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Heat transfer response of free convection flow from a vertical heated plate to an oscillating surface heat flux

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Summary. An investigation is undertaken of the unsteady response of two-dimensional laminar free convection boundary layer flow of a viscous incompressible fluid along a semi-infinite vertical heated plate where the mean surface heat flux oscillates with a small amplitude about a steady profile. The buoyancy forces are favourable, resulting from a positive flux of heat from the surface of the plate into the fluid. The interaction of the time-periodic heat flux with the usual boundary-layer flow is examined by using a linearized theory. Solutions are obtained using three distinct methods, namely an extended series expansion method for low frequencies, an asymptotic series expansion method for high frequencies and a fully numerical finite difference method for general frequencies. Calculations have been carried out for a wide range of parameters to examine the solutions in terms of the amplitude and phase angle of the fluctuating parts of the surface shear stress and the surface temperature. It has been found that the amplitude and phase angle of both the shear stress and the surface temperature predicted by these three methods are in very good agreement in their respective ranges of validity.

1 Introduction

In the area of laminar boundary layer theory, Lighthill [1] was the first to study the unsteady forced flow of a viscous incompressible fluid past both a flat plate and a circular cylinder with the free stream having small amplitude oscillations. The similarity solutions for free convection flow from a vertical plate with a uniform surface heat flux was originally studied by Sparrow and Gregg [2] who obtained solutions valid near the leading edge. The corresponding problem of unsteady free convection flow along a vertical plate with an oscillating surface temperature was studied by Nanda and Sharma [3] and Eshghy et al. [4]. Muhuri and Maiti [5] and Verma [6] have analyzed the effect of an oscillating surface temperature on the unsteady free convection flow from a horizontal plate. All of these investigations were based on the assumption that the surface temperature executes temporal oscillations with a small amplitude about the uniform mean temperature, and they were carried out by employing the Karman-Pohlhausen approximate integral method. To obtain perturbation solutions valid near the leading edge and in the far down-stream region Roy [7] considered the same type of problem for high Prandtl numbers and Wilks [8] studied free convection flow over a vertical plate with a uniform surface heat flux. The case of prescribed surface heat flux on a vertical plate was studied by Merkin and Mahmood [9], Merkin et al. [10] and Chaudhary and Merkin [11]. These works were confined to steady flow, and solutions were obtained which are valid at large distances from the leading edge, and solutions were also given for intermediate regions. Based on a linearized theory, Kelleher and

Yang [12] have studied the heat transfer responses of a laminar free convection boundary layer along a vertical heated plate to surface temperature oscillations and, recently, Hossain et al. [13] have investigated the same type of problem in detail, where the mean surface temperature, $\theta_w(x)$, is proportional to x^n, where x is the streamwise distance from the leading edge of the plate. Hossain et al. [13] presented solutions in terms of the amplitude and phase of the surface heat transfer rate for small and large values of ξ, the streamwise distribution of frequency of oscillation. An attempt to match the low and high-frequency oscillations was also made.

The corresponding case of the free convection boundary layer flow over a vertical heated plate with a non-uniform surface heat flux has not been treated previously, and this is what is considered here. It should be noted that, since non-uniform surface heat flux variations are more likely to occur physically than uniform surface heat fluxes, it is important to determine the extent to which a non-uniform surface heat flux will affect the boundary layer response. In the present paper we consider an unsteady free convection flow of a viscous incompressible fluid along a vertical heated plate when the surface heat flux of the plate oscillates with a small amplitude about a mean flux which itself varies as the power of n of the distance from the leading edge. We investigate this general problem by employing (i) an extended series expansion method in the low-frequency range, (ii) an asymptotic series expansion method in the high-frequency range, and (iii) a finite difference method to find the nature of solutions for general frequencies. Calculations have been carried out for a wide range of the parameters to determine the effects of unsteadiness on the surface temperature and shear stress.

2 Mathematical formulation

In a Cartesian coordinate system a semi-infinite vertical plate is placed at $y = 0$, $x \geqq 0$ so that x measures the distance along the plate from the leading edge, and y is measured normally outwards from the plate into the fluid. Far from the surface the ambient fluid temperature is T_∞. Favourable buoyancy forces arise as a result of a positive surface heat flux, q_w, from the plate. Under the Boussinesq approximation, the usual Navier-Stokes and energy equations for two dimensional incompressible flow, for the case where the surface rate of heat flux is time-dependent, reduce to the following boundary layer equations (Kramer and Pai [14]):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty), \tag{2}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

where u, v are respectively the x and y-components of the velocity field, ν is the kinematic viscosity, T and T_∞ being the temperature of the fluid in the boundary layer and the temperature of the ambient fluid, g is the acceleration due to gravity, β is the coefficient of volume expansion, and α is the thermal diffusivity.

Equations (1) – (3) are to be solved subject to the boundary conditions

$$\left.\begin{array}{l} y = 0\colon\ u = 0,\ v = 0,\ -k\partial T/\partial y = q_w(x)\,[1 + \varepsilon\cos\omega t] \\ y \to \infty\colon\ u \to 0,\ T \to T_\infty \end{array}\right\}, \tag{4}$$

where ω is the frequency of oscillation of the surface heat flux of the plate and $\varepsilon \ll 1$ is a measure of its amplitude.

