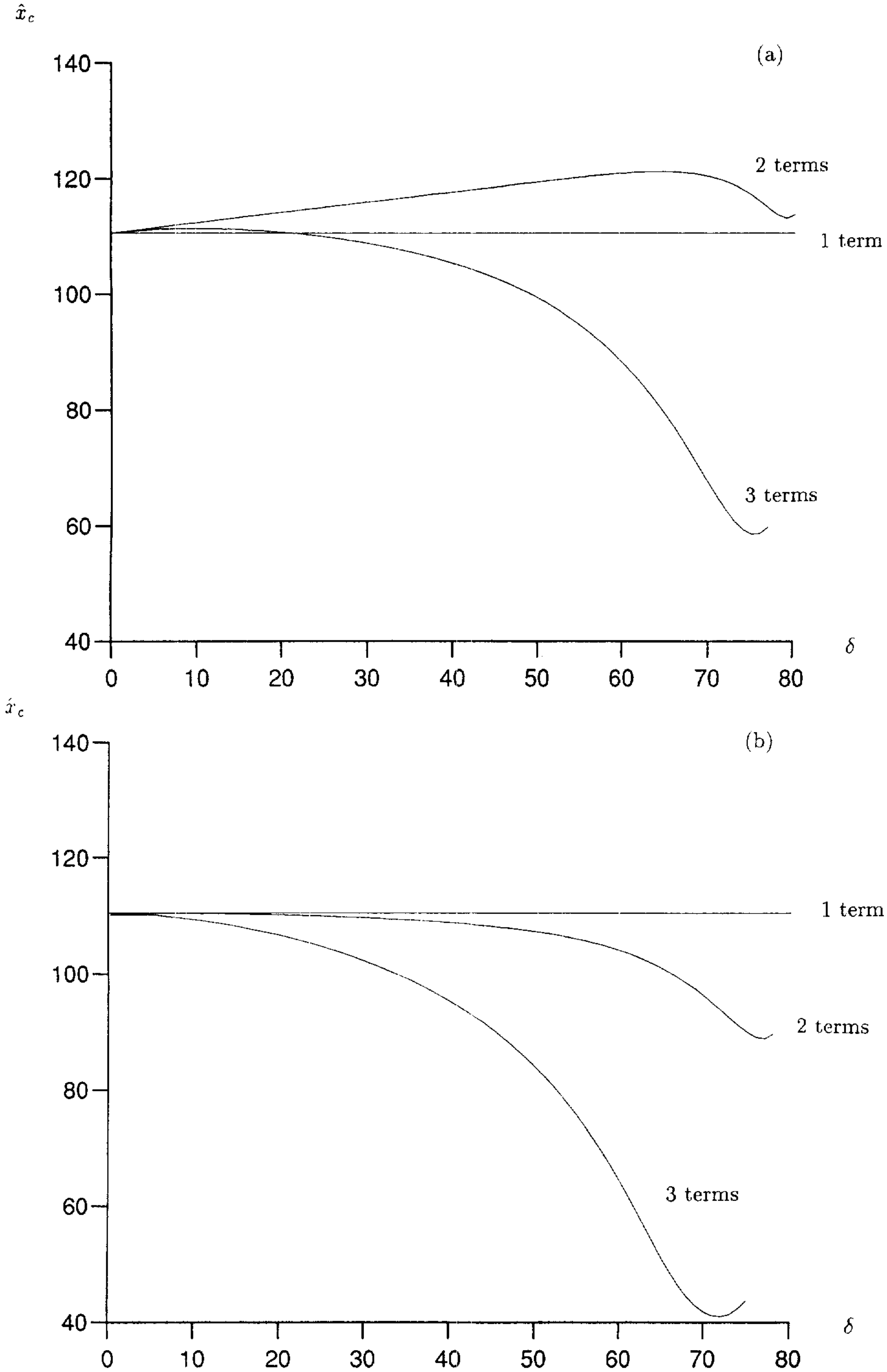

Fig. 3. Variation of the critical distance $\hat{x}_c$ with inclination angle δ and the number of terms in the asymptotic series for the basic flow. (a) $\alpha = 90°$; (b) $\alpha = 180°$; and (c) $\alpha = 270°$.

and wave disturbances are also presented there. Again, the asymptotic series admits eigensolutions the first of which arises at the same point in the expansion as for the inclined case. For the horizontal boundary layer, however, the second term in the asymptotic series can be shown to be zero, and therefore the leading order flow is supplemented only by the third term in the series. When $\alpha = 270°$ the third term is also zero so that the leading order boundary layer flow is the most accurate representation we have for this value of the wedge angle.