The boundary conditions (4) suggest that the solutions of Eqs. (1) – (3) may be found as the real parts of the following expressions (Ishigaki [15]):

$$u = u_0 + \varepsilon \exp(i\omega t)\, u_1,$$
$$v = v_0 + \varepsilon \exp(i\omega t)\, v_1, \tag{5}$$
$$T - T_\infty = q_w(x)\,(\bar{\theta}_0 + \varepsilon \exp(i\omega t)\, \theta_1),$$

where the components u_0, v_0 and $\bar{\theta}_0$ represent the basic steady mean flow satisfying the differential equations

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0, \tag{6}$$

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} = g\beta q_w(x)\, \bar{\theta}_0 + \nu \frac{\partial^2 u_0}{\partial y^2}, \tag{7}$$

$$\frac{\partial \bar{\theta}_0}{\partial t} + u_0 \frac{\partial \bar{\theta}_0}{\partial x} + v_0 \frac{\partial \bar{\theta}_0}{\partial y} = \alpha \frac{\partial^2 \bar{\theta}_0}{\partial y^2}, \tag{8}$$

with the boundary conditions

$$u_0 = 0,\ v_0 = 0,\ \bar{\theta}_0{}' = -q_w(x) \ \text{at}\ y = 0,$$
$$u_0 \to 0,\ \bar{\theta}_0 \to 0 \ \text{as}\ y \to \infty, \tag{9}$$

and u_1, v_1 and θ_1 are the components of the unsteady flow, which satisfy the differential equations

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \tag{10}$$

$$u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_1}{\partial y} + v_1 \frac{\partial u_0}{\partial y} + i\omega u_1 = g\beta q_w(x)\, \theta_1 + \nu \frac{\partial^2 u_1}{\partial y^2}, \tag{11}$$

$$u_0 \frac{\partial \theta_1}{\partial x} + u_1 \frac{\partial \bar{\theta}_0}{\partial x} + v_0 \frac{\partial \theta_1}{\partial y} + v_1 \frac{\partial \bar{\theta}_0}{\partial y} + i\omega\theta_1 = \alpha \frac{\partial^2 \theta_1}{\partial y^2}, \tag{12}$$

subject to the boundary conditions

$$u_1 = 0,\ v_1 = 0,\ \theta'_1 = -q_w(x) \quad \text{at} \quad y = 0,$$
$$u_1 \to 0,\ \theta_1 \to 0 \quad \text{as} \quad y \to \infty. \tag{13}$$

In order to get the similarity equations for the steady state equations (6) – (8), we introduce the following group of transformations:

$$\psi_0 = C_2 x^{4/5} F(\eta),\quad \bar{\theta}_0 = q_w(x)\, \Theta(\eta),\quad \eta = \frac{C_1 y}{x^{1/5}},\quad q_w(x) = q_0 x^n, \tag{14.1}$$

where

$$C_1 = \left(\frac{g\beta q_w}{k\nu^2}\right)^{1/5},\quad C_2 = \left(\frac{g\beta q_w \nu^3}{k}\right)^{1/5}. \tag{14.2}$$

In the above $\psi_0(x, y)$ satisfies the equation of continuity for the steady state flow, and q_0 is a constant related to the mean surface heat flux.

Thus we obtain the equations

$$F''' + \frac{(n+4)}{5} FF'' - \frac{(2n+3)}{5} F'^2 + \Theta = 0, \tag{15}$$

$$\frac{1}{Pr}\Theta'' + \frac{(n+4)}{5} F\Theta' - \frac{(4n+1)}{5} F'\Theta = 0. \tag{16}$$

The boundary conditions to be satisfied by the above equations are

$$F(0) = F'(0) = 0, \quad \Theta'(0) = -1,$$

$$F'(\infty) = 0, \quad \Theta(\infty) = 0. \tag{17}$$

Here primes denote differentiation with respect to η, and $Pr = \nu/\alpha$ is the Prandtl number.

The transformations (14) lead us to the following group of transformations for Eqs. (10) – (13) for the fluctuating part of the problem:

$$\psi_1 = C_2 x^{4/5} f(\eta, \xi), \quad \theta_1 = q_w(x)\, \theta(\eta, \xi), \; \xi = \omega\left(\frac{kx}{g\beta q_w \nu}\right)^{2/5}. \tag{18}$$

Equation (11) and (12) then reduce to

$$f''' + \frac{(n+4)}{5} Ff'' - \left[i\xi + \frac{(4n+6)}{5} F'\right] f' + \frac{(n+4)}{5} F'' f + \theta$$

$$= \frac{2(1-n)}{5}\,\xi\left[F'\frac{\partial f'}{\partial \xi} - F''\frac{\partial f}{\partial \xi}\right], \tag{19}$$

$$\frac{1}{Pr}\theta'' + \frac{(n+4)}{5} F\theta' - \left[i\xi + \frac{(4n+1)}{5} F'\right]\theta + \frac{(n+4)}{5}\Theta' f - \frac{(4n+1)}{5}\Theta f'$$

$$= \frac{2(1-n)}{5}\,\xi\left[F'\frac{\partial \theta}{\partial \xi} - \Theta'\frac{\partial f}{\partial \xi}\right], \tag{20}$$

with the boundary conditions

$$\left.\begin{aligned} &f(\xi, 0) = f'(\xi, 0) = 0, \quad \theta'(\xi, 0) = -1, \\ &f'(\xi, \infty) = \theta(\xi, \infty) = 0. \end{aligned}\right\} \tag{21}$$

The unsteady shear stress and surface temperature are important quantities to find, and these may be obtained from the solutions of Eqs. (15) – (17) and (19) – (21). Here we present the solutions in terms of amplitude and phase of the shear stress and the rate of surface heat flux. These are defined according to

$$A_u = \sqrt{(f_r'')^2 + (f_i'')^2}\,|_{\eta=0}, \quad A_t = \sqrt{(\theta_r)^2 + (\theta_i)^2}\,|_{\eta=0}, \tag{22}$$

and

$$\phi_u = \tan^{-1}\left(\frac{f_i''}{f_r''}\right)|_{\eta=0}, \quad \phi_t = \tan^{-1}\left(\frac{\theta_i}{\theta_r}\right)|_{\eta=0}, \tag{23}$$

where (f_r, f_i) and (θ_r, θ_i) represent the real and imaginary part of $f(\xi, \eta)$ and $\theta(\xi, \eta)$ respectively.

Equations (15) – (17) describe the steady mean flow and temperature field. The solutions of these equations have already been obtained by Chaudhary and Merkin [11] for different values of the associated physical parameters Pr and n. Equations (19) – (21) describe the $O(\varepsilon)$ fluctuating component of the solution, and these shall be solved by a variety of methods. In Section 3 we detail the solution for small values of ξ using a series solution. Given that ξ is proportional to both ω and $x^{2/5}$, see (18), such a series solution is valid both for small values of x while $\omega = O(1)$, and for very low frequencies ($\omega \ll 1$) with $x = O(1)$. Section 4 discusses the asymptotic solution for large values of ξ – again this could be interpreted either as a large distance limit or as a high frequency limit. Solutions for intermediate values of ξ, where the equations are, in general, nonsimilar, were obtained using the Keller-box method (see [17]). Details of the implementation of this method are now quite standard and have been discussed in [13]. It is also worth noting that, when $n = 1$, Eqs. (19) and (20) reduce to a pair of linear ordinary differential equations where solutions may be obtained via a straightforward shooting method.

3 Series solution for small ξ

Clearly, a description of the effect of unsteadiness on the flow near the leading edge using results based on a finite number of terms in the series will only be valid in a very small range of frequencies. Since small values of ξ also correspond to very low frequencies, ω, we expect the flow to adjust quasi-statically to the fluctuating rate of heat transfer at the boundary. We expand f and θ according to

$$f(\xi,\eta) = \sum_{m=0}^{\infty} (2i\xi)^m f_m(\eta), \quad \theta(\xi,\eta) = \sum_{m=0}^{\infty} (2i\xi)^m \theta_m(\eta). \tag{24}$$

Substituting these into Eqs. (19) – (20) and then equating the terms of like powers of $(2i\xi)$ to zero, we obtain the following pairs of ordinary differential equations for the functions f_m and θ_m:

$$f_0''' + \frac{(n+4)}{5} F f_0'' - \frac{(4n+6)}{5} F' f_0' + \frac{(n+4)}{5} F'' f_0 + \theta_0 = 0, \tag{25}$$

$$\frac{1}{Pr}\theta_0'' + \frac{(n+4)}{5} F\theta'_0 - \frac{(4n+1)}{5}\theta_0 F' - \frac{(4n+1)}{5}\Theta f'_0 + \frac{(n+4)}{5}\Theta' f_0 = 0, \tag{26}$$

$$\begin{aligned} f_m''' &+ \frac{(n+4)}{5} F f_m'' + \left[\frac{2m(n-1)}{5} - \frac{(4n+6)}{5}\right] F' f_m' \\ &+ \left[\frac{(n+4)}{5} - \frac{2m(n-1)}{5}\right] F'' f_m + \theta_m = \frac{1}{2} f_{m-1}', \end{aligned} \tag{27}$$

$$\begin{aligned} \frac{1}{Pr}\theta_m'' &+ \frac{(n+4)}{5} F\theta_m' + \left[\frac{2m(n-1)}{5} - \frac{(4n+1)}{5}\right] F'\theta_m - \frac{(4n+1)}{5}\Theta f'_m \\ &+ \left[\frac{(n+4)}{5} - \frac{2m(n-1)}{5}\right]\Theta' f_m = \frac{1}{2}\theta_{m-1}, \end{aligned} \tag{28}$$

where $m = 1, 2, 3, \ldots$. The respective boundary conditions are

$$\left.\begin{aligned} &f_0(0) = f'_0(0) = 0, \quad \theta'_0(0) = -1, \quad f'_0(\infty) = \theta_0(\infty) = 0, \\ &f_m(0) = f_m'(0) = \theta_m(0) = 0, \quad f_m'(\infty) = \theta_m(\infty) = 0, \end{aligned}\right\} \tag{29}$$

where primes again refer to derivatives with respect to η.