Our numerical results are summarised in Figs. 4

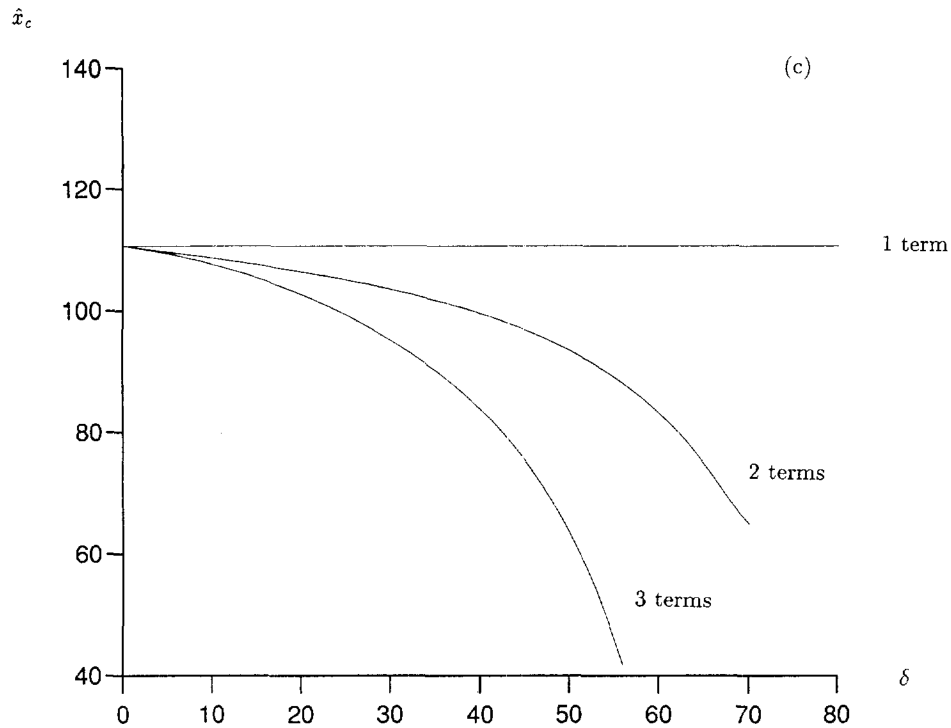

Fig. 3—*continued.*

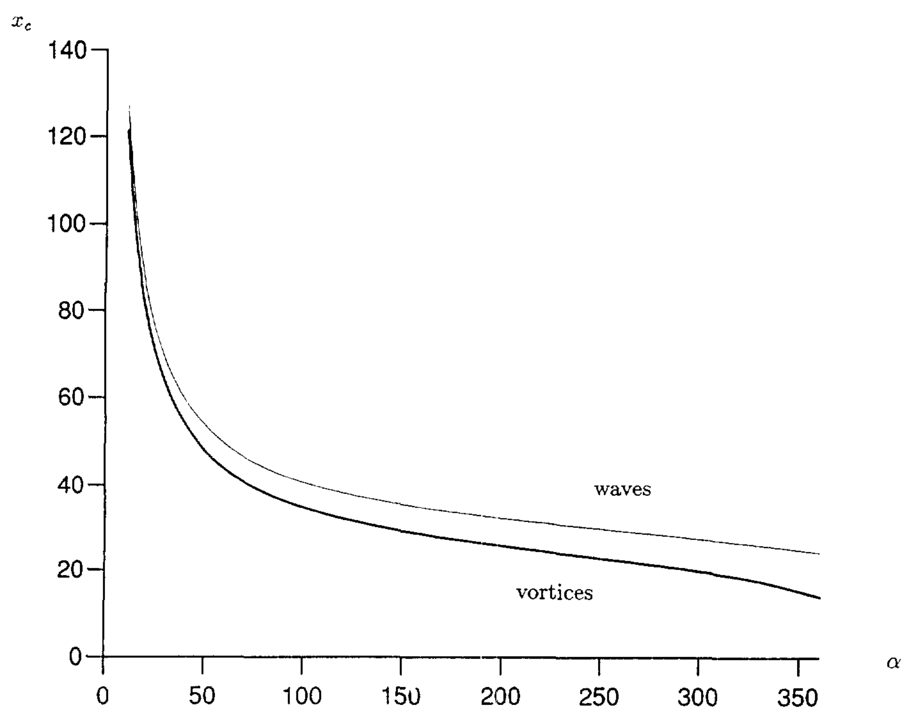

Fig. 4. Variation of $\hat{x}_c$ with α for the horizontal thermal boundary layer. The thick line denotes vortices and the thin line, waves.

and 5. In Fig. 4 we display the variation of x_c with wedge angle for both vortices and waves. In all cases the curve for vortices lies below that for waves and this suggests that vortices might be the preferred mode of instability. However, the wide variation in x_c with α, and in particular the variation relative to the one-

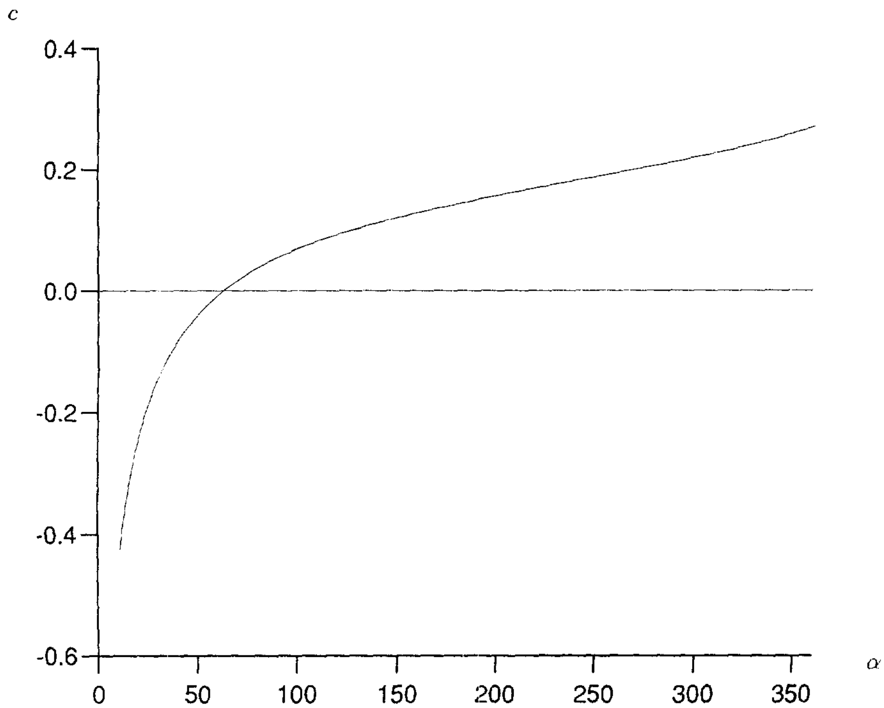

Fig. 5. Variation of the wavespeed c with α for the horizontal thermal boundary layer.

term basic flow result (given by the $\alpha = 270^\circ$ solution), means that we cannot give any firm conclusions about the accuracy of the present results. Partial proof that the basic flow is not sufficiently accurate may be inferred from Fig. 5, which shows the variation of the disturbance wavespeed c with α. Given that the steady boundary layer flow is in the positive x-direction one would expect wave disturbances to have a positive wavespeed. However, when α is less than about 62° the wavespeed is negative, which is an unphysical result. We therefore cannot have confidence in the accuracy of the basic flow for the purposes of determining stability criteria for horizontal thermal boundary layer flow in porous media.