It can be seen that Eqs. (25) – (28) are linear, but coupled, and may be solved independently pairwise one after another. In the present analysis, the implicit Runge-Kutta-Butcher [18] initial value problem solver together with the Nachtsheim-Swigert [19] iteration scheme is employed to solve the system of Eqs. (25) – (28) up to $O(\xi^{10})$. Here the Pade's approximant [20] is also used to obtain a more accurate approximation for the local amplitude and phase of the fluctuating parts of both the shear stress and the surface heat flux. Detailed numerical results are discussed in Section 5.

4 Asymptotic solution for large ξ

Away from the leading edge buoyancy forces become increasingly important until far down-stream the flow will be predominantly one of free convection perturbed only slightly by the presence of the free stream. Therefore, in this Section attention has been given to the behaviour of the solutions to Eqs. (19) and (20) when ξ is large. Again, we emphasize that this limit corresponds either to large values of x for finite ω or to large values of ω when x is finite. Actually, a detailed examination of the numerical results obtained by means of the Keller box method shows that, for large values of ξ, the unsteady response is confined to a thin region adjacent to the surface. We note, however, that this conclusion may not be true at higher orders in the ε-expansion. Therefore we seek a series solution in the high frequency range, utilizing the limiting solution as the zeroth-order approximation. For this reason, the following transformations are introduced:

$$Y = \xi^{1/2}\eta, \quad \varphi(\xi, Y) = \xi^{3/2} f(\xi, \eta), \quad \theta(\xi, Y) = \theta(\xi, \eta). \tag{30}$$

These scalings were motivated by an order-of-magnitude analysis of (19).

Equations (19) and (20) then become

$$\begin{aligned} &\frac{\partial^3\varphi}{\partial Y^3} + \frac{(n+4)}{5} F\xi^{-1/2}\frac{\partial^2\varphi}{\partial Y^2} - i\frac{\partial\varphi}{\partial Y} \\ &\quad - \frac{(6n+4)}{5} F'\xi^{-1}\frac{\partial\varphi}{\partial Y} + \frac{(4n+1)}{5} F''\xi^{-3/2}\varphi + \theta \\ &\quad = \frac{2(n-1)}{5}\left[F'\left(\frac{\partial^2\varphi}{\partial Y\partial\xi} + \frac{Y}{2\xi}\frac{\partial^2\varphi}{\partial Y^2}\right) - F''\xi^{-1/2}\left(\frac{\partial\varphi}{\partial\xi} + \frac{Y}{2\xi}\frac{\partial\varphi}{\partial Y}\right)\right], \end{aligned} \tag{31}$$

and

$$\begin{aligned} &\frac{1}{Pr}\frac{\partial^2\theta}{\partial Y^2} + \frac{(n+4)}{5} F\xi^{-1/2}\frac{\partial\theta}{\partial Y} - i\theta \\ &\quad - \frac{(4n+1)}{5} F'\xi^{-1}\theta - \frac{(4n+1)}{5}\Theta\xi^{-2}\frac{\partial\varphi}{\partial Y} + \frac{(4n+1)}{5}\Theta'\xi^{-5/2}\varphi \\ &\quad = \frac{2(1-n)}{5}\left[F'\left(\frac{\partial\theta}{\partial\xi} + \frac{Y}{2\xi}\frac{\partial\theta}{\partial Y}\right) - \Theta'\xi^{-3/2}\left(\frac{\partial\varphi}{\partial\xi} + \frac{Y}{2\xi}\frac{\partial\varphi}{\partial Y}\right)\right], \end{aligned} \tag{32}$$

respectively. Since these equations correspond to a thin near-wall layer, the leading order functions, F and Θ, in this region can be represented with good accuracy by the following power series:

$$F = a_2\eta^2 + a_3\eta^3 + a_4\eta^4 + a_5\eta^2 + \ldots, \tag{33}$$

$$\Theta = b_1\eta + b_2\eta^2 + b_3\eta^3 + b_4\eta^4 + b_5\eta^5 + \ldots, \tag{34}$$

where, according to Eqs. (15) – (17)

$$a_2 = \frac{1}{2} F''(0), \ldots, b_1 = \Theta'(0), \ldots.$$

Based on the above expansions, solutions to Eqs. (31) and (32) may be obtained in the following form:

$$f(\xi, Y) = \sum_{m=0}^{\infty} \xi^{-m/2} \bar{f}_m(Y), \quad \theta(\xi, Y) = \sum_{m=0}^{\infty} \xi^{-m/2} \bar{\Theta}_m(Y). \tag{35}$$

When Eqs. (35) are substituted into Eqs. (31) and (32) and terms of like powers of ξ are collected, one obtains

$$\bar{f}_0''' - i\bar{f}_0' = -\bar{\Theta}_0, \tag{36.1}$$

$$\bar{f}_1''' - i\bar{f}_1' = -\bar{\Theta}_1, \tag{36.2}$$

$$\bar{f}_2''' - i\bar{f}_2' = -\bar{\Theta}_2, \tag{36.3}$$

$$\bar{f}_3''' - i\bar{f}_3' = -\frac{(3n+1)}{5} a_2 Y^2 \bar{f}_0'' + \frac{2(7n+3)}{5} a_2 Y \bar{f}'_0 - \frac{2(4n+1)}{5} a_2 \bar{f}_0 - \bar{\Theta}_3, \tag{36.4}$$

and

$$\frac{1}{Pr} \bar{\Theta}_0'' - i\bar{\Theta}_0 = 0, \tag{37.1}$$

$$\frac{1}{Pr} \bar{\Theta}_1'' - i\bar{\Theta}_1 = 0, \tag{37.2}$$

$$\frac{1}{Pr} \bar{\Theta}_2'' - i\bar{\Theta}_2 = 0, \tag{37.3}$$

$$\frac{1}{Pr} \bar{\Theta}_3'' - i\bar{\Theta}_3 = -\frac{(3n+2)}{5} a_2 Y^2 \bar{\Theta}'_0 + \frac{(8n+2)}{5} a_2 Y \bar{\Theta}_0, \tag{37.4}$$

where primes now denote differentiation with respect to Y. The associated boundary conditions are

$$\left.\begin{array}{l} \bar{f}_m(0) = \bar{f}_m'(0) = \bar{f}_m'(\infty) = 0, \quad \text{for} \quad m = 0, 1, 2, 3, 4, \ldots \\ \bar{\Theta}'_0(0) = -1, \quad \bar{\Theta}_m(0) = \bar{\Theta}_m(\infty) = 0, \quad \text{for} \quad m = 1, 2, 3, 4, \ldots \end{array}\right\}. \tag{38}$$

Now solving Eqs. (36) and (37), subject to the boundary conditions (38), we find the following expressions for $f''(\xi, 0)$ and $\theta(\xi, 0)$:

$$f''(\xi, 0) = \left(\frac{i}{1 + \sqrt{Pr}}\right) \xi^{-1/2} + (A_{10} + A_{11} + A_{12})\, \xi^{-5/2} + O(\xi^{-7/2}) \tag{39}$$

and

$$\theta(\xi, 0) = \frac{1}{s\sqrt{Pr}} \xi^{-1/2} + \frac{(11n+4)\, a_2}{20Pr} \xi^{-5/2} + O(\xi^{-7/2}) \tag{40}$$

where $s = \sqrt{i}$ is evaluated in the first quadrant (i.e. $s = (1 + i)/\sqrt{2}$), and

$$P_0 = \frac{1}{\sqrt{Pr}(Pr - 1)}, \quad A_0 = -\frac{1}{Pr(Pr - 1)} - P_0, \quad A_1 = -\frac{(11n + 4)\, a_2}{20Pr}, \quad A_2 = \frac{(3n + 2)\sqrt{Pr}a_2}{10s},$$

$$A_3 = -\frac{s(8n + 2)\, a_2}{10}, \quad A_4 = -\frac{(3n + 1)\, a_2}{5}, \quad A_5 = \frac{2(7n + 3)\, a_2}{5}, \quad A_6 = -\frac{(8n + 2)\, a_2}{5},$$

$$A_7 = \frac{sA_1}{Pr - 1} + \frac{4A_2\sqrt{Pr}(3Pr + 1)}{s(Pr - 1)^4}, \quad A_8 = \frac{8A_3Pr}{(Pr - 1)^3} - \frac{2sA_4(3Pr + 1)}{(Pr - 1)^4}, \quad A_9 = \frac{2sA_5}{(Pr - 1)^3}$$

$$- sA_6A_0 - \frac{sA_6}{Pr(Pr - 1)^2}, \quad A_{10} = -sP_1 + \frac{A_1\sqrt{Pr}}{s(Pr - 1)} + \frac{A_2(15Pr^3 + 19Pr^2 - 3Pr + 1)}{2Pr(Pr - 1)^4},$$

$$P_1 = A_7 + A_8 + A_9, A_{11} = \frac{sA_3(11Pr^2 + 6Pr - 1)}{2\sqrt{Pr}(Pr - 1)^3} - \frac{A_4P_0}{4s} - \frac{2sA_4\sqrt{Pr}(3Pr + 1)}{(Pr - 1)^4}$$

$$- \frac{4A_4\sqrt{Pr}}{s(Pr - 1)^3}, \quad A_{12} = \frac{A_5P_0}{4s} - \frac{A_5(Pr + 1)}{s\sqrt{Pr}(Pr - 1)^3} - \frac{A_6P_0}{2s} + \frac{A_6}{s\sqrt{Pr}(Pr - 1)^2}.$$

It is to be noted that the complex expressions (39) and (40) are valid for all Prandtl numbers, $Pr \neq 1$. However, when calculations for $Pr = 1$ are needed, it is necessary to take the limits of these solutions as $Pr \to 1$.