6. CONCLUSION

In this paper we have considered the instability of thermal boundary layer flow in porous media for both inclined and horizontal heated surfaces. Although other analyses exist which study the same problem, the novelty of our approach is that we have undertaken to find a more accurate stability criterion by using asymptotic methods to obtain a better approximation to the basic steady boundary layer flow. However, the results we have presented show that instability occurs too close to the leading edge for the basic flow to be represented adequately either by the leading order boundary layer flow used by previous authors, or even by the higher order theory described here. Thus confident assertions cannot be made about where the boundary layer becomes unstable. The one exception to this rather bleak picture is when the inclined surface is close to the vertical and the wedge angle is not too close to either 0° or 360°; in this case x_c is sufficiently large that higher order corrections to the basic boundary layer flow are small, and the corresponding change in x_c as the number of terms in the asymptotic series increases is also very small.

It is necessary, therefore, that the instability of thermal boundary layers in porous media should be undertaken using more powerful techniques. One possibility lies in the use of direct numerical simulation where the full nonlinear time-dependent equations are solved; the first papers using this technique for porous media convection have appeared only recently [13, 14]. A second possibility would be to compute the exact basic flow and to solve the full (rather than the approximate) linearised disturbance equations.

In view of the difficulties associated with the type of analysis used here, it is also of interest to examine other boundary layer flows, such as the fluid analogue of the present problem, to determine the effect of the outer flowfield and to investigate the validity of previous work.

Acknowledgement—L.S. gratefully acknowledges financial support from the Norwegian Research Council during his sabbatical leave at the University of Bath, Autumn 1994.

REFERENCES

1. Cheng, P. and Chang, I. D., Buoyancy induced flows in a porous medium adjacent to impermeable horizontal

boundaries. *International Journal of Heat and Mass Transfer*, 1976, **19,** 1267–1272.
2. Cheng, P. and Minkowycz, W. J., Free convection about a vertical flat plate imbedded in a porous medium with application to heat transfer from a dike. *Journal of Geophysical Research*, 1977, **82,** 2040–2044.
3. Chang, I. D. and Cheng, P., Matched asymptotic expansions for free convection about an impermeable horizontal surface in a porous medium. *International Journal of Heat and Mass Transfer*, 1983, **26,** 163–174.
4. Daniels, P. G. and Simpkins, P. G., The flow induced by a heated vertical wall in a porous medium. *Quarterly Journal of Mechanics and Applied Mathematics*, 1984, **37,** 339–354.
5. Cheng, P. and Hsu, C. T., Higher order approximations for Darcian free convection about a semi-infinite vertical flat plate. *Transactions of ASME Journal of Heat Transfer*, 1984, **106,** 143–151.
6. Riley, D. S. and Rees, D. A. S., Non-Darcy natural convection from arbitrarily inclined heated surfaces in saturated porous media. *Quarterly Journal of Mechanics and Applied Mathematics*, 1985, **38,** 277–295.
7. Hsu, C. T. and Cheng, P., Effects of upstream geometry on natural convection of a Darcian fluid about a semi-infinite inclined heated surface. *Transactions of ASME Journal of Heat Transfer*, 1985, **107,** 283–292.
8. Hsu, C. T., Cheng, P. and Homsy, G. M., Instability of free convection flow over a horizontal impermeable surface in a porous medium. *International Journal of Heat and Mass Transfer*, 1978, **21,** 1221–1228.
9. Hsu, C. T. and Cheng, P., Vortex instability in buoyancy-induced flow over inclined heated surfaces in porous media. *Transactions of ASME Journal of Heat Transfer*, 1979, **101,** 660–665.
10. Rees, D. A. S. and Bassom, A. P., The linear wave instability of boundary layer flow induced by a horizontal heated surface in porous media. *International Communications in Heat and Mass Transfer*, 1994, **21,** 143–150.
11. Rees, D. A. S. and Storesletten, L., The effect of anisotropic permeability on free convective boundary layers in porous media. *Transport in Porous Media*, 1995, **19,** 79–92.
12. Rees, D. A. S., Convection in fluid-saturated porous media. Ph.D. thesis, School of Mathematics, University of Bristol, U.K., 1985.
13. Rees, D. A. S. and Bassom, A. P., The nonlinear non-parallel wave instability of free convection induced by a horizontal heated surface in fluid-saturated porous media. *Journal of Fluid Mechanics*, 1993, **253,** 267–296.
14. Rees, D. A. S., Nonlinear wave stability of vertical thermal boundary layer flow in a porous medium. *Journal of Applied Mathematics and Physics (Z.A.M.P.)*, 1993, **44,** 306–313.