5 Results and discussion

In the present analysis, solutions for the fluctuating free convection flow of a viscous incompressible fluid along a vertical heated surface with a small amplitude oscillation in the surface heat flux about a non-uniform steady heat-flux which varies in power of the distance measured from the leading edge are investigated. Solutions of the (leading order) steady part of the problem have already been discussed by Chaudhary and Merkin [11]. The fluctuating part of the problem has been analyzed by the Keller box method in the entire frequency regime. The solutions have also been obtained in the small frequency regime using the perturbation method and in the large frequency regime by an asymptotic method. The results thus obtained are expressed in terms of amplitude and phase of the fluctuating parts of shear stress as well as those of surface temperature showing the effects of varying both the surface heat flux-gradient parameter n and the Prandtl number, Pr. Here, some of the values of the Prandtl number are chosen to represent the fluid as a liquid metal which is currently used as coolant in nuclear engineering (Wilks [8]), e.g., 0.05 for lithium and 0.01 for mercury. We have also obtained solutions for $Pr = 1.0, 0.7, 0.1, 0.05, 0.01$.

Numerical values of the amplitude and phase of fluctuating parts of the shear stress and the surface temperature obtained by the three methods mentioned above are given in Tables 1 and 2 for $Pr = 1.0$ and $n = 1.0$. The comparison shows that the perturbation solutions and the asymptotic solutions are in excellent agreement with finite difference solutions. For $n = 0.0, 0.25, 0.5$ and 0.75 with $Pr = 0.7$, which corresponds to air, the numerical values obtained by the methods mentioned above for the amplitude and phase of the fluctuating shear stress are presented graphically in Figs. 1 and 2. The corresponding values of the amplitude and phase of the fluctuating surface temperature are shown in Figs. 3 and 4. In these figures the thick curves

Table 1. Amplitude and phase of the fluctuating part of the shear stress for $Pr = 1.0$ and $n = 1.0$

ξ	A_u		$-\phi_u$	
	Series and asymp.	Keller	Series and asymp.	Keller
.00	.6061[a]	.6059	.0000[a]	.0000
.10	.6040[a]	.6063	5.3640[a]	5.3637
.20	.5980[a]	.5979	10.6776[a]	10.7173
.30	.5883[a]	.5904	15.8945[a]	15.8893
.40	.5752[a]	.5752	20.9769[a]	21.0539
.50	.5591[a]	.5615	25.8986[a]	25.8855
.60	.5395[a]	.5412	30.6509[a]	30.7364
.70	.5153[a]	.5231	35.2556[a]	35.1104
.80	.4827[a]	.4998	39.7982[a]	39.5363
.90	.4346[a]	.4793	44.5220[a]	43.3488
1.00	.3588[a]	.4551	50.1771[a]	47.2693
1.10	.2366[a]	.4343	60.1781[a]	50.4661
1.20	.0814[a]	.4109	55.9940[a]	53.8628
1.30	.3369[a]	.3912	31.8783[a]	56.4534
2.00		.2730		59.6674
3.00	.3967[b]	.1812	62.2827[b]	62.7032
4.00	.2329[b]	.1336	67.3015[b]	67.3071
5.00	.1600[b]	.1054	71.2366[b]	71.2458
6.00	.1204[b]	.0869	74.2890[b]	74.2815
7.00	.0962[b]	.0739	76.6676[b]	76.6675
8.00	.0800[b]	.0643	78.5420[b]	78.5421
9.00	.0684[b]	.0569	80.0388[b]	80.0386
10.00	.0598[b]	.0511	81.2505[b]	81.2508
15.00	.0368[b]	.0338	84.8519[b]	84.8120
20.00	.0267[b]	.0252	86.5420[b]	86.5420
25.00	.0209[b]	.0202	87.4810[b]	87.4805
30.00	.0173[b]	.0168	88.0628[b]	88.0628
40.00	.0128[b]	.0126	88.7264[b]	88.7264
50.00	.0102[b]	.0101	89.0828[b]	89.0828
60.00	.0084[b]	.0084	89.2996[b]	89.2997
70.00	.0072[b]	.0072	89.4427[b]	89.4427
75.00	.0067[b]	.0068	89.4970[b]	89.4879

[a] Series solution for low frequency
[b] Asymptotic solution for high frequency

represent the Keller box solutions, the circled and solid circled curves represent the perturbation and asymptotic solutions, respectively. As before, the comparisons between these curves ascertained that the perturbation solutions and the asymptotic solutions are in excellent agreement with the Keller box solutions at every selected value of the surface heat flux exponent. Figures 5 – 6 and 7 – 8 represent the numerical values of amplitude and phase, respectively, of the fluctuating parts of the shear stress as well as the surface temperature at $Pr = 0.7, 0.1, 0.05$, and 0.01 while $n = 0.5$.

From Table 1 and Table 2, we may see that the amplitude of the shear stress and the surface heat transfer decrease monotonically as the frequency increases regardless of the surface temperature-gradient for the prescribed Prandtl number (which can be seen from Figs. 1 and 3). This is due to the temperature lag in the fluid layer adjacent to the surface, the amount of lag varying with the frequency. It is seen that, in the entire frequency range, the surface temperature

Table 2. Amplitude and phase of the fluctuating part of the surface temperature for $Pr = 1.0$ and $n = 1.0$

ξ	A_t		$-\phi_t$	
	Series and asymp.	Keller	Series and asymp.	Keller
.00	1.2116[a]	1.2117	.0000[a]	.0000
.10	1.2088[a]	1.2112	3.1757[a]	3.1871
.20	1.2005[a]	1.2004	6.3120[a]	6.3427
.30	1.1871[a]	1.1892	9.3721[a]	9.3910
.40	1.1693[a]	1.1688	12.3257[a]	12.3656
.50	1.1476[a]	1.1493	15.1529[a]	15.1577
.60	1.1227[a]	1.1221	17.8505[a]	17.8635
.70	1.0951[a]	1.0971	20.4396[a]	20.3317
.80	1.0646[a]	1.0658	22.9761[a]	22.7117
.90	1.0299[a]	1.0377	25.5645[a]	24.8090
1.00	.9881[a]	1.0049	28.3801[a]	26.8298
1.10	.9343[a]	.9763	31.7156[a]	28.5418
1.20	.8608[a]	.9441	36.0979[a]	30.2082
1.30	.7578[a]	.9169	42.6369[a]	31.5594
2.00		.7461		33.9817
3.00	.8186[b]	.5999	29.9148[b]	34.8627
4.00	.6313[b]	.5134	34.0575[b]	35.0530
5.00	.5295[b]	.4557	36.6715[b]	36.6152
6.00	.4646[b]	.4140	38.4168[b]	38.4168
7.00	.4189[b]	.3820	39.6403[b]	39.6398
8.00	.3847[b]	.3565	40.5332[b]	40.5333
9.00	.3578[b]	.3355	41.2065[b]	41.2094
10.00	.3359[b]	.3178	41.7281[b]	41.7287
15.00	.2669[b]	.2585	43.1700[b]	43.1702
20.00	.2285[b]	.2234	43.7977[b]	43.7977
25.00	.2031[b]	.1996	44.1345[b]	44.1345
30.00	.1847[b]	.1820	44.3392[b]	44.3392
40.00	.1593[b]	.1574	44.5690[b]	44.5696
50.00	.1422[b]	.1407	44.6909[b]	44.6909
60.00	.1296[b]	.1283	44.7646[b]	44.7646
70.00	.1199[b]	.1187	44.8130[b]	44.8134
75.00	.1158[b]	.1147	44.8314[b]	44.8314

[a] Series solution for low frequency
[b] Asymptotic solution for high frequency

oscillations always lead that of the fluctuating surface temperature. The phase angles, ϕ_u and ϕ_t, are zero under quasi-steady conditions, and they decrease monotonically towards the respective asymptotic values $-90°$ and $-45°$ respectively as $\xi \to \infty$. We further observe that the phase angle decreases in the low-frequency range as the value of the surface temperature-gradient decreases for given Prandtl number (which can be seen from Figs. 2 and 4).

From Figs. 1 and 3 it can be seen that, in the low-frequency regime, the amplitudes of the shear stress and the fluctuating surface temperature decrease when the exponent of the surface heat-flux increases. Similarly, from Figs. 2 and 4 it can be seen that the phase angle increases in the low-frequency range as the value of the exponent increases, while the value of Pr is kept constant.

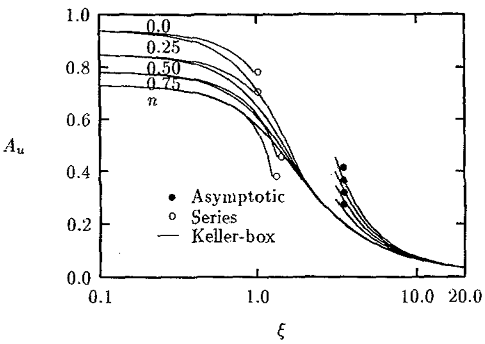

Fig. 1. Amplitude of fluctuating shear-stress for different values of n while $Pr = 0.7$

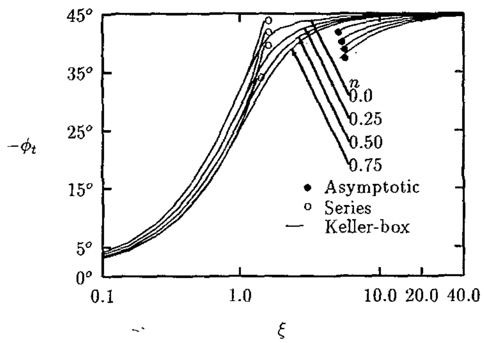

Fig. 2. Phase of fluctuating shear-stress for different values of n while $Pr = 0.7$

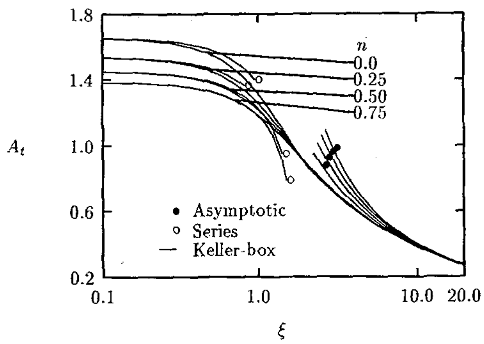

Fig. 3. Amplitude of fluctuating surface temperature for different values of n while $Pr = 0.7$