APPENDIX

In this appendix we present briefly the equations for the basic flow and for both vortex and wave disturbances for flow induced above an isothermal horizontal surface in a porous medium.

When the surface is horizontal, the basic boundary layer expansion equation (5) breaks down, and the following expansion has to be used instead:

$$\psi_0 = x^{1/3}f_0(\eta)+f_1(\eta)+x^{-1/3}\ln x\bar{f}_2(\eta)+x^{-1/3}f_2(\eta)+\cdots \tag{A1}$$

$$\theta_0 = g_0(\eta)+x^{-1/3}g_1(\eta)+x^{-2/3}\ln x\bar{g}_2(\eta)+x^{-2/3}g_2(\eta)+\cdots \tag{A2}$$

where the similarity variable η is given by

$$\eta = \frac{y}{x^{2/3}}. \tag{A3}$$

Note that ψ_0, θ_0, x and y are the same nondimensional terms which appear in equations (2) and (3), but that the coefficient functions and η are different from those used in Sections 2 and 3. Here, the coefficient functions, f_0, g_0, f_1, etc. satisfy ordinary differential equations and boundary conditions analogous to equations (7) and (8); for further details see ref. [12].

For x-independent vortex disturbances the stability equations take the form:

$$\bar{\psi}''-a^2x^{4/3}\bar{\psi} = -iax^{4/3}\bar{\theta} \tag{A4a}$$

$$\begin{aligned}\bar{\theta}''-a^2x^{4/3}\bar{\theta} = &-ia[x^{2/3}g_0'+x^{1/3}g_1'+\ln x\bar{g}_2']\bar{\psi}+b\bar{\theta}\\ &-[\tfrac{1}{3}(f_0-2\eta f_0')-\tfrac{2}{3}x^{-1/3}\eta f_1'\\ &+\tfrac{1}{3}x^{-2/3}(3\bar{f}_2-\ln x(\bar{f}_2+2\eta\bar{f}_2'))]\bar{\theta}'.\end{aligned} \tag{A4b}$$

For wave disturbances we perturb about the basic profile in the usual way by setting

$$\psi = \psi_0+\bar{\psi}(y)e^{ia(x-ct)} \tag{A5a}$$

$$\theta = \theta_0+\bar{\theta}(y)e^{ia(x-ct)} \tag{A5b}$$

where $\bar{\psi}$ and $\bar{\theta}$ are infinitesimally small, a is the wavenumber, and c is the wavespeed. For neutral stability the imaginary part of c is zero. The wave disturbance equations are given by:

$$\bar{\psi}''-a^2x^{4/3}\bar{\psi} = -iax^{4/3}\bar{\theta} \tag{A6a}$$

$$\begin{aligned}\bar{\theta}''-a^2x^{4/3}\bar{\theta} = &-ia[x^{2/3}g_0'+x^{1/3}g_1'+\ln x\bar{g}_2']\bar{\psi}\\ &+ia[-cx^{4/3}+xf_0'+x^{2/3}f_1'+x^{1/3}\ln x\bar{f}_2']\bar{\theta}\\ &-[\tfrac{2}{3}\eta g_0'+\tfrac{1}{3}x^{-2/3}(g_1+2\eta g_2')\\ &-\tfrac{1}{3}x^{-1}(3\bar{g}_2-2\ln x(\bar{g}_2+\eta\bar{g}_2'))]\bar{\psi}'\\ &+[\tfrac{1}{3}(f_0-2\eta f_0')-\tfrac{2}{3}x^{-1/3}\eta f_1'\\ &+\tfrac{1}{3}x^{-2/3}(3\bar{f}_2-\ln x(\bar{f}_2+2\eta\bar{f}_2'))]\bar{\theta}'.\end{aligned} \tag{A6b}$$

Once more, equations (A4) and (A6) need to be supplemented by those obtained by differentiating with respect to the wavenumber in order to find the value of x_c for vortices and waves, respectively.