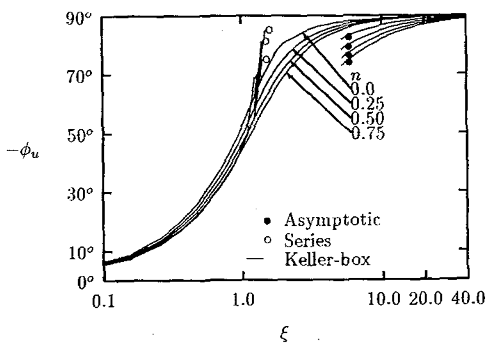

Fig. 4. Phase of fluctuating surface temperature for different values of n while $Pr = 0.7$

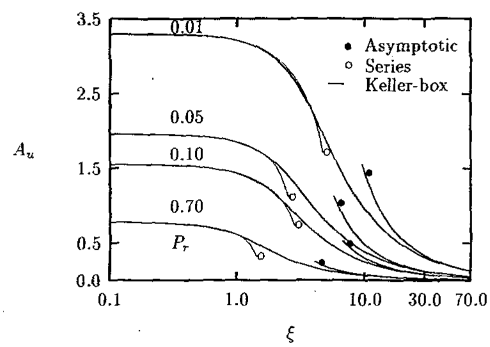

Fig. 5. Amplitude of fluctuating shear-stress for different values of Pr while $n = 0.5$

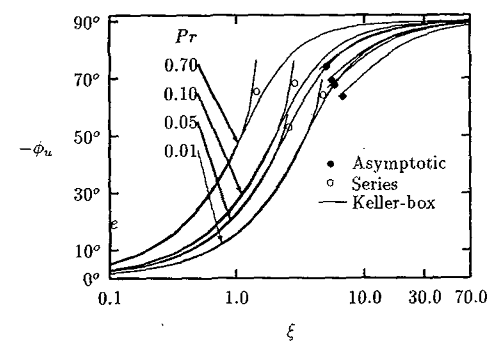

Fig. 6. Phase of fluctuating shear-stress for different values of Pr while $n = 0.5$

From Figs. 5 and 7 it can be seen that, in the low-frequency regime, the amplitudes of the shear stress and the surface temperature decrease as the Prandtl number increases. From Figs. 6 and 8 it may be observed that the phase angle decreases as the Prandtl number increases, while n is fixed at 0.5.

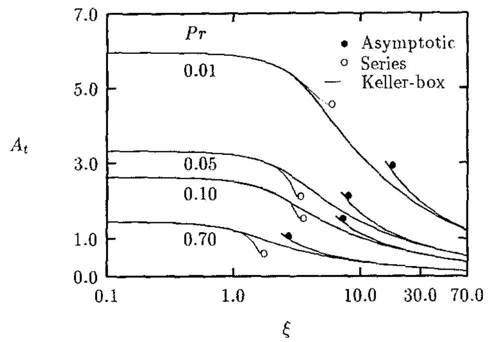

Fig. 7. Amplitude of fluctuating surface temperature for different values of Pr while $n = 0.5$

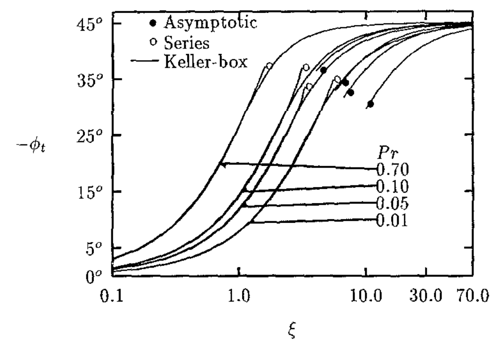

Fig. 8. Phase of fluctuating surface temperature for different values of Pr while $n = 0.5$

6 Conclusions

A linearized theory has been used to study the unsteady response of a laminar free convection boundary layer flow of viscous incompressible fluid along a vertical heated plate to time-periodic surface heat flux oscillations, when the mean surface heat flux varies as a power of x. Three distinct methodologies, namely, a perturbation method for low frequencies, an asymptotic method for high frequencies and the Keller box method for intermediate frequencies, have been used to obtain solutions. Detailed calculations were carried out to obtain numerical values of amplitude and phase of the fluctuating shear stress and the surface temperature for different values of the surface heat-flux exponent n and the Prandtl number Pr. It has been found that the amplitude and phase of the fluctuating shear stress and the surface temperature predicted by these three methods are in very good agreement in the entire frequency range. It may further be concluded that the amplitudes of the shear stress and surface temperature decrease as the frequency increases regardless of the Prandtl number and the value of the surface heat flux exponent. The phase angles of the shear stress and the surface temperature also decrease monotonically towards the respective asymptotic values $-90°$ and $-45°$ as $\xi \to \infty$ regardless of the value of Pr and n.

